STEINER SYSTEMS AND LARGE NON-HAMILTONIAN HYPERGRAPHS

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From Steiner systems $S(k - 2, 2k - 3, v)$, we construct $k$-uniform hypergraphs of large size without Hamiltonian cycles. This improves previous estimates due to G. Y. Katona and H. Kierstead [J. Graph Theory 30 (1999), pp. 205–212].

1. The Results.

In this short note we study an extremal hypergraph problem raised by G. Y. Katona and H. Kierstead [4]. Assume that the hypergraph (set system) $\mathcal{H}$ on vertex set $X$ is $k$-uniform for a given $k \geq 2$; i.e., $H \subseteq X$ and $|H| = k$ hold for all edges $H \in \mathcal{H}$. We denote by $v = |X|$ the number of vertices. Let $v > k$. An ordering $x_1 x_2 \cdots x_v$ of $X$ is called a Hamiltonian cycle if $\{x_i, x_{i+1}, \ldots, x_{i+k-1}\} \in \mathcal{H}$ holds for all $i = 1, 2, \ldots, v$, where subscript addition is taken modulo $v$. Hence, if $k = 2$, this corresponds to a Hamiltonian cycle in the usual graph-theoretic sense.

The problem we consider here is to determine the largest possible number $f(v, k)$ of edges in a $k$-uniform hypergraph $\mathcal{H}$ of order $v$ under...
the assumption that $\mathcal{H}$ does not admit any Hamiltonian cycles. It is well-known that for graphs,

$$f(v, 2) = \binom{v}{2} - v + 2$$

holds for every $v \geq 3$. The extremal graph consists of $K_{v-1}$ together with a pendant vertex of degree 1. Generalizing this construction, Katona and Kierstead [4] observed that the lower bound

$$f(v, k) \geq \binom{v}{k} - \binom{v - 2}{k - 1} = \binom{v - 1}{k} + \binom{v - 2}{k - 2}$$

remains valid for every $k \geq 3$ and every $v > k$. To obtain this inequality, two vertices $x, x'$ are selected and, from the family of all $k$-element subsets of $X$, those sets are deleted which contain $x$ and do not contain $x'$.

The constructs we are going to apply in connection with non-hamiltonicity are some kinds of Steiner systems. A Steiner system $S(t, b, v)$, of order $v$ and block size $b$, is a $b$-uniform hypergraph on $v$ vertices, such that every $t$-element vertex subset is contained in precisely one edge. A partial Steiner system, denoted $PS(t, b, v)$, is a $b$-uniform hypergraph on $v$ vertices, such that no $t$-element vertex subset is contained in more than one edge.

In this note we derive a stronger lower bound on $f(v, k)$ from Steiner systems when they exist, improving on (1) with the term $\binom{v - 2}{k - 3}$ as follows:

**Theorem 1.** If a Steiner system $S(k - 2, 2k - 3, v - 1)$ exists, and $v > 2k - 2$, then

$$f(v, k) \geq \binom{v - 1}{k} + \binom{v - 1}{k - 2}.$$  

Also, for values $v$ for which the corresponding Steiner system does not exist but a fairly large partial Steiner system is available, a lower bound can be established:

**Theorem 2.** Let $k > 2$ and $v > 2k - 3$ be integers such that there exists a partial Steiner system $PS(k - 2, 2k - 3, v - 1)$ of order $v - 1$ with $p \binom{v - 1}{k - 2} / \binom{2k - 3}{k - 2}$ blocks, for some real $p$ whose value possibly depends on $v$ and is near to 1. Then

$$f(v, k) \geq \binom{v - 1}{k} + p \binom{v - 1}{k - 2}.$$
Some related problems and results, also concerning minimum-degree conditions, are surveyed in [3].

2. The Construction.

The two theorems will be handled together. Let \( v \) and \( k \) be given, and suppose that \( S_{v-1,k} \) is a partial Steiner system \( PS(k-2, 2k-3, v-1) \) with block size \( 2k - 3 \). We choose the value of \( p \) so that the number of blocks is exactly

\[
|S_{v-1,k}| = p \left( \frac{v-1}{k-2} \right) \left( \frac{2k-3}{k-2} \right).
\]

Applying \( S_{v-1,k} \), we construct a non-hamiltonian \( k \)-uniform hypergraph \( H_{v,k} \) on a \( v \)-element vertex set \( X \).

Let us fix an element \( x \in X \), and assume that \( S_{v-1,k} \) is given on the vertex set \( X \setminus \{x\} \). The edges of \( H_{v,k} \) are now defined as:

- all \( k \)-element subsets of \( X \setminus \{x\} \), and
- all sets of the form \( Y \cup \{x\} \), where \( |Y| = k-1 \) and \( Y \subseteq S \) for some \( S \in S_{v-1,k} \).

Theorems 1 and 2 will be deduced from the following two assertions:

**Lemma 3.** The number of edges in \( H_{v,k} \) is equal to \( \left( \frac{v-1}{k} \right) + p \left( \frac{v-1}{k-2} \right) \).

**Proof.** Since no \( (k-1) \)-tuple of \( X \setminus \{x\} \) appears in more than one block of \( S_{v-1,k} \), we obtain \( |H_{v,k}| = \left( \frac{v-1}{k} \right) + |S_{v-1,k}| \left( \frac{2k-3}{k-1} \right) \). Thus, the lemma follows by the assumption on the size of \( S_{v-1,k} \).

**Lemma 4.** The hypergraph \( H_{v,k} \) does not have any Hamiltonian cycles.

**Proof.** Suppose for a contradiction that \( x_1 x_2 \cdots x_v \) is a Hamiltonian cycle of \( H_{v,k} \). Assume, without loss of generality, that the selected vertex \( x \) occurs as \( x_k \) in this cyclic order.

We concentrate on the set \( X' = \{x_i \mid 1 \leq i \leq 2k-1 \} \). Note that \( |X'| = 2k - 1 \), since we have assumed \( v > 2k-2 \). By assumption, for each \( i \) in the range \( 1 \leq i \leq k \), there exists a block \( S_i \in S_{v-1,k} \) such that

\[
\{x_j \mid i \leq j \leq i + k - 1\} \subseteq S_i \cup \{x_k\}.
\]
We cannot have $S_i = S_k$, because $2k - 2 = |X \setminus \{x_k\}| \leq |S_i \cup S_k|$ and $|S_i| = |S_k| = 2k - 3$. Thus, there exists a subscript $i$ ($1 \leq i \leq k$) such that $S_i \neq S_{i+1}$. On the other hand,

$$\{x_j \mid i + 1 \leq j \leq i + k - 1\}\setminus \{x_k\} \subseteq S_i \cap S_{i+1},$$

i.e., we have found two distinct members of $S_{v-1,k}$ that should share at least $k - 2$ vertices. This contradicts the assumption that $S_{v-1,k}$ is a $PS(k - 2, 2k - 3, v - 1)$.

\[\square\]

**Proof of Theorems 1 and 2.** Lemmas 3 and 4 together imply (3). Moreover, if $S_{v-1,k}$ is an exact $S(k - 2, 2k - 3, v - 1)$ system, then each of the $\binom{v-1}{k-2}$ $(k - 2)$-element subsets of $X \setminus \{x\}$ occurs in precisely one $S \in S_{v-1,k}$. Thus, the particular case of $p = 1$ applies, and (2) follows.

\[\square\]

3. Concluding remarks.

1. The applicability of the exact construction in Theorem 1 is subject to the existence of Steiner systems with particular parameters, a major open problem in Design Theory for large block size. See, e.g., [1] for details.

2. On applying known results (cf. [1, p. 42]), we obtain an improvement on (1) for an infinite sequence of $v$ for 4-uniform hypergraphs, namely for every $v \geq 22$ such that $v - 1 \equiv 1 \text{ or } 5 \pmod{20}$, which is the necessary and sufficient condition for the existence of $S(2, 5, v - 1)$. Theorem 2 may then be considered for the intermediate values of $v$.

3. For general $k$, the existence of asymptotically tight partial Steiner systems — i.e., with $p = 1 - o(1)$ in Theorem 2 — has been proved by Rödl in [5], for any fixed $k$ as $v$ gets large. In order to obtain explicit improvements on (1), however, one would need more precise estimates on the tightness of possible constructions of the $PS(t, b, v)$.

4. The currently known best upper bound on $f(v, k)$ seems to be

$$f(v, k) \leq \binom{v}{k} - \frac{4}{4k - 1} \left(\frac{v - 1}{k - 1}\right),$$

proven in [2]. This is slightly better than the estimate $\binom{v}{k} - \frac{1}{k} \left(\frac{v - 1}{k - 1}\right)$ published in [4]. It remains an open problem to determine tight asymptotics
for the complementary function, \( \binom{v}{k} - f(v, k) \).

5. In Theorems 1 and 2, the condition \( v > 2k - 2 \) cannot be omitted. Indeed, though an \( S(k-2, 2k-3, 2k-3) \) obviously exists — and consists of one single block — Inequality (2) for \( v = 2k-2 \) would state \( f(2k-2, k) \geq \binom{2k-3}{k} + \binom{2k-3}{k-2} = \binom{2k-2}{k} \), which is certainly false, because in the complete hypergraph having all \( k \)-element vertex subsets as edges, every cyclic permutation of the vertex set is a Hamiltonian cycle.

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REFERENCES


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