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DIFFERENTIAL SANDWICH THEOREMS FOR HIGHER-ORDER DERIVATIVES OF p-VALENT FUNCTIONS INVOLVING A GENERALIZED DIFFERENTIAL OPERATOR

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In the present article, we obtain some applications of first order differential subordination, superordination and sandwich results for higher-order derivatives of p-valent functions involving a generalized differential operator. Some of our results improve and generalize previously known results.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} \dots \ (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, \dots\}).$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}(p)$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1}$$

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which are $p-$valent in U. We write $\mathcal{A}(1) = \mathcal{A}$.

If $f, g \in H(U)$, we say that f is subordinate to g or g is superordinate to f, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g.,[12], [21] and [22]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h be univalent function in U. If β is analytic function in U and satisfies the first order differential subordination:

$$\phi\left(\beta(z), z\beta'(z); z\right) \prec h(z), \tag{2}$$

then β is a solution of the differential subordination (2). The univalent function q is called a dominant of the solutions of the differential subordination (2) if $\beta(z) \prec q(z)$ for all β satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (2) is called the best dominant. If β and ϕ are univalent functions in U and if satisfies first order differential superordination:

$$h(z) \prec \phi\left(\beta(z), z\beta'(z); z\right), \tag{3}$$

then β is a solution of the differential superordination (3). An analytic function q is called a subordinant of the solutions of the differential superordination (3) if $q(z) \prec \beta(z)$ for all β satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants of (3) is called the best subordinant.

Using the results of Miller and Mocanu [22], Bulboaca [11] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [12]. Ali et al. [1], have used the results of Bulboaca [11] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [30] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}$ in terms of the quantity $\dfrac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [28] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

For functions $f \in \mathcal{A}(p)$ given by (1) and $g \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \tag{4}$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{5}$$

Upon differentiating both sides of (5) $j-$times with respect to z, we have

$$(f * g)^{(j)}(z) = \delta(p;j) z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k;j) a_k b_k z^{k-j}, \tag{6}$$

where

$$\delta(p;j) = \frac{p!}{(p-j)!} \; (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{7}$$

For functions $f, g \in \mathcal{A}(p)$, Aouf et al. [6] (see also [7]) define the linear operator $D^n_{\lambda,p}(f * g)^{(j)} : \mathcal{A}(p) \to \mathcal{A}(p)$ by

$$\begin{aligned} D^n_{\lambda,p}&(f * g)^{(j)}(z) \\ &= \delta(p;j) z^{p-j} + \sum_{k=p+1}^{\infty} \left(\frac{p-j+\lambda(k-p)}{p-j} \right)^n \delta(k;j) a_k b_k z^{k-j} \\ &\qquad (\lambda \geq 0;\, p > j;\, p \in \mathbb{N};\, j,n \in \mathbb{N}_0;\, z \in U). \end{aligned} \tag{8}$$

From (8), we can easily deduce that

$$\frac{\lambda z}{p-j} \left(D^n_{\lambda,p}(f * g)^{(j)}(z) \right)' = D^{n+1}_{\lambda,p}(f * g)^{(j)}(z) - (1-\lambda) D^n_{\lambda,p}(f * g)^{(j)}(z)$$
$$(\lambda > 0; p > j; p \in \mathbb{N}; n,j \in \mathbb{N}_0; z \in U). \tag{9}$$

We observe that the linear operator $D^n_{\lambda,p}(f * g)^{(j)}(z)$ reduces to several interesting many other linear operators considered earlier for different choices of j, n, λ and the function g:

(i) For $j=0$, $D^n_{\lambda,p}(f*g)^{(j)} = D^n_{\lambda,p}(f*g)$, where the operator $D^n_{\lambda,p}(f*g)$ $(\lambda \geq 0,\ p \in \mathbb{N}, n \in \mathbb{N}_0)$ was introduced and studied by Selvaraj et al. [26] (see also [10]) and $D^n_{\lambda,1}(f*g)(z) = D^n_{\lambda}(f*g)(z)$, where the operator $D^n_{\lambda}(f*g)$ was introduced by Aouf and Mostafa [9];

(ii) For

$$g(z) = \frac{z^p}{1-z} \quad (p \in \mathbb{N}; z \in U) \tag{10}$$

we have $D^n_{\lambda,p}(f*g)^{(j)}(z) = D^n_{\lambda,p} f^{(j)}(z)$, $D^n_{\lambda,p} f^{(0)}(z) = D^n_{\lambda,p} f(z)$, where the operator $D^n_{\lambda,p}$ is the $p-$valent Al-Oboudi operator which was introduced by El-Ashwah and Aouf [17], $D^n_{1,p} f^{(j)}(z) = D^n_p f^{(j)}(z)$, where the operator $D^n_p f^{(j)}$ $(p > j, p \in \mathbb{N}, n, j \in \mathbb{N}_0)$ was introduced and studied by Aouf [3,4] and $D^n_{1,p} f^{(0)}$ $= D^n_p f$, where the operator D^n_p is the $p-$valent Sălăgean operator which was introduced and studied by Kamali and Orhan [18] (see also [8]);

(iii) For

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \ldots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \ldots (\beta_s)_{k-p}} \frac{z^k}{(1)_{k-p}} \qquad (z \in U), \tag{11}$$

(for complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ $(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$, $j = 1, \ldots, s)$; $q \leq s+1; p \in \mathbb{N}$; $q,\ s \in \mathbb{N}_0)$ where $(\nu)_k$ is the Pochhammer symbol defined in terms to the Gamma function Γ, by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & (k=0), \\ \nu(\nu+1)(\nu+2)\ldots(\nu+k-1) & (k \in \mathbb{N}). \end{cases}$$

we have $D^n_{\lambda,p}(f*g)^{(j)}(z) = D^n_{\lambda,p}(H_{p,q,s}(\alpha_1)f)^{(j)}(z)$ and $D^0_{\lambda,p}(f*g)^{(0)}(z) = H_{p,q,s}(\alpha_1)f(z)$, where the operator $H_{p,q,s}(\alpha_1)$ is the Dziok-Srivastava operator which was introduced and studied by Dziok and Srivastava [15,16] and which contains in turn many interesting operators such as, $H_{1,2,1}(a,1;c) = L(a,c)$, where the operator $L(a,c)$ was introduced by Carlson and Shaffer [13];

(iv) For

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l+\alpha(k-p)}{p+l} \right)^m z^k \tag{12}$$

$$(\alpha \geq 0;\ l \geq 0;\ p \in \mathbb{N};\ m \in \mathbb{N}_0; z \in U),$$

we have $D^n_{\lambda,p}(f*g)^{(j)}(z) = D^n_{\lambda,p}(I_p(m,\alpha,l)f)^{(j)}(z)$ and $D^0_{\lambda,p}(f*g)^{(0)}(z) = I_p(m,\alpha,l)f(z)$, where the operator $I_p(m,\alpha,l)$ was introduced and studied by Cătas [14] which contains in turn many interesting operators such as, $I_p(m,1,l)$ $= I_p(m,l)$, where the operator $I_p(m,l)$ was investigated by Kumar et al. [19];

(v) For

$$g(z)=z^{p}+\frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)}\sum_{k=p+1}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)}z^{k} \tag{13}$$

$$(\ \alpha\geq 0;\ p\in\mathbb{N};\ \beta>-1;z\in U\ )$$

we have $D_{\lambda,p}^{n}(f*g)^{(j)}=D_{\lambda,p}^{n}\left(Q_{\beta,p}^{\alpha}f\right)^{(j)}$ and $D_{\lambda,p}^{0}(f*g)^{(0)}=Q_{\beta,p}^{\alpha}f$, where the operator $Q_{\beta,p}^{\alpha}$ was introduced and studied by Liu and Owa [20];

(vi) For $j=0$ and g of the form (11) with $p=1$, we have $D_{\lambda,1}^{n}(f*g)(z)=D_{\lambda}^{n}(\alpha_{1},\ldots,\alpha_{q};\beta_{1},\ldots,\beta_{s})(z)$, where the operator $D_{\lambda}^{n}(\alpha_{1},\ldots,\alpha_{q};\beta_{1},\ldots,\beta_{s})$ was introduced and studied by Selvaraj and Karthikeyan [25];

(vii) For $j=0$, $p=1$ and

$$g(z)=z+\sum_{k=2}^{\infty}\left[\frac{\Gamma(k+1)\Gamma(2-m)}{\Gamma(k+1-m)}\right]^{n}z^{k}$$

$$(n\in\mathbb{N}_{0};0\leq\ m<1;z\in U\ )$$

we have $D_{\lambda,1}^{n}(f*g)(z)=D_{\lambda}^{n,m}f(z)$, where the operator $D_{\lambda}^{n,m}$ was introduced and studied by Al-Oboudi and Al-Amoudi [2].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator $D_{\lambda,p}^{n}(f*g)^{(j)}$.

2. Definitions and preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 2.1 ([22])**.** Denote by Q, the set of all functions f that are analytic and injective on $\overline{U}\backslash E(f)$, where

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and are such that $f^{'}(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(f)$.

Lemma 2.2 ([22])**.** *Let q be univalent in U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w)\neq 0$ when $w\in q(U)$. Set*

$$\psi(z)=zq^{'}(z)\,\varphi(q(z))\quad\textit{and}\quad h(z)=\theta(q(z))+\psi(z). \tag{14}$$

Suppose that

(i) $\psi(z)$ is starlike univalent in U,

(ii) $\Re\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ *for* $z \in U$.

If β *is analytic with* $\beta(0) = q(0)$, $\beta(U) \subset D$ *and*

$$\theta(\beta(z)) + z\beta'(z)\varphi(\beta(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{15}$$

then $\beta(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 2.3 ([11]). *Let* q *be convex univalent in* U *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$. *Suppose that* (i) $\Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$, (ii) $\Psi(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U. *If* $\beta(z) \in H[q(0),1] \cap Q$, *with* $\beta(U) \subseteq D$, *and* $\theta(\beta(z)) + z\beta'(z)\phi(\beta(z))$ *is univalent in* U *and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(\beta(z)) + zp'(z)\phi(\beta(z)), \tag{16}$$

then $q(z) \prec \beta(z)$ *and* q *is the best subordinant.*

Lemma 2.4 ([24]). *The function* $q(z) = (1-z)^{-2ab}$ $(a, b \in \mathbb{C}^* (\mathbb{C}\setminus\{0\}))$ *is univalent in* U *if and only if* $|2ab-1| \leq 1$ *or* $|2ab+1| \leq 1$.

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that η, $\gamma_i \in \mathbb{C}$ $(i = 1,2,3)$, γ_4, $\mu \in \mathbb{C}^*$, $\lambda > 0$, $\delta(p; j)$ is given by (7), $p > j$, $p \in \mathbb{N}$, $n, j \in \mathbb{N}_0$ and the powers are understood as the principle values.

Theorem 3.1. *Let* $q(z)$ *be univalent in* U *with* $q(0) = 1$, $q(z) \neq 1$ *and let* $\frac{zq'(z)}{q(z)}$ *be starlike in* U. *Let* $f \in \mathcal{A}(p)$ *and assume that* f *and* q *satisfy the following conditions:*

$$\left[\frac{D^n_{\lambda,p}(f * g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p}(f * g_2)^{(j)}(z)}\right]^{\eta} \neq 0, \tag{17}$$

and

$$\Re\left\{1 + \frac{\gamma_2}{\gamma_4}q(z) + \frac{2\gamma_3}{\gamma_4}[q(z)]^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in U). \tag{18}$$

If

$$\gamma_1+\gamma_2\left[\frac{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]^{\eta}$$
$$+\gamma_3\left[\frac{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{2\mu}\left[\frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]^{2\eta}$$
$$+\gamma_4\mu\left(\frac{p-j}{\lambda}\right)\left[\frac{D_{\lambda,p}^{n+1}(f*g_1)^{(j)}(z)}{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}-1\right]$$
$$+\gamma_4\eta\left(\frac{p-j}{\lambda}\right)\left[1-\frac{D_{\lambda,p}^{n+1}(f*g_2)^{(j)}(z)}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]$$

$$\prec\gamma_1+\gamma_2 q(z)+\gamma_3[q(z)]^2+\gamma_4\frac{zq'(z)}{q(z)}, \tag{19}$$

then

$$\left[\frac{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]^{\eta}\prec q(z) \tag{20}$$

and $q(z)$ is the best dominant.

Proof. Define a function ρ by

$$\rho(z)=\left[\frac{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]^{\eta}\ (z\in U). \tag{21}$$

Then the function ρ is analytic in U and $\rho(0)=1$. Therefore, differentiating (21) logarithmically with respect to z and using the identity (9) in the resulting equation, we have

$$\gamma_1+\gamma_2\left[\frac{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]^{\eta}$$
$$+\gamma_3\left[\frac{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{2\mu}\left[\frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]^{2\eta}+$$

$$+\gamma_4\mu\left(\frac{p-j}{\lambda}\right)\left[\frac{D_{\lambda,p}^{n+1}(f*g_1)^{(j)}(z)}{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}-1\right]$$
$$+\gamma_4\eta\left(\frac{p-j}{\lambda}\right)\left[1-\frac{D_{\lambda,p}^{n+1}(f*g_2)^{(j)}(z)}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]$$
$$=\gamma_1+\gamma_2\rho(z)+\gamma_3[\rho(z)]^2+\gamma_4\frac{z\rho'(z)}{\rho(z)},$$

that is,

$$\gamma_1+\gamma_2\rho(z)+\gamma_3[\rho(z)]^2+\gamma_4\frac{z\rho'(z)}{\rho(z)} \prec \gamma_1+\gamma_2 q(z)+\gamma_3[q(z)]^2+\gamma_4\frac{zq'(z)}{q(z)}.$$

By setting

$$\theta(w)=\gamma_1+\gamma_2 w+\gamma_3 w^2 \text{ and } \varphi(w)=\frac{\gamma_4}{w},$$

it can be easily observed that θ is analytic function in $\mathbb{C}$, φ is analytic function in $\mathbb{C}^*$ and $\varphi(w)\neq 0$. Also we see that

$$\psi(z)=zq'(z)\varphi(q(z))=\gamma_4\frac{zq'(z)}{q(z)}$$

and

$$h(z)=\theta(q(z))+\psi(z)=\gamma_1+\gamma_2 q(z)+\gamma_3[q(z)]^2+\gamma_4\frac{zq'(z)}{q(z)},$$

it is clear that $\psi(z)$ is starlike univalent in U and

$$\Re\left\{\frac{zh'(z)}{\psi(z)}\right\}=\Re\left\{1+\frac{\gamma_2}{\gamma_4}q(z)+\frac{2\gamma_3}{\gamma_4}[q(z)]^2-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}>0$$

Therefore, Theorem 3.1 now follows by applying Lemma 2.2. □

Taking $p=\lambda=1$, $n=j=0$, $g_1=z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k$, $g_2=z+\sum_{k=2}^{\infty}\frac{(a+1)_{k-1}}{(c)_{k-1}}z^k$, and $c\notin\mathbb{Z}_0^-$ in Theorem 3.1, we obtain the following corollary which improves the result of Shanmugam et al. [27, Theorem 3.1].

Corollary 3.2. . *Let $q(z)$ be univalent in U with $q(0)=1$, $q(z)\neq 1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. Let $f\in\mathcal{A}(p)$ such that*

$$\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\neq 0, \tag{22}$$

and suppose q satisfies (18). If

$$\begin{aligned}
&\gamma_1+\gamma_2\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\\
&+\gamma_3\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta}\\
&+\gamma_4\mu a\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-1\right]+\gamma_4\eta(a+1)\left[1-\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\\
&\prec \gamma_1+\gamma_2 q(z)+\gamma_3[q(z)]^2+\gamma_4\frac{zq'(z)}{q(z)},
\end{aligned}$$

then

$$\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\prec q(z)$$

and $q(z)$ is the best dominant.

Putting $q(z)=\frac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$ in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.2].

Corollary 3.3. *Let $-1\leq B<A\leq 1$ and assume that*

$$\Re\left\{\frac{\gamma_2}{\gamma_4}\left[\frac{1+Az}{1+Bz}\right]+\frac{2\gamma_3}{\gamma_4}\left[\frac{1+Az}{1+Bz}\right]^2+\frac{1-ABz^2}{(1+Az)(1+Bz)}\right\}>0 \quad (z\in U),$$

holds. If $f\in\mathcal{A}$ such that (22) holds and satisfy the following subordination condition:

$$\begin{aligned}
&\gamma_1+\gamma_2\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\\
&+\gamma_3\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta}\\
&+\gamma_4\mu a\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-1\right]+\gamma_4\eta(a+1)\left[1-\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\\
&\prec \gamma_1+\gamma_2\left[\frac{1+Az}{1+Bz}\right]+\gamma_3\left[\frac{1+Az}{1+Bz}\right]^2+\frac{\gamma_4(A-B)z}{(1+Az)(1+Bz)},
\end{aligned}$$

then

$$\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\prec\frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^{\vartheta}$ $(0 < \vartheta \leq 1)$ in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.3].

Corollary 3.4. *Assume that*

$$\Re\left\{\frac{\gamma_2}{\gamma_4}\left(\frac{1+z}{1-z}\right)^{\vartheta} + \frac{2\gamma_3}{\gamma_4}\left(\frac{1+z}{1-z}\right)^{2\vartheta} + \frac{1-3z^2}{1-z^2}\right\} > 0 \quad (z \in U),$$

holds. If $f \in \mathcal{A}$ such that (22) holds and satisfy the following subordination condition:

$$\begin{aligned}&\gamma_1 + \gamma_2\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\\&+\gamma_3\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta}\\&+\gamma_4\mu a\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right] + \gamma_4\eta\,(a+1)\left[1 - \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\\&\prec \gamma_1 + \gamma_2\left(\frac{1+z}{1-z}\right)^{\vartheta} + \gamma_3\left(\frac{1+z}{1-z}\right)^{2\vartheta} + \frac{2\gamma_4\vartheta z}{(1-z)^2},\end{aligned}$$

then

$$\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta} \prec \left(\frac{1+z}{1-z}\right)^{\vartheta}$$

and the function $\left(\frac{1+z}{1-z}\right)^{\vartheta}$ is the best dominant.

Putting $q(z) = e^{\mu Az}$ $(|\mu A| < \pi)$ in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.4].

Corollary 3.5. *Assume that*

$$\Re\left\{1 + \frac{\gamma_2}{\gamma_4}e^{\mu Az} + \frac{2\gamma_3}{\gamma_4}e^{2\mu Az}\right\} > 0 \quad (z \in U),$$

holds. If $f \in \mathcal{A}$ such that (22) holds and satisfy the following subordination condition:

$$\begin{aligned}&\gamma_1 + \gamma_2\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\\&+\gamma_3\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta}\\&+\gamma_4\mu a\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right] + \gamma_4\eta\,(a+1)\left[1 - \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\\&\prec \gamma_1 + \gamma_2 e^{\mu Az} + \gamma_3 e^{2\mu Az} + \gamma_4\mu Az,\end{aligned}$$

then

$$\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\prec e^{\mu Az}$$

and the function $e^{\mu Az}$ *is the best dominant.*

Taking $\gamma_1=p=\lambda=1$, $\gamma_2=\gamma_3=n=j=\eta=0$, $g_1=z+\sum\limits_{k=2}^{\infty}z^k$, $q(z)=\frac{1}{(1-z)^{2ab}}$ $(a,b\in\mathbb{C}^*)$, $\mu=a$ and $\gamma_4=\frac{1}{ab}$ in Theorem 3.1, then combining this to gather with Lemma 2.4 we obtain the following corollary obtained by Obradovič et al. [23, Theorem 1].

Corollary 3.6. *Let* $a,b\in\mathbb{C}^*$ *such that* $|2ab-1|\le 1$ *or* $|2ab+1|\le 1$. *Let* $f\in\mathcal{A}$ *and suppose that* $\frac{f(z)}{z}\ne 0$ $(z\in U)$. *If*

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z}\right)^{a}\prec\frac{1}{(1-z)^{2ab}}$$

and the function $\frac{1}{(1-z)^{2ab}}$ *is the best dominant.*

Remark 3.7. For $a=1$, Corollary 3.6 reduces to the recent result obtained by Srivastava and Lashin [29, Theorem 3].

Taking $\gamma_1=p=\lambda=1$, $\gamma_2=\gamma_3=n=j=\eta=0$, $g_1=z+\sum\limits_{k=2}^{\infty}z^k$, $q(z)=(1-z)^{-2ab\cos\tau e^{-i\tau}}$ $\left(a,b\in\mathbb{C}^*,|\tau|<\frac{\pi}{2}\right)$, $\mu=a$ and $\gamma_4=\frac{e^{i\tau}}{ab\cos\tau}$ in Theorem 3.1, then combining this to gather with Lemma 2.4 we obtain the following corollary obtained by Aouf et al. [5, Theorem 1].

Corollary 3.8. *Let* $a,b\in\mathbb{C}^*,|\tau|<\frac{\pi}{2}$ *and suppose that* $\left|2ab\cos\tau e^{-i\tau}-1\right|\le 1$ *or* $\left|2ab\cos\tau e^{-i\tau}+1\right|\le 1$. *Let* $f\in\mathcal{A}$ *and suppose that* $\frac{f(z)}{z}\ne 0$ $(z\in U)$. *If*

$$1+\frac{e^{i\tau}}{b\cos\tau}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z}\right)^{a}\prec(1-z)^{-2ab\cos\tau e^{-i\tau}}$$

and the function $(1-z)^{-2ab\cos\tau e^{-i\tau}}$ *is the best dominant.*

Theorem 3.9. *Let q be convex univalent in U with $q(0)=1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. Further assume that*

$$\Re\left(\left(\gamma_2+2\gamma_3 q(z)\right)\frac{q(z)\,q'(z)}{\gamma_4}\right)>0. \tag{23}$$

Let $f\in\mathcal{A}(p)$ such that

$$0\neq\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)\,z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{\eta}\in H[q(0),1]\cap Q. \tag{24}$$

If

$$\begin{aligned}&\gamma_1+\gamma_2\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)\,z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{\eta}\\&+\gamma_3\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right]^{2\mu}\left[\frac{\delta(p;j)\,z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{2\eta}\\&+\gamma_4\mu\left(\frac{p-j}{\lambda}\right)\left[\frac{D^{n+1}_{\lambda,p}(f*g_1)^{(j)}(z)}{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}-1\right]\\&+\gamma_4\eta\left(\frac{p-j}{\lambda}\right)\left[1-\frac{D^{n+1}_{\lambda,p}(f*g_2)^{(j)}(z)}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]\end{aligned} \tag{25}$$

is univalent in U and satisfies the following superordination condition

$$\begin{aligned}&\gamma_1+\gamma_2 q(z)+\gamma_3[q(z)]^2+\gamma_4\frac{zq'(z)}{q(z)}\\&\prec\gamma_1+\gamma_2\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)\,z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{\eta}\\&+\gamma_3\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right]^{2\mu}\left[\frac{\delta(p;j)\,z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{2\eta}\\&+\gamma_4\mu\left(\frac{p-j}{\lambda}\right)\left[\frac{D^{n+1}_{\lambda,p}(f*g_1)^{(j)}(z)}{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}-1\right]\end{aligned}$$

$$+\gamma_4\eta\left(\frac{p-j}{\lambda}\right)\left[1-\frac{D_{\lambda,p}^{n+1}(f*g_2)^{(j)}(z)}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right], \tag{26}$$

holds, then

$$q(z)\prec\left[\frac{D_{\lambda,p}^{n}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^{n}(f*g_2)^{(j)}(z)}\right]^{\eta} \tag{27}$$

and q is the best subordinant.

Proof. By setting

$$\theta(w)=\gamma_1+\gamma_2 w+\gamma_3 w^2 \text{ and } \varphi(w)=\frac{\gamma_4}{w},$$

it can be easily observed that θ is analytic function in $\mathbb{C}$, φ is analytic function in $\mathbb{C}^*$ and $\varphi(w)\neq 0$. From the assumption of Theorem 3.9, we see that

$$\Re\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\}=\Re\left((\gamma_2+2\gamma_3 q(z))\frac{q(z)q'(z)}{\gamma_4}\right)>0 \text{ for } z\in U,$$

Therefore, Theorem 3.9 now follows by applying Lemma 2.3. □

Taking $p=\lambda=1$, $n=j=0, g_1=z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k$, $g_2=z+\sum_{k=2}^{\infty}\frac{(a+1)_{k-1}}{(c)_{k-1}}z^k$, and $c\notin\mathbb{Z}_0^-$ in Theorem 3.9, we obtain the following corollary which improves the result of Shanmugam et al. [27, Theorem 3.11].

Corollary 3.10. *Let q be convex univalent in U with $q(0)=1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. Further assume that (23) holds. Let $f\in\mathcal{A}(p)$ such that*

$$0\neq\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\in H[q(0),1]\cap Q.$$

If

$$\begin{aligned}&\gamma_1+\gamma_2\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\\&+\gamma_3\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta}\\&+\gamma_4\mu a\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-1\right]+\gamma_4\eta(a+1)\left[1-\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\end{aligned}$$

is univalent in U and satisfies the following superordination condition

$$\begin{aligned}
&\gamma_1+\gamma_2 q(z)+\gamma_3 [q(z)]^2+\gamma_4 \frac{zq'(z)}{q(z)} \\
&\prec \gamma_1+\gamma_2\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta} \\
&+\gamma_3\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta} \\
&+\gamma_4\mu a\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-1\right]+\gamma_4\eta(a+1)\left[1-\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right],
\end{aligned}$$

holds, then

$$q(z)\prec\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}$$

and q is the best subordinant.

Combining Theorems 3.1 and 3.9, we get the following sandwich theorem for the linear operator $D^n_{\lambda,p}(f*g)^{(j)}$.

Theorem 3.11. *Let q_i be convex univalent in U with $q_i(0)=1$ and let $\frac{zq_i'(z)}{q_i(z)}$ $(i=1,2)$ be starlike in U. Suppose that q_1 satisfies (23) and q_2 satisfies (18). Let $f\in\mathcal{A}(p)$ such that*

$$0\neq\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{\eta}\in H[q(0),1]\cap Q. \quad (28)$$

If

$$\begin{aligned}
&\gamma_1+\gamma_2\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{\eta} \\
&+\gamma_3\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{2\mu}\left[\frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{2\eta} \\
&+\gamma_4\mu\left(\frac{p-j}{\lambda}\right)\left[\frac{D^{n+1}_{\lambda,p}(f*g_1)^{(j)}(z)}{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}-1\right] \\
&+\gamma_4\eta\left(\frac{p-j}{\lambda}\right)\left[1-\frac{D^{n+1}_{\lambda,p}(f*g_2)^{(j)}(z)}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right] \quad (29)
\end{aligned}$$

is univalent in U and

$$\gamma_1+\gamma_2 q_1(z)+\gamma_3[q_1(z)]^2+\gamma_4\frac{zq_1'(z)}{q_1(z)}$$
$$\prec \gamma_1+\gamma_2\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{\eta}$$
$$+\gamma_3\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{2\mu}\left[\frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{2\eta}$$
$$+\gamma_4\mu\left(\frac{p-j}{\lambda}\right)\left[\frac{D^{n+1}_{\lambda,p}(f*g_1)^{(j)}(z)}{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}-1\right]$$
$$+\gamma_4\eta\left(\frac{p-j}{\lambda}\right)\left[1-\frac{D^{n+1}_{\lambda,p}(f*g_2)^{(j)}(z)}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]$$
$$\prec \gamma_1+\gamma_2 q_2(z)+\gamma_3[q_2(z)]^2+\gamma_4\frac{zq_2'(z)}{q_2(z)} \tag{30}$$

holds, then

$$q_1(z)\prec\left[\frac{D^n_{\lambda,p}(f*g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}}\right]^{\mu}\left[\frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p}(f*g_2)^{(j)}(z)}\right]^{\eta}\prec q_2(z) \tag{31}$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Taking $p=\lambda=1$, $n=j=0, g_1=z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k$, $g_2=z+\sum_{k=2}^{\infty}\frac{(a+1)_{k-1}}{(c)_{k-1}}z^k$, and $c\notin\mathbb{Z}_0^-$ in Theorem 3.11, we obtain the following corollary which improves the result of Shanmugam et al. [27, Theorem 3.12].

Corollary 3.12. *Let q_i be convex univalent in U with $q_i(0)=1$ and $\frac{zq_i'(z)}{q_i(z)}$ be starlike in U for $i=1,2$. Suppose that q_1 satisfies (23) and q_2 satisfies (18). Let $f\in\mathcal{A}(p)$ such that*

$$0\neq\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\in H[q(0),1]\cap Q.$$

If

$$\begin{aligned}&\gamma_1+\gamma_2\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\\&+\gamma_3\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta}\\&+\gamma_4\mu a\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-1\right]+\gamma_4\eta\,(a+1)\left[1-\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\end{aligned}$$

is univalent in U and

$$\begin{aligned}&\gamma_1+\gamma_2 q_1(z)+\gamma_3\left[q_1(z)\right]^2+\gamma_4\frac{zq_1'(z)}{q_1(z)}\\&\prec\gamma_1+\gamma_2\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\\&+\gamma_3\left[\frac{L(a,c)f(z)}{z}\right]^{2\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta}\\&+\gamma_4\mu a\left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-1\right]+\gamma_4\eta\,(a+1)\left[1-\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\\&\prec\gamma_1+\gamma_2 q_2(z)+\gamma_3\left[q_2(z)\right]^2+\gamma_4\frac{zq_2'(z)}{q_2(z)},\end{aligned}$$

holds, then

$$q_1(z)\prec\left[\frac{L(a,c)f(z)}{z}\right]^{\mu}\left[\frac{z}{L(a+1,c)f(z)}\right]^{\eta}\prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

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