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STEINER SYSTEMS AND LARGE NON-HAMILTONIAN HYPERGRAPHS

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From Steiner systems $S(k-2, 2k-3, v)$, we construct k-uniform hypergraphs of large size without Hamiltonian cycles. This improves previous estimates due to G. Y. Katona and H. Kierstead [*J. Graph Theory* **30** (1999), pp. 205–212].

1. The Results.

In this short note we study an extremal hypergraph problem raised by G. Y. Katona and H. Kierstead [4]. Assume that the hypergraph (set system) $\mathcal{H}$ on vertex set X is *k-uniform* for a given $k \geq 2$; i.e., $H \subseteq X$ and $|H| = k$ hold for all edges $H \in \mathcal{H}$. We denote by $v = |X|$ the number of vertices. Let $v > k$. An ordering $x_1x_2 \cdots x_v$ of X is called a *Hamiltonian cycle* if $\{x_i, x_{i+1}, \ldots, x_{i+k-1}\} \in \mathcal{H}$ holds for all $i = 1, 2, \ldots, v$, where subscript addition is taken modulo v. Hence, if $k = 2$, this corresponds to a Hamiltonian cycle in the usual graph-theoretic sense.

The problem we consider here is to determine the largest possible number $f(v, k)$ of edges in a k-uniform hypergraph $\mathcal{H}$ of order v under

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the assumption that $\mathcal{H}$ does not admit any Hamiltonian cycles. It is well-known that for graphs,

$$f(v, 2) = \binom{v}{2} - v + 2$$

holds for every $v \geq 3$. The extremal graph consists of K_{v-1} together with a pendant vertex of degree 1. Generalizing this construction, Katona and Kierstead [4] observed that the lower bound

$$f(v, k) \geq \binom{v}{k} - \binom{v-2}{k-1} = \binom{v-1}{k} + \binom{v-2}{k-2} \tag{1}$$

remains valid for every $k \geq 3$ and every $v > k$. To obtain this inequality, two vertices x, x' are selected and, from the family of all k-element subsets of X, those sets are deleted which contain x and do not contain x'.

The constructs we are going to apply in connection with non-hamiltonicity are some kinds of Steiner systems. A *Steiner system* $S(t, b, v)$, of order v and block size b, is a b-uniform hypergraph on v vertices, such that every t-element vertex subset is contained in precisely one edge. A *partial Steiner system*, denoted $PS(t, b, v)$, is a b-uniform hypergraph on v vertices, such that no t-element vertex subset is contained in more than one edge.

In this note we derive a stronger lower bound on $f(v, k)$ from Steiner systems when they exist, improving on (1) with the term $\binom{v-2}{k-3}$ as follows:

Theorem 1. *If a Steiner system $S(k-2, 2k-3, v-1)$ exists, and $v > 2k-2$, then*

$$f(v, k) \geq \binom{v-1}{k} + \binom{v-1}{k-2}. \tag{2}$$

Also, for values v for which the corresponding Steiner system does not exists but a fairly large partial Steiner system is available, a lower bound can be established:

Theorem 2. *Let $k > 2$ and $v > 2k-3$ be integers such that there exists a partial Steiner system $PS(k-2, 2k-3, v-1)$ of order $v-1$ with $p\binom{v-1}{k-2}/\binom{2k-3}{k-2}$ blocks, for some real p whose value possibly depends on v and is near to 1. Then*

$$f(v, k) \geq \binom{v-1}{k} + p\binom{v-1}{k-2}. \tag{3}$$

Some related problems and results, also concerning minimum-degree conditions, are surveyed in [3].

2. The Construction.

The two theorems will be handled together. Let v and k be given, and suppose that $\mathcal{S}_{v-1,k}$ is a partial Steiner system $PS(k-2, 2k-3, v-1)$ with block size $2k-3$. We choose the value of p so that the number of blocks is exactly

$$|\mathcal{S}_{v-1,k}| = p\binom{v-1}{k-2}\Big/\binom{2k-3}{k-2}.$$

Applying $\mathcal{S}_{v-1,k}$, we construct a non-hamiltonian k-uniform hypergraph $\mathcal{H}_{v,k}$ on a v-element vertex set X.

Let us fix an element $x \in X$, and assume that $\mathcal{S}_{v-1,k}$ is given on the vertex set $X\setminus \{x\}$. The edges of $\mathcal{H}_{v,k}$ are now defined as:

- all k-element subsets of $X\setminus\{x\}$, and
- all sets of the form $Y \cup \{x\}$, where $|Y| = k-1$ and $Y \subseteq S$ for some $S \in \mathcal{S}_{v-1,k}$.

Theorems 1 and 2 will be deduced from the following two assertions:

Lemma 3. *The number of edges in* $\mathcal{H}_{v,k}$ *is equal to* $\binom{v-1}{k} + p\binom{v-1}{k-2}$.

Proof. Since no $(k-1)$-tuple of $X\setminus\{x\}$ appears in more than one block of $\mathcal{S}_{v-1,k}$, we obtain $|\mathcal{H}_{v,k}| = \binom{v-1}{k} + |\mathcal{S}_{v-1,k}|\binom{2k-3}{k-1}$. Thus, the lemma follows by the assumption on the size of $\mathcal{S}_{v-1,k}$. □

Lemma 4. *The hypergraph* $\mathcal{H}_{v,k}$ *does not have any Hamiltonian cycles.*

Proof. Supose for a contradiction that $x_1x_2\cdots x_v$ is a Hamiltonian cycle of $\mathcal{H}_{v,k}$. Assume, without loss of generality, that the selected vertex x occurs as x_k in this cyclic order.

We concentrate on the set $X' = \{x_i \mid 1 \le i \le 2k-1\}$. Note that $|X'| = 2k-1$, since we have assumed $v > 2k-2$. By assumption, for each i in the range $1 \le i \le k$, there exists a block $S_i \in \mathcal{S}_{v-1,k}$ such that

$$\{x_j \mid i \le j \le i+k-1\} \subseteq S_i \cup \{x_k\}.$$

We cannot have $S_1 = S_k$, because $2k - 2 = |X'\backslash\{x_k\}| \leq |S_1 \cup S_k|$ and $|S_1| = |S_k| = 2k - 3$. Thus, there exists a subscript i $(1 \leq i \leq k)$ such that $S_i \neq S_{i+1}$. On the other hand,

$$\{x_j \mid i + 1 \leq j \leq i + k - 1\}\backslash\{x_k\} \subseteq S_i \cap S_{i+1},$$

i.e., we have found two distinct members of $\mathcal{S}_{v-1,k}$ that should share at least $k - 2$ vertices. This contradicts the assumption that $\mathcal{S}_{v-1,k}$ is a $PS(k - 2, 2k - 3, v - 1)$. □

Proof of Theorems 1 and 2. Lemmas 3 and 4 together imply (3). Moreover, if $\mathcal{S}_{v-1,k}$ is an exact $S(k - 2, 2k - 3, v - 1)$ system, then each of the $\binom{v-1}{k-2}$ $(k - 2)$-element subsets of $X\backslash\{x\}$ occurs in precisely one $S \in \mathcal{S}_{v-1,k}$. Thus, the particular case of $p = 1$ applies, and (2) follows. □

3. Concluding remarks.

1. The applicability of the exact construction in Theorem 1 is subject to the existence of Steiner systems with particular parameters, a major open problem in Design Theory for large block size. See, e.g., [1] for details.

2. On applying known results (cf. [1, p. 42]), we obtain an improvement on (1) for an infinite sequence of v for 4-uniform hypergraphs, namely for every $v \geq 22$ such that $v - 1 \equiv 1$ or $5 \pmod{20}$, which is the necessary and sufficient condition for the existence of $S(2, 5, v - 1)$. Theorem 2 may then be considered for the intermediate values of v.

3. For general k, the existence of asymptotically tight partial Steiner systems — i.e., with $p = 1 - o(1)$ in Theorem 2 — has been proved by Rödl in [5], for any fixed k as v gets large. In order to obtain explicit improvements on (1), however, one would need more precise estimates on the tightness of possible constructions of the $PS(t, b, v)$.

4. The currently known best upper bound on $f(v, k)$ seems to be

$$f(v, k) \leq \binom{v}{k} - \frac{4}{4k - 1}\binom{v - 1}{k - 1},$$

proven in [2]. This is slightly better than the estimate $\binom{v}{k} - \frac{1}{k}\binom{v - 1}{k - 1}$ published in [4]. It remains an open problem to determine tight asymptotics for the complementary function, $\binom{v}{k} - f(v,k)$.

5. In Theorems 1 and 2, the condition $v > 2k-2$ cannot be omitted. Indeed, though an $S(k-2, 2k-3, 2k-3)$ obviously exists — and consists of one single block — Inequality (2) for $v = 2k-2$ would state $f(2k-2, k) \geq \binom{2k-3}{k} + \binom{2k-3}{k-2} = \binom{2k-2}{k}$, which is certainly false, because in the complete hypergraph having all k-element vertex subsets as edges, every cyclic permutation of the vertex set is a Hamiltonian cycle.

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