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FIBONACCI DIFFERENCE SEQUENCE SPACES FOR MODULUS FUNCTIONS

KULDIP RAJ - SURUCHI PANDOH - SEEMA JAMWAL

In the present paper we introduce the Fibonacci difference sequence spaces $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$ by using a sequence of modulus functions and a new band matrix $\hat{F}$. We also make an effort to study some inclusion relations, topological and geometric properties of these spaces. Furthermore, the $\alpha-,\beta-,\gamma-$ duals and matrix transformation of the space $l(\hat{F},\mathcal{F},p,u)$ are determined.

1. Introduction and Preliminaries

Let w be the space of all real or complex-valued sequences. By l_∞, c, c_0 and l_p $(1 \leq p < \infty)$, we denote the sets of all bounded, convergent, null sequences and p-absolutely convergent series, respectively. The notion of difference sequence spaces was introduced by Kızmaz [18], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [11] by introducing the spaces $l_\infty(\Delta^m), c(\Delta^m)$ and $c_0(\Delta^m)$. In 1981, Kızmaz [18] defined the sequence spaces

$$X(\Delta) = \{x = (x_k) \in w : (x_k - x_{k+1}) \in X\}$$

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for $X = l_\infty, c$ and c_0. The difference space bv_p consisting of all sequences (x_k) such that $(x_k - x_{k-1})$ is in the sequence space l_p, was studied in the case $0 < p < 1$ by Altay and Başar [4] and in the case $1 \le p \le \infty$ by Başar and Altay [7] and Çolak et al. [9]. The paranormed difference sequence space

$$\Delta\lambda(p) = \{x = (x_k) \in w : (x_k - x_{k+1}) \in \lambda(p)\}$$

was examined by Ahmad and Mursaleen [6] and Malkowsky [21], where $\lambda(p)$ is any of the paranormed spaces $l_\infty(p)$, $c(p)$ and $c_0(p)$ defined by Simons [29] and Maddox [22]. Recently, Altay et al. [5] have defined the sequence spaces $bv(u,p)$ and $bv_\infty(u,p)$ by

$$bv(u,p) = \Big\{x = (x_k) \in w : \sum_k |u_k(x_k - x_{k-1})|^{p_k} < \infty\Big\}$$

and

$$bv_\infty(u,p) = \Big\{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |u_k(x_k - x_{k-1})|^{p_k} < \infty\Big\},$$

where $u = (u_k)$ is an arbitrary fixed sequence and $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. These spaces are generalization of the space bv_p for $1 \le p \le \infty$.

Definition 1.1. A modulus function is a function $f : [0,\infty) \to [0,\infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x+y) \le f(x) + f(y)$, for all $x, y \ge 0$,
3. f is increasing,
4. f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$ then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([2], [3], [23], [24], [28]) and references therein.

Definition 1.2. Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

$(P1)$ $p(x) \ge 0$ for all $x \in X$,

$(P2)$ $p(-x) = p(x)$ for all $x \in X$,

$(P3)$ $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$,

$(P4)$ if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [30, Theorem 10.4.2, p. 183]).

Definition 1.3. Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$ and the A-transform of x is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1}$$

By (X, Y) we denote the class of all matrices A such that $A : X \to Y$. Thus $A \in (X, Y)$ if and only if the series on the right-hand side of (1) converges for each $n \in \mathbb{N}$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$.

Definition 1.4. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{2}$$

which is a sequence space.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors (see [1], [19], [26]). In [17] Kara introduce the Fibonacci difference sequence spaces $l_p(\hat{F})$ and $l_\infty(\hat{F})$ as

$$l_p(\hat{F}) = \left\{ x = (x_n) \in w : \sum_n \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^p < \infty \right\}, \quad 1 \leq p < \infty,$$

and

$$l_\infty(\hat{F}) = \left\{ x = (x_n) \in w : \sup_{n \in \mathbb{N}} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right| < \infty \right\}.$$

The sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers is given by the linear recurrence relations $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$. Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also, in [14] some basic properties of Fibonacci numbers are given as follows:

$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha$ (golden ratio),

$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad (n \in \mathbb{N}),$

$\sum_k \frac{1}{f_k}$ converges,

$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1)$ (Cassini formula).

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$. Let f_n be the nth Fibonacci number for every $n \in \mathbb{N}$. Then we define the infinite matrix $\hat{F} = (\hat{f}_{nk})$ by

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & (k = n-1), \\ \frac{f_n}{f_{n+1}} & (k = n), \\ 0 & (0 \leq k < n-1 \text{ or } k > n) \end{cases}$$

where $n,k \in \mathbb{N}$ (see [17]).

Define the sequence $y = (y_n)$ by the $\hat{F}$ transform of a sequence $x = (x_n)$, i.e.,

$$y_n = \hat{F}_n(x) = \begin{cases} \frac{f_0}{f_1}x_0 = x_0 & (n = 0), \\ \frac{f_n}{f_{n+1}}x_n - \frac{f_{n+1}}{f_n}x_{n-1} & (n \geq 1) \end{cases} \quad (n \in \mathbb{N}). \tag{3}$$

Definition 1.5. A sequence space X with a linear topology is called a K-space, provided each of the maps $p_n : X \to \mathbb{R}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$. A K-space X is called an FK-space provided X is complete linear metric space. An FK-space whose topology is normable is called a BK-space. The space $l_p (1 \leq p < \infty)$ is a BK-space with $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{\frac{1}{p}}$ and c_0, c and l_∞ are BK-spaces with $\|x\|_\infty = \sup_k |x_k|$.

Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. $\hat{F} = (\hat{f}_{nk})$ denotes a Fibonacci band matrix and f_k is the kth Fibonacci number for every $k \in \mathbb{N}$. In this paper we define the following sequence spaces:

$$l(\hat{F}, \mathcal{F}, p, u) = \Big\{x = (x_k) \in w : \sum_k \Big[u_k F_k\Big(\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\Big)\Big]^{p_k} < \infty\Big\},$$

and

$$l_\infty(\hat{F},\mathcal{F},p,u)=\Big\{x=(x_k)\in w:\sup_{k\in\mathbb{N}}\left[u_kF_k\left(\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right)\right]^{p_k}<\infty\Big\}.$$

With the notation of (2), the sequence spaces $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$ may be redefined as follows

$$l(\hat{F},\mathcal{F},p,u)=\{l(\mathcal{F},p,u)\}_{\hat{F}}\ \ (1\le p<\infty)\ \text{ and }\ l_\infty(\hat{F},\mathcal{F},p,u)=\{l_\infty(\mathcal{F},p,u)\}_{\hat{F}}. \tag{4}$$

The following inequality will be used throughout the paper. Let $p=(p_k)$ be a sequence of positive real numbers with $0<p_k\le \sup_k p_k=H$, and let $D=\max\{1,2^{H-1}\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$|a_k+b_k|^{p_k}\le D(|a_k|^{p_k}+|b_k|^{p_k}). \tag{5}$$

In this paper, we define Fibonacci difference sequence spaces defined by a sequence of modulus functions and Fibonacci matrix $\hat{F}$. We investigate some topological properties of these new sequence spaces and establish some inclusion relations concerning these spaces. Also we determine the $\alpha-,\beta-$ and $\gamma-$duals of the space $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$ in third section of this paper. In the fourth section of the paper we construct the matrix transformation of the space $(l(\hat{F},\mathcal{F},p,u),X)$ and $(l_\infty(\hat{F},\mathcal{F},p,u),X)$, where $1\le p<\infty$ and X is any of the spaces l_∞, l_1, c and c_0. In the last section, we characterize some geometric properties of the space $l(\hat{F},\mathcal{F},p,u)$.

2. Some topological properties of the spaces $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$

Theorem 2.1. *Let $\mathcal{F}=(F_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. Let $x,y\in l(\hat{F},\mathcal{F},p,u)$. Then

$$\sum_k\left[u_kF_k\left(\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right)\right]^{p_k}<\infty$$

and

$$\sum_k\left[u_kF_k\left(\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right)\right]^{p_k}<\infty.$$

For $\lambda,\mu \in \mathbb{C}$, there exist integers M_λ and N_μ such that $|\lambda| \le M_\lambda$ and $|\mu| \le N_\mu$. Using inequality (5) and definition of modulus function, we have

$$\sum_k \left[u_k F_k\left(\left|\lambda\Big(\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big) + \mu\Big(\frac{f_k}{f_{k+1}}y_k - \frac{f_{k+1}}{f_k}y_{k-1}\Big)\right|\right)\right]^{p_k}$$
$$\le \sum_k \left[u_k F_k\left(|\lambda|\left|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\right|\right)\right]^{p_k}$$
$$+\sum_k \left[u_k F_k\left(|\mu|\left|\frac{f_k}{f_{k+1}}y_k - \frac{f_{k+1}}{f_k}y_{k-1}\right|\right)\right]^{p_k}$$
$$\le DM_\lambda^H \sum_k \left[u_k F_k\left(\left|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\right|\right)\right]^{p_k}$$
$$+DN_\mu^H \sum_k \left[u_k F_k\left(\left|\frac{f_k}{f_{k+1}}y_k - \frac{f_{k+1}}{f_k}y_{k-1}\right|\right)\right]^{p_k}$$
$$< \infty$$

so that $\lambda x + \mu y \in l(\hat{F},\mathcal{F},p,u)$. This proves that $l(\hat{F},\mathcal{F},p,u)$ is a linear space. Similarly we can prove that $l_\infty(\hat{F},\mathcal{F},p,u)$ is a linear space. □

Theorem 2.2. *Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $l(\hat{F},\mathcal{F},p,u)$ is a paranormed space with*

$$g(x) = \sup_k \left(\sum_k \left[u_k F_k\left(\left|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\right|\right)\right]^{p_k}\right)^{\frac{1}{K}}$$

where $0 < p_k \le \sup p_k = H < \infty$ and $K = \max(1,H)$.

Proof. Clearly $g(x) = g(-x)$, for all $x \in l(\hat{F},\mathcal{F},p,u)$. It is trivial that $\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1} = 0$, for $x = 0$. Since $\frac{p_k}{K} \le 1$, using Minkowsky inequality, we have

$$\left(\sum_k \left[u_k F_k\left(\left|\Big(\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big) + \Big(\frac{f_k}{f_{k+1}}y_k - \frac{f_{k+1}}{f_k}y_{k-1}\Big)\right|\right)\right]^{p_k}\right)^{\frac{1}{K}}$$
$$\le \left(\sum_k \left[u_k F_k\left(\left|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\right|\right) + u_k F_k\left(\left|\frac{f_k}{f_{k+1}}y_k - \frac{f_{k+1}}{f_k}y_{k-1}\right|\right)\right]^{p_k}\right)^{\frac{1}{K}}$$
$$\le \left(\sum_k \left[u_k F_k\left(\left|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\right|\right)\right]^{p_k}\right)^{\frac{1}{K}}$$

$$+\left(\sum_k\left[u_kF_k\left(\left|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\right|\right)\right]^{p_k}\right)^{\frac{1}{K}}.$$

Hence $g(x)$ is subadditive. For the continuity of multiplication let us take any complex number α. By definition, we have

$$\begin{aligned}g(\alpha x)&=\sup_k\left(\sum_k\left[u_kF_k\left(\left|\alpha\left(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right)\right|\right)\right]^{p_k}\right)^{\frac{1}{K}}\\&\leq C_\alpha^{\frac{H}{K}}g(x)\end{aligned}$$

where C_α is a positive integer such that $|\alpha|\leq C_\alpha$. Now, Let $\alpha\to 0$ for any fixed x with $g(x)=0$. By definition for $|\alpha|<1$, we have

$$\sum_k\left[u_kF_k\left(\left|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right|\right)\right]^{p_k}<\varepsilon \text{ for } n>n_0(\varepsilon). \tag{6}$$

Also for $1\leq n<n_0$, taking α small enough. Since F_k is continuous, we have

$$\sum_k\left[u_kF_k\left(\left|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right|\right)\right]^{p_k}<\varepsilon. \tag{7}$$

Now from equation (6) and (7), we have

$$g(\alpha x)\to 0 \text{ as } \alpha\to 0.$$

This completes the proof. □

Theorem 2.3. *Let $\mathcal{F}=(F_k)$ be a sequence of modulus functions. If $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers with $0\leq p_k\leq q_k<\infty$ for each k, then $l(\hat{F},\mathcal{F},p,u)\subseteq l(\hat{F},\mathcal{F},q,u)$.*

Proof. Let $x\in l(\hat{F},\mathcal{F},p,u)$. Then

$$\sum_k\left[u_kF_k\left(\left|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right|\right)\right]^{p_k}<\infty.$$

This implies that

$$\left[u_kF_k\left(\left|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right|\right)\right]^{p_k}\leq 1,$$

for sufficiently large values of k (say) $k \geq k_0$, for some fixed $k_0 \in \mathbb{N}$. Since F_k is increasing and $p_k \leq q_k$ we have

$$\sum_{k \geq k_0} \left[u_k F_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{q_k} \leq \sum_{k \geq k_0} \left[u_k F_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k} < \infty.$$

Hence $x \in l(\hat{F}, \mathcal{F}, q, u)$. This completes the proof. □

Theorem 2.4. *Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions and $\beta = \lim\limits_{t \to \infty} \frac{F_k(t)}{t} > 0$. Then $l(\hat{F}, \mathcal{F}, p, u) \subseteq l(\hat{F}, p, u)$.*

Proof. In order to prove that $l(\hat{F}, \mathcal{F}, p, u) \subseteq l(\hat{F}, p, u)$. Let $\beta > 0$. By definition of β, we have $F_k(t) \geq \beta(t)$, for all $t > 0$. Since $\beta > 0$, we have $t \leq \frac{1}{\beta} F_k(t)$ for all $t > 0$.
Let $x = (x_k) \in l(\hat{F}, \mathcal{F}, p, u)$. Thus, we have

$$\sum_k \left[u_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k} \leq \frac{1}{\beta} \sum_k \left[u_k F_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k}$$

which implies that $x = (x_k) \in l(\hat{F}, p, u)$. This completes the proof. □

Theorem 2.5. *Let $\mathcal{F}' = (F'_k)$ and $\mathcal{F}'' = (F''_k)$ are sequences of modulus functions, then*

$$l(\hat{F}, \mathcal{F}', p, u) \cap l(\hat{F}, \mathcal{F}'', p, u) \subseteq l(\hat{F}, \mathcal{F}' + \mathcal{F}'', p, u).$$

Proof. Let $x = (x_k) \in l(\hat{F}, \mathcal{F}', p, u) \cap l(\hat{F}, \mathcal{F}'', p, u)$. Therefore

$$\sum_k \left[u_k F'_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k} < \infty$$

and

$$\sum_k \left[u_k F''_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k} < \infty.$$

Then, we have

$$\begin{aligned} &\sum_k \left[u_k (F'_k + F''_k) \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k} \\ &\leq K \left\{ \sum_k \left[u_k F'_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k} \right\} \\ &+ K \left\{ \sum_k \left[u_k F''_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k} \right\}. \end{aligned}$$

Thus, $\sum_k \left[u_k(F'_k + F''_k)\left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]^{p_k} < \infty.$

Therefore, $x = (x_k) \in l(\hat{F}, \mathcal{F}' + \mathcal{F}'', p, u)$ and this completes the proof. □

Theorem 2.6. *Let $\mathcal{F} = (F_k)$ and $\mathcal{F}' = (F'_k)$ be two sequences of modulus functions, then*

$$l(\hat{F}, \mathcal{F}', p, u) \subseteq l(\hat{F}, \mathcal{F} o \mathcal{F}', p, u).$$

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $F_k(t) < \varepsilon$ for $0 \le t \le \delta$.

Write $y_k = \left[u_k F'_k \left(\left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| \right) \right]$ and consider

$$\sum_k [F_k(y_k)]^{p_k} = \sum_1 [F_k(y_k)]^{p_k} + \sum_2 [F_k(y_k)]^{p_k}$$

where the first summation is over $y_k \le \delta$ and second summation is over $y_k > \delta$. Since F_k is continuous, we have

$$\sum_1 [F_k(y_k)]^{p_k} < \varepsilon^H \tag{8}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}.$$

By the definition, we have for $y_k > \delta$

$$F_k(y_k) < 2F_k(1)\frac{y_k}{\delta}.$$

Hence

$$\sum_2 [F_k(y_k)]^{p_k} \le \max\left(1, (2F_k(1)\delta^{-1})^H\right) \sum_k [y_k]^{p_k}. \tag{9}$$

From equation (8) and (9), we have

$$l(\hat{F}, \mathcal{F}', p, u) \subseteq l(\hat{F}, \mathcal{F} o \mathcal{F}', p, u).$$

This completes the proof. □

Theorem 2.7. *Let $1 \le p_k \le H \le \infty$ for all $k \in \mathbb{N}$. Then $l(\hat{F}, \mathcal{F}, p, u)$ is a BK-space with the norm $\|x\|_{l(\hat{F}, \mathcal{F}, p, u)} = \|\hat{F}x\|_p$, i.e.,*

$$\|x\|_{l(\hat{F}, \mathcal{F}, p, u)} = \left(\sum_n \left[u_k F_k(|\hat{F}_n(x)|) \right]^{p_k} \right)^{\frac{1}{K}}$$

and

$$\|x\|_{l_\infty(\hat{F}, \mathcal{F}, p, u)} = \sup_{n \in \mathbb{N}} \left[u_k F_k(|\hat{F}_n(x)|) \right]^{p_k}.$$

Proof. Since (4) holds, l_p and l_∞ are BK-spaces with respect to their natural norms and the matrix $\hat{F}$ is a triangle; Theorem 4.3.12 of Wilansky [30, p.63] gives the fact that the spaces $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$ are BK-spaces with the given norms, where $1 \leq p_k \leq H < \infty$ for all $k \in \mathbb{N}$. This completes the proof. □

Remark 2.8. One can easily check that the absolute property does not hold on the spaces $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$, that is $\|x\|_{l(\hat{F},\mathcal{F},p,u)} \neq \||x|\|_{l(\hat{F},\mathcal{F},p,u)}$ and $\|x\|_{l_\infty(\hat{F},\mathcal{F},p,u)} \neq \||x|\|_{l_\infty(\hat{F},\mathcal{F},p,u)}$ for at least one sequence in both the spaces $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$; this shows that $l(\hat{F},\mathcal{F},p,u)$ and $l_\infty(\hat{F},\mathcal{F},p,u)$ are the sequence spaces of non-absolute type, where $|x| = (|x_k|)$ and $1 \leq p_k \leq H < \infty$ for all $k \in \mathbb{N}$.

Theorem 2.9. *The sequence space $l(\hat{F},\mathcal{F},p,u)$ of non absolute type is linearly isomorphic to the space l_p, that is, $l(\hat{F},\mathcal{F},p,u) \cong l_p$, for $1 \leq p_k \leq H < \infty$ for all $k \in \mathbb{N}$.*

Proof. It is enough to show that the existence of a linear bijection between the spaces $l(\hat{F},\mathcal{F},p,u)$ and l_p for $1 \leq p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$. Consider the transformation T defined with the notation of (3), from $l(\hat{F},\mathcal{F},p,u)$ to l_p by $x \to y = Tx$. Then $Tx = y = \hat{F}x \in l_p$, for every $x \in l(\hat{F},\mathcal{F},p,u)$. Also, the linearity of T is clear. Further it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.

We assume that $y = (y_k) \in l_p$, for $1 \leq p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j, \;\; k \in \mathbb{N}.$$

Then in the case $1 \leq p_k \leq H < \infty$ for all $k \in \mathbb{N}$ and $p = \infty$, we get

$$\begin{aligned}
\|x\|_{l(\hat{F},\mathcal{F},p,u)} &= \left(\sum_k \left[u_k F_k \left(\Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right) \right]^{p_k} \right)^{\frac{1}{K}} \\
&= \left(\sum_k \left[u_k F_k \left(\Big| \frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j - \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{f_k^2}{f_j f_{j+1}} y_j \Big| \right) \right]^{p_k} \right)^{\frac{1}{K}} \\
&= \left(\sum_k |y_k|^{p_k} \right)^{\frac{1}{K}} \\
&= \|y\|_p < \infty
\end{aligned}$$

and

$$\|x\|_{l_\infty(\hat{F},\mathcal{F},p,u)} = \sup_{k \in \mathbb{N}} \Big[u_k F_k (|\hat{F}_k(x)|) \Big]^{p_k} = \|y\|_\infty < \infty,$$

respectively. Hence T is linear bijection which shows that the spaces l_p and $l(\hat{F},\mathcal{F},p,u)$ are linearly isomorphic for $1 \leq p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$. This concludes the proof. □

We wish to exhibit some inclusion relations concerning the space $l(\hat{F},\mathcal{F},p,u)$.

Theorem 2.10. *The inclusion $l_p \subset l(\hat{F},\mathcal{F},p,u)$ strictly holds for $1 \leq p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$.*

Proof. To prove the validity of the inclusion $l_p \subset l(\hat{F},\mathcal{F},p,u)$ for $1 \leq p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$, it suffices to show the existence of a number $M > 0$ such that $\|x\|_{l(\hat{F},\mathcal{F},p,u)} \leq M\|x\|_p$ for every $x \in l_p$.
Let $x \in l_p$ and $1 < p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$. Since the inequalities $F_k\left(\frac{f_k}{f_{k+1}}\right) \leq 1$ and $F_k\left(\frac{f_{k+1}}{f_k}\right) \leq 2$ hold for every $k \in \mathbb{N}$, we obtain with the notation of (3),

$$\begin{aligned}\sum_k \left[u_k F_k(|\hat{F}_k(x)|)\right]^{p_k} &\leq \sum_k 2^{p_k-1}\left[u_k F_k\Big(|x_k| + |2x_{k-1}|\Big)\right]^{p_k} \\ &\leq 2^{2p_k-1}\sum_k \left[u_k F_k|x_k|\right]^{p_k} + \sum_k \left[u_k F_k|2x_{k-1}|\right]^{p_k}\Big)\end{aligned}$$

and

$$\sup_{k\in\mathbb{N}} \left[u_k F_k(|\hat{F}_k(x)|)\right]^{p_k} \leq 3\sup_{k\in\mathbb{N}} \left[u_k F_k(|x_k|)\right]^{p_k},$$

which together yield, as expected,

$$\|x\|_{l(\hat{F},\mathcal{F},p,u)} \leq 4\|x\|_p \qquad \text{for } 1 < p \leq \infty. \tag{10}$$

Further, since the sequence $x = (x_k) = (f_{k+1}^2) = (1, 2^2, 3^2, 5^2, \dots)$ belongs to $l(\hat{F},\mathcal{F},p,u) - l_p$, the inclusion $l_p \subset l(\hat{F},\mathcal{F},p,u)$ is strict for $1 < p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$. Similarly, one can easily prove that the inequality (10) also holds in the case $p = 1$, and so we omit the details. This completes the proof. □

3. The $\alpha-, \beta-$ and $\gamma-$ duals of the space $l(\hat{F},\mathcal{F},p,u)$

The $\alpha-, \beta-$ and $\gamma-$ duals of the sequence space X are respectively defined by

$$X^\alpha = \{a = (a_k) \in w : ax = (a_k x_k) \in l_1 \text{ for all } x = (x_k) \in X\},$$

$$X^\beta = \{a = (a_k) \in w : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}$$

and

$$X^\gamma = \{a = (a_k) \in w : ax = (a_k x_k) \in bs \text{ for all } x = (x_k) \in X\},$$

where *cs* and *bs* are the sequence spaces of all convergent and bounded series, respectively [8]. We assume throughout that $p,q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{H}$.
In this section, we determine $\alpha-, \beta-$ and $\gamma-$ duals of the sequence space $l(\hat{F},\mathcal{F},p,u)$ of non-absolute type. Since the case $p=1$ can be proved by same analogy, we omit the proof of that case and consider only the case $1 < p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$ in the proof of Theorem 3.5. In [27] the following known results are fundamental for our investigation.

Lemma 3.1. *$A=(a_{nk}) \in (l_p, l_1)$ if and only if*

$$\sup_{K\in\mathcal{H}} \sum_k \Big| \sum_{n\in K} a_{nk} \Big| < \infty, \;\; 1 < p \leq \infty.$$

Lemma 3.2. *$A=(a_{nk}) \in (l_p, c)$ if and only if*

$$\lim_{n\to\infty} a_{nk} \;\; \textit{exists for all } k \in \mathbb{N}, \tag{11}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|^q < \infty, \;\; 1 < p < \infty. \tag{12}$$

Lemma 3.3. *$A=(a_{nk}) \in (l_\infty, c)$ if and only if (11) holds and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \Big| \lim_{n\to\infty} a_{nk} \Big|. \tag{13}$$

Lemma 3.4. *$A=(a_{nk}) \in (l_p, l_\infty)$ if and only if (12) holds with $1 < p \leq \infty$.*

Theorem 3.5. *The $\alpha-$ dual of the space $l(\hat{F},\mathcal{F},p,u)$ is the set*

$$\hat{d}_1 = \left\{ a=(a_k) \in w : \sup_{K\in\mathcal{H}} \sum_k \left[u_k F_k \left(\Big| \sum_{n\in K} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \Big|^q \right) \right]^{p_k} < \infty \right\},$$

where $1 < p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$.

Proof. Let $1 < p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$. For any fixed sequence $a=(a_n) \in w$, we define the matrix $B=(b_{nk})$ by

$$b_{nk} = \begin{cases} \sum_k \left[u_k F_k \left(\Big| \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \Big| \right) \right]^{p_k} & (0 \leq k \leq n), \\ 0 & (k > n) \end{cases}$$

for all $n,k \in \mathbb{N}$. Also, for every $x=(x_n) \in w$, we put $y=\hat{F}x$. Then it follows by (3) that

$$a_n x_n = \sum_k \left[u_k F_k \left(\Big| \sum_{k=0}^{n} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n y_k \Big| \right) \right]^{p_k} = B_n(y) \;\;\; (n \in \mathbb{N}). \tag{14}$$

Thus, we observe by (14) that $ax = a_n x_n \in l_1$ whenever $x \in l(\hat{F}, \mathcal{F}, p, u)$ if and only if $B(y) \in l_1$ whenever $y \in l_p$. Therefore, we drive by using Lemma 3.1 that

$$\sup_{K \in \mathcal{H}} \sum_k \left[u_k F_k \left(\Big| \sum_{n \in K} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \Big|^q \right) \right]^{p_k} < \infty,$$

which implies that $[l(\hat{F}, \mathcal{F}, p, u)]^\alpha = \hat{d}_1$. □

Theorem 3.6. *Define the sets $\hat{d}_2, \hat{d}_3$ and $\hat{d}_4$ by*

$$\hat{d}_2 = \left\{ a = (a_k) \in w : \sum_k \left[u_k F_k \left(\Big| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \Big| \right) \right]^{p_k} \text{ exists for all } k \in \mathbb{N} \right\},$$

$$\hat{d}_3 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left[u_k F_k \left(\Big| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \Big|^q \right) \right]^{p_k} < \infty \right\}$$

and

$$\hat{d}_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} \left[u_k F_k \left(\Big| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \Big| \right) \right]^{p_k} = \sum_k \left[u_k F_k \left(\Big| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \Big| \right) \right]^{p_k} < \infty \right\}.$$

Then $[l(\hat{F}, \mathcal{F}, p, u)]^\beta = \hat{d}_2 \cap \hat{d}_3$ and $[l_\infty(\hat{F}, \mathcal{F}, p, u)]^\beta = \hat{d}_2 \cap \hat{d}_4$ where $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.

Theorem 3.7. *$[l(\hat{F}, \mathcal{F}, p, u)]^\gamma = \hat{d}_3$, where $1 < p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$.*

Proof. The result can be obtained from Lemma 3.4. □

4. Matrix transformations related to the sequence space $l(\hat{F}, \mathcal{F}, p, u)$

In this section, we characterize the classes $(l(\hat{F}, \mathcal{F}, p, u), X)$, where $1 \leq p_k \leq H \leq \infty$ for all $k \in \mathbb{N}$ and X is any of the spaces l_∞, l_1, c and c_0.
For simplicity in notation, we write

$$\tilde{a}_{nk} = \sum_k \left[u_k F_k \left(\Big| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \Big| \right) \right]^{p_k} \quad \text{for all } k, n \in \mathbb{N}.$$

The following lemma is essential for our results.

Lemma 4.1 ([15], Theorem 4.1)**.** *Let λ be an FK-space, U be a triangle, V be its inverse and μ be an arbitrary subset of w. Then we have $A = (a_{nk}) \in (\lambda_U, \mu)$ if and only if*

$$C^{(n)} = \left(c_{mk}^{(n)}\right) \in (\lambda, c) \quad \textit{for all } n \in \mathbb{N}$$

and

$$C = (c_{nk}) \in (\lambda, \mu),$$

where

$$c_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} a_{nj} v_{jk} & (0 \leq k \leq m), \\ 0 & (k > m) \end{cases}$$

and

$$c_{nk} = \sum_{j=k}^{\infty} a_{nj} v_{jk} \quad \textit{for all } k, m, n \in \mathbb{N}.$$

Now, we list the following conditions:

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^{m} \left[u_k F_k \left(\Big| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \Big|^q \right) \right]^{p_k} < \infty, \tag{15}$$

$$\lim_{m \to \infty} \left[u_k F_k \left(\Big| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \Big| \right) \right]^{p_k} = \tilde{a}_{nk}, \quad \forall n, k \in \mathbb{N}, \tag{16}$$

$$\lim_{m \to \infty} \sum_{k=0}^{m} \left[u_k F_k \left(\Big| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \Big| \right) \right]^{p_k} = \sum_k |\tilde{a}_{nk}| \quad \text{for each } n \in \mathbb{N}, \tag{17}$$

$$\sup_{n \in \mathbb{N}} \sum_k |\tilde{a}_{nk}|^q < \infty, \tag{18}$$

$$\sup_{N \in \mathcal{H}} \sum_k \Big| \sum_{n \in \mathbb{N}} \tilde{a}_{nk} \Big|^q < \infty, \tag{19}$$

$$\lim_{n \to \infty} \tilde{a}_{nk} = \tilde{a}_k; \quad k \in \mathbb{N}, \tag{20}$$

$$\lim_{n \to \infty} \sum_k |\tilde{a}_{nk}| = \sum_k |\tilde{a}_k|, \tag{21}$$

$$\lim_{n \to \infty} \sum_k \tilde{a}_{nk} = 0, \tag{22}$$

$$\sup_{n,k \in \mathbb{N}} |\tilde{a}_{nk}| < \infty, \tag{23}$$

$$\sup_{k,m \in \mathbb{N}} \left[u_k F_k \left(\Big| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \Big| \right) \right]^{p_k} < \infty, \tag{24}$$

$$\sup_{k\in\mathbb{N}}\sum_{k}|\tilde{a}_{nk}|<\infty, \tag{25}$$

$$\sup_{N,K\in\mathcal{H}}\left|\sum_{n\in N}\sum_{k\in K}\tilde{a}_{nk}\right|<\infty. \tag{26}$$

Then by combining Lemma 4.1 with the results in (see [27]), we immediately derive the following results.

Theorem 4.2. *(a)* $A=(a_{nk})\in(l_1(\hat{F},\mathcal{F},p,u),l_\infty)$ *if and only if (16), (23) and (24) hold.*
(b) $A=(a_{nk})\in(l_1(\hat{F},\mathcal{F},p,u),c)$ *if and only if (16), (20), (23) and (24) hold.*
(c) $A=(a_{nk})\in(l_1(\hat{F},\mathcal{F},p,u),c_0)$ *if and only if (16), (20) with* $\tilde{a}_k=0$*, (23) and (24) hold.*
(d) $A=(a_{nk})\in(l_1(\hat{F},\mathcal{F},p,u),l_1)$ *if and only if (16), (24) and (25) hold.*

Theorem 4.3. *Let* $1<p_k\le H<\infty$ *for all* $k\in\mathbb{N}$*. Then we have*
(a) $A=(a_{nk})\in(l(\hat{F},\mathcal{F},p,u),l_\infty)$ *if and only if (15), (16) and (18) hold.*
(b) $A=(a_{nk})\in(l(\hat{F},\mathcal{F},p,u),c)$ *if and only if (15), (16), (18) and (20) hold.*
(c) $A=(a_{nk})\in(l(\hat{F},\mathcal{F},p,u),c_0)$ *if and only if (15), (16), (18) and (20) with* $\tilde{a}_k=0$ *hold.*
(d) $A=(a_{nk})\in(l(\hat{F},\mathcal{F},p,u),l_1)$ *if and only if (15), (16) and (19) hold.*

Theorem 4.4. *(a)* $A=(a_{nk})\in(l_\infty(\hat{F},\mathcal{F},p,u),l_\infty)$ *if and only if (16), (17) and (18) with* $q=1$ *hold.*
(b) $A=(a_{nk})\in(l_\infty(\hat{F},\mathcal{F},p,u),c)$ *if and only if (16), (17), (20) and (21) hold.*
(c) $A=(a_{nk})\in(l_\infty(\hat{F},\mathcal{F},p,u),c_0)$ *if and only if (16), (17) and (22) hold.*
(d) $A=(a_{nk})\in(l_\infty(\hat{F},\mathcal{F},p,u),l_1)$ *if and only if (16), (17) and (26) hold.*

5. Some geometric properties of the space $l(\hat{F},\mathcal{F},p,u)$

In this section, we study some geometric properties of the space $l(\hat{F},\mathcal{F},p,u)$ for $(1<p_k\le H<\infty)$ for all $k\in\mathbb{N}$. For these properties, (see [10], [20], [25]).
A Banach space X is said to have the Banach-Saks property if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $\{t_k(z)\}$ is convergent in the norm in X [19], where

$$t_k(z)=\frac{1}{k}(z_0+z_1+\ldots+z_k)\ (k\in\mathbb{N}). \tag{27}$$

A Banach space X is said to have the weak Banach-Saks property whenever, given any weakly null sequence $(x_n)\subset X$, there exists a subsequence (z_n) of

(x_n) such that the sequence $\{t_k(z)\}$ is strongly convergent to zero.
In [12], García-Falset introduces the following coefficient:

$$R(X) = \sup\big\{ \liminf_{n\to\infty} \|x_n - x\| : (x_n) \subset B(X), x_n \xrightarrow{w} 0,\ x \in B(X)\big\}, \tag{28}$$

where $B(X)$ denotes the unit ball of X.

Remark 5.1. A Banach space X with $R(X) < 2$ has the weak fixed point property [13].

Let $1 < p < \infty$. A Banach space is said to have the Banach-Saks type p or the property $(BS)_p$ if every weakly null sequence (x_k) has a subsequence (x_{k_l}) such that for some $C > 0$,

$$\left\| \sum_{l=0}^{n} x_{k_l} \right\| < C(n+1)^{\frac{1}{p}} \tag{29}$$

for all $n \in \mathbb{N}$ (See [16]).
Now, we may give the following results related to some geometric properties of the space $l(\hat{F}, \mathcal{F}, p, u)$, where $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.

Theorem 5.2. *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then the space $l(\hat{F}, \mathcal{F}, p, u)$ has the Banach-Saks type p.*

Proof. Let (ε_n) be a sequence of positive numbers for which $\sum \varepsilon_n \leq \frac{1}{2}$, and also let (x_n) be a weakly null sequence in $B(l(\hat{F}, \mathcal{F}, p, u))$. Set $a_0 = x_0 = 0$ and $a_1 = x_{n_1} = x_1$. Then there exists $m_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=m_1+1}^{\infty} a_1(i) e^{(i)} \right\|_{l(\hat{F}, \mathcal{F}, p, u)} < \varepsilon_1. \tag{30}$$

Since (x_n) being a weakly null sequence implies $x_n \to 0$ coordinatewise, there is an $n_2 \in \mathbb{N}$ such that

$$\left\| \sum_{i=0}^{m_1} x_n(i) e^{(i)} \right\|_{l(\hat{F}, \mathcal{F}, p, u)} < \varepsilon_1,$$

when $n \geq n_2$. Set $a_2 = x_{n_2}$. Then there exists an $m_2 > m_1$ such that

$$\left\| \sum_{i=m_2+1}^{\infty} a_2(i) e^{(i)} \right\|_{l(\hat{F}, \mathcal{F}, p, u)} < \varepsilon_2.$$

Again using the fact that $x_n \to 0$ coordinatewise, there exists an $n_3 \geq n_2$ such that

$$\left\| \sum_{i=0}^{m_2} x_n(i) e^{(i)} \right\|_{l(\hat{F}, \mathcal{F}, p, u)} < \varepsilon_2,$$

when $n \geq n_3$.
If we continue this process, we can find two increasing subsequences (m_i) and (n_i) such that

$$\left\| \sum_{i=0}^{m_j} x_n(i) e^{(i)} \right\|_{l(\hat{F},\mathcal{F},p,u)} < \varepsilon_j$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=m_j+1}^{\infty} a_j(i) e^{(i)} \right\|_{l(\hat{F},\mathcal{F},p,u)} < \varepsilon_j,$$

where $b_j = x_{n_j}$. Hence,

$$\begin{aligned}
&\left\| \sum_{j=0}^{n} a_j \right\|_{l(\hat{F},\mathcal{F},p,u)} \\
=&\left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} a_j(i) e^{(i)} + \sum_{i=m_{j-1}+1}^{m_j} a_j(i) e^{(i)} + \sum_{i=m_j+1}^{\infty} a_j(i) e^{(i)} \right) \right\|_{l(\hat{F},\mathcal{F},p,u)} \\
\leq&\left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} a_j(i) e^{(i)} \right) \right\|_{l(\hat{F},\mathcal{F},p,u)} + \left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} a_j(i) e^{(i)} \right) \right\|_{l(\hat{F},\mathcal{F},p,u)} \\
+&\left\| \sum_{j=0}^{n} \left(\sum_{i=m_j+1}^{\infty} a_j(i) e^{(i)} \right) \right\|_{l(\hat{F},\mathcal{F},p,u)} \\
\leq&\left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} a_j(i) e^{(i)} \right) \right\|_{l(\hat{F},\mathcal{F},p,u)} + 2\sum_{j=0}^{n} \varepsilon_j.
\end{aligned}$$

On the other hand, it can be seen that $\|x\|_{l(\hat{F},\mathcal{F},p,u)} < 1$. Therefore, we have that

$$\begin{aligned}
&\left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} a_j(i) e^{(i)} \right) \right\|_{l(\hat{F},\mathcal{F},p,u)}^{p_k} \\
&= \sum_{j=0}^{n} \sum_{i=m_{j-1}+1}^{m_j} \left[u_k F_k \left(\left| \frac{f_i}{f_{i+1}} a_j(i) - \frac{f_{i+1}}{f_i} a_j(i-1) \right| \right) \right]^{p_k} \\
&\leq \sum_{j=0}^{n} \sum_{i=0}^{\infty} \left[u_k F_k \left(\left| \frac{f_i}{f_{i+1}} a_j(i) - \frac{f_{i+1}}{f_i} a_j(i-1) \right| \right) \right]^{p_k} \leq (n+1).
\end{aligned}$$

Hence, we obtain

$$\left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} a_j(i) e^{(i)} \right) \right\| \leq (n+1)^{\frac{1}{p_k}}.$$

By using the fact $1 \leq (n+1)^{\frac{1}{p_k}}$ for all $n \in \mathbb{N}$ and $1 < p_k < \infty$, we have

$$\left\| \sum_{j=0}^{n} a_j(i) \right\|_{l(\hat{F},\mathcal{F},p,u)} \leq (n+1)^{\frac{1}{p_k}} + 1 \leq 2(n+1)^{\frac{1}{p_k}}.$$

Hence, $l(\hat{F},\mathcal{F},p,u)$ has the Banach-Saks type p. This concludes the proof. □

Remark 5.3. Note that $R(l(\hat{F},\mathcal{F},p,u)) = R(l_p) = 2^{\frac{1}{p}}$ since $l(\hat{F},\mathcal{F},p,u)$ is linearly isomorphic to l_p.

Hence by Remarks 5.1 and 5.3, we have the following theorem.

Theorem 5.4. *The space $l(\hat{F},\mathcal{F},p,u)$ has the weak fixed point property, where $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.*

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KULDIP RAJ
School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
e-mail: kuldeepraj68@rediffmail.com

SURUCHI PANDOH
School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
e-mail: suruchi.pandoh87@gmail.com

SEEMA JAMWAL
School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
e-mail: seemajamwal8@gmail.com