QUADRUPLE FIXED POINT THEOREM FOR HYBRID PAIR OF MAPPINGS UNDER GENERALIZED NONLINEAR CONTRACTION

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We establish a quadruple coincidence and common quadruple fixed point theorem for hybrid pair of mappings under generalized nonlinear contraction on a non-complete metric space, which is not partially ordered. It is to be noted that to find quadruple coincidence point, we do not employ the condition of continuity of any mapping involved therein. We also give an example to validate our result. We improve, extend and generalize various known results.

1. Introduction and Preliminaries

Let \((X, d)\) be a metric space and \(CB(X)\) be the set of all non-empty closed bounded subsets of \(X\). Let \(D(x, A)\) denote the distance from \(x\) to \(A \subset X\) and \(H\) denote the Hausdorff metric induced by \(d\), that is,

\[
D(x, A) = \inf_{a \in A} d(x, a)
\]

and \(H(A, B) = \max \left\{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \right\}\), for all \(A, B \in CB(X)\).

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The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was initiated by Markin [31]. The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to ([4], [14], [15], [26], [35], [43]) and the reference therein. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics. Nadler [39] extended the famous Banach Contraction Principle [6] from single-valued mapping to multivalued mapping.

Bhaskar and Lakshmikantham [10] established some coupled fixed point theorems and applied these results to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [27] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results of Bhaskar and Lakshmikantham [10].

Bercinde and Borcut [8] introduced the concept of tripled fixed point for single-valued mappings in partially ordered metric spaces and established the existence of tripled fixed point for single-valued mappings in partially ordered metric spaces.

Karapinar [25] introduced the concept of quadruple fixed point for single valued mappings in partially ordered metric spaces and established the existence of quadruple fixed point for single-valued mappings in partially ordered metric spaces.

Many researchers have studied coupled, tripled and quadruple fixed point theorems for single valued mappings including([3], [5], [7], [9], [11], [16], [17], [18], [19], [23], [24], [28], [30], [36], [37], [38], [40], [42], [45]).

Recently Samet et al. [41] claimed that most of the coupled fixed point theorems for single valued mappings on ordered metric spaces are consequences of well-known fixed point theorems. Some of our basic references are ([12], [13], [32], [33], [34]).

Coupled fixed point theory were extended by Abbas et al. [2] to multivalued mappings and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces.

The concepts of tripled fixed point theory were extended by Deshpande et al. [20] in the settings of multivalued mappings and obtained tripled coincidence point and common tripled fixed point theorems involving hybrid pair of mappings under generalized nonlinear contraction.

Further quadruple fixed point theory were extended to multivalued mappings by Deshpande and Handa [21]: they obtained quadruple coincidence point and common quadruple fixed point theorems involving hybrid pair of mappings.
under $\varphi - \psi$ contraction.

In [21], Deshpande and Handa introduced the following for multivalued mappings:

**Definition 1.1.** Let $X$ be a non-empty set, $F : X^4 \to 2^X$ (a collection of all non-empty subsets of $X$) be a multivalued mapping and $g$ be a self-mapping on $X$. An element $(x, y, z, w) \in X^4$ is called

1. a quadruple fixed point of $F$ if $x \in F(x, y, z, w), y \in F(y, z, w, x), z \in F(z, w, x, y)$ and $w \in F(w, x, y, z)$.
2. a quadruple coincidence point of hybrid pair $\{F, g\}$ if $g(x) \in F(x, y, z, w), g(y) \in F(y, z, w, x), g(z) \in F(z, w, x, y)$ and $g(w) \in F(w, x, y, z)$.
3. a common quadruple fixed point of hybrid pair $\{F, g\}$ if $x = g(x) \in F(x, y, z, w), y = g(y) \in F(y, z, w, x), z = g(z) \in F(z, w, x, y)$ and $w = g(w) \in F(w, x, y, z)$.

We denote the set of quadruple coincidence points of mappings $F$ and $g$ by $C\{F, g\}$. Note that if $(x, y, z, w) \in C\{F, g\}$, then $(y, z, w, x), (z, w, x, y)$ and $(w, x, y, z)$ are also in $C\{F, g\}$.

**Definition 1.2.** Let $F : X^4 \to 2^X$ be a multivalued mapping and $g$ be a self-mapping on $X$. The hybrid pair $\{F, g\}$ is called $w$-compatible if $g(F(x, y, z, w)) \subseteq F(gx, gy, gz, gw)$ whenever $(x, y, z, w) \in C\{F, g\}$.

**Definition 1.3.** Let $F : X^4 \to 2^X$ be a multivalued mapping and $g$ be a self-mapping on $X$. The mapping $g$ is called $F$-weakly commuting at some point $(x, y, z, w) \in X^4$ if $g^2x \in F(gx, gy, gz, gw), g^2y \in F(gy, gz, gw, gx), g^2z \in F(gz, gw, gx, gy)$ and $g^2w \in F(gw, gx, gy, gz)$.

**Lemma 1.4.** Let $(X, d)$ be a metric space. Then, for each $a \in X$ and $B \in CB(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b \in B} d(a, b)$.

Very few papers were devoted to coupled, tripled and quadruple fixed point problems for hybrid pair of mappings including ([11], [2], [20], [21], [22], [29], [44]).

In this paper, we prove a quadruple coincidence and common quadruple fixed point for hybrid pair of mappings under generalized nonlinear contraction on a non-complete metric space, which is not partially ordered. It is to be noted that to find quadruple coincidence point, we do not employ the condition of continuity of any mapping involved therein. We improve, extend and generalize the results of Abbas et al. [2], Bhaskar and Lakshmikantham [10], Ding et al. [23] and Lakshmikantham and Ciric [27]. An example is also given to demonstrate the effectiveness of our result.
2. Main results

Let $\Phi$ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i) $\varphi$ is non-decreasing,

(ii) $\lim_{n \to \infty} \varphi^n(t) = 0$ for all $t > 0$, where $\varphi^{n+1}(t) = \varphi^n(\varphi(t))$.

It is clear that $\varphi(t) < t$ for each $t > 0$. In fact, if $\varphi(t_0) \geq t_0$ for some $t_0 > 0$, then, since $\varphi$ is non-decreasing, $\varphi^n(t_0) \geq t_0$ for all $n \in \mathbb{N}$, which contradicts with $\lim_{n \to \infty} \varphi^n(t_0) = 0$. In addition, it is easy to see that $\varphi(0) = 0$.

**Theorem 2.1.** Let $(X, d)$ be a metric space, $F : X^4 \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exists $\varphi \in \Phi$ such that

$$H(F(x, y, z, w), F(p, q, r, s)) \leq \varphi \left[ \max \left\{ \frac{d(gx, gp)}{D(gx, F(x, y, z, w))}, \frac{D(gp, F(p, q, r, s))}{D(gp, F(x, y, z, w))}, \frac{D(gy, gq)}{D(gy, F(y, z, w, x))}, \frac{D(gq, F(q, r, s, p))}{D(gq, F(y, z, w, x))}, \frac{d(gz, gr)}{D(gz, F(z, w, x, y))}, \frac{D(gr, F(r, s, p, q))}{D(gr, F(z, w, x, y))}, \frac{d(gw, gs)}{D(gw, F(w, x, y, z))}, \frac{D(gs, F(s, p, q, r))}{D(gs, F(w, x, y, z))} \right\} + \frac{1}{2} D(gp, F(x, y, z, w)) \right]$$

for all $x, y, z, w, p, q, r, s \in X$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of $X$. Then $F$ and $g$ have a quadruple coincidence point. Moreover, $F$ and $g$ have a common quadruple fixed point if one of the following conditions holds:

(a) $F$ and $g$ are $w$-compatible.

$$\lim_{n \to \infty} g^n x = p, \quad \lim_{n \to \infty} g^n y = q, \quad \lim_{n \to \infty} g^n z = r, \quad \lim_{n \to \infty} g^n w = s$$

for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and $g$ is continuous at $p, q, r$ and $s$.

(b) $g$ is $F$-weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and $gx, gy, gz$ and $gw$ are fixed points of $g$, that is, $g^2x = gx, g^2y = gy, g^2z = gz$ and $g^2w = gw$.

(c) $g$ is continuous at $x, y, z$ and $w$.

$$\lim_{n \to \infty} g^n p = x, \quad \lim_{n \to \infty} g^n q = y, \quad \lim_{n \to \infty} g^n r = z, \quad \lim_{n \to \infty} g^n s = w$$

for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.

(d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$. 
Proof. Let \( x_0, y_0, z_0, w_0 \in X \) be arbitrary. Then \( F(x_0, y_0, z_0, w_0), F(y_0, z_0, w_0, x_0), F(z_0, w_0, x_0, y_0) \) and \( F(w_0, x_0, y_0, z_0) \) are well defined. Choose \( g_1 \in F(x_0, y_0, z_0, w_0), g_2 \in F(y_0, z_0, w_0, x_0), g_3 \in F(z_0, w_0, x_0, y_0) \) and \( g_4 \in F(w_0, x_0, y_0, z_0) \), because \( F(X^4) \subseteq g(X) \). Since \( F : X^4 \to CB(X) \), therefore by Lemma 1.4, there exist \( u_1 \in F(x_1, y_1, z_1, w_1), u_2 \in F(y_1, z_1, w_1, x_1), u_3 \in F(z_1, w_1, x_1, y_1) \) and \( u_4 \in F(w_1, x_1, y_1, z_1) \) such that

\[
\begin{align*}
&d(g_1, u_1) \leq H(F(x_0, y_0, z_0, w_0), F(x_1, y_1, z_1, w_1)), \\
d(g_2, u_2) \leq H(F(y_0, z_0, w_0, x_0), F(y_1, z_1, w_1, x_1)), \\
d(g_3, u_3) \leq H(F(z_0, w_0, x_0, y_0), F(z_1, w_1, x_1, y_1)), \\
d(g_4, u_4) \leq H(F(w_0, x_0, y_0, z_0), F(w_1, x_1, y_1, z_1)).
\end{align*}
\]

Since \( F(X^4) \subseteq g(X) \), therefore there exist \( x_2, y_2, z_2, w_2 \in X \) such that \( u_1 = gx_2, u_2 = gy_2, u_3 = gz_2 \) and \( u_4 = gw_2 \). Thus

\[
\begin{align*}
&d(g_1, gx_2) \leq H(F(x_0, y_0, z_0, w_0), F(x_1, y_1, z_1, w_1)), \\
d(g_2, gy_2) \leq H(F(y_0, z_0, w_0, x_0), F(y_1, z_1, w_1, x_1)), \\
d(g_3, gz_2) \leq H(F(z_0, w_0, x_0, y_0), F(z_1, w_1, x_1, y_1)), \\
d(g_4, gw_2) \leq H(F(w_0, x_0, y_0, z_0), F(w_1, x_1, y_1, z_1)).
\end{align*}
\]

Continuing this process, we obtain sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) and \( \{w_n\} \) in \( X \) such that for all \( n \in \mathbb{N} \), we have \( gx_{n+1} \in F(x_n, y_n, z_n, w_n), gy_{n+1} \in F(y_n, z_n, w_n, x_n), gz_{n+1} \in F(z_n, w_n, x_n, y_n) \) and \( gw_{n+1} \in F(w_n, x_n, y_n, z_n) \) such that

\[
\begin{align*}
d(gx_n, gx_{n+1}) & \leq H(F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), F(x_n, y_n, z_n, w_n)) \\
& \leq \varphi \max \left\{ \begin{array}{l}
\frac{1}{2} d(gx_{n-1}, gy_n), D(gx_n, F(x_n, y_n, z_n, w_n)), \\
\frac{1}{2} d(gx_{n-1}, gz_n), D(gx_n, F(x_n, y_n, z_n, w_n)), \\
\frac{1}{2} d(gy_{n-1}, gy_n), D(gy_n, F(y_n, z_n, w_n, x_n)), \\
\frac{1}{2} d(gz_{n-1}, gy_n), D(gz_n, F(z_n, w_n, x_n, y_n)), \\
\frac{1}{2} d(gy_{n-1}, gw_n), D(gw_n, F(w_n, x_n, y_n, z_n)), \\
\frac{1}{2} d(gz_{n-1}, gw_n), D(gw_n, F(w_n, x_n, y_n, z_n)), \\
\frac{1}{2} d(gy_{n-1}, gw_n), D(gw_n, F(w_n, x_n, y_n, z_n)), \\
\frac{1}{2} d(gz_{n-1}, gw_n), D(gw_n, F(w_n, x_n, y_n, z_n)) \end{array} \right\}
\end{align*}
\]
Combining them, we get

Thus

Similarly

Combining them, we get

\[
\begin{align*}
\max & \quad \left\{ d(g_{x_{n-1}}, g_{x_n}), d(g_{x_n}, g_{x_{n+1}}), d(g_{y_{n-1}}, g_{y_n}), \\
& \quad \frac{d(gw_{n-1}, gw_n)}{2}, \frac{d(gw_{n-1}, gw_{n+1})}{2} \right\}, \\
\max & \quad \left\{ d(g_{y_{n-1}}, g_{y_{n+1}}), d(g_{y_{n}}, g_{y_{n+1}}), d(g_{z_{n-1}}, g_{z_n}), d(g_{z_{n}}, g_{z_{n+1}}), \\
& \quad \frac{d(gw_{n-1}, gw_n)}{2}, \frac{d(gw_{n-1}, gw_{n+1})}{2} \right\}, \\
\max & \quad \left\{ d(g_{z_{n-1}}, g_{z_{n}}), d(g_{z_{n}}, g_{z_{n+1}}), d(gw_{n-1}, gw_n), d(gw_{n}, gw_{n+1}), \\
& \quad \frac{d(gw_{n-1}, gw_{n+1})}{2} \right\}. \\
\end{align*}
\]
Thus

\[
\max \left\{ \begin{array}{ll}
& d(g_{x^{n-1}}, g_{x^n}), d(d(g_{x^n}, g_{x^{n+1}})), d(g_{y^{n-1}}, g_{y^n}), \\
& d(g_{y^n}, g_{y^{n+1}}), d(g_{z^{n-1}}, g_{z^n}), d(g_{z^n}, g_{z^{n+1}}), \\
& d(g_{w^{n-1}}, g_{w^n}), d(g_{w^n}, g_{w^{n+1}}), \\
& d(g_{x^{n-1}}, g_{y^n}) + d(g_{x^n}, g_{y^{n+1}}), d(g_{y^{n-1}}, g_{y^n}) + d(g_{y^n}, g_{y^{n+1}}), \\
& \frac{1}{2} d(g_{z^{n-1}}, g_{z^n}) + d(g_{z^n}, g_{z^{n+1}}), d(g_{w^{n-1}}, g_{w^n}) + d(g_{w^n}, g_{w^{n+1}})
\end{array} \right\} \]

\[
\leq \varphi \max \left\{ \begin{array}{ll}
& d(g_{x^{n-1}}, g_{x^n}), d(g_{y^{n-1}}, g_{y^n}), d(g_{z^{n-1}}, g_{z^n}), \\
& d(g_{w^{n-1}}, g_{w^n}), d(g_{x^n}, g_{x^{n+1}}), d(g_{y^n}, g_{y^{n+1}}), \\
& d(g_{z^n}, g_{z^{n+1}}), d(g_{w^n}, g_{w^{n+1}})
\end{array} \right\} .
\]

If we suppose that

\[
\max \left\{ \begin{array}{ll}
& d(g_{x^{n-1}}, g_{x^n}), d(g_{y^{n-1}}, g_{y^n}), d(g_{z^{n-1}}, g_{z^n}), \\
& d(g_{w^{n-1}}, g_{w^n}), d(g_{x^n}, g_{x^{n+1}}), d(g_{y^n}, g_{y^{n+1}}), \\
& d(g_{z^n}, g_{z^{n+1}}), d(g_{w^n}, g_{w^{n+1}})
\end{array} \right\} = \max \left\{ d(g_{x^n}, g_{x^{n+1}}), d(g_{y^n}, g_{y^{n+1}}), d(g_{z^n}, g_{z^{n+1}}), d(g_{w^n}, g_{w^{n+1}}) \right\} .
\]

Then, by (2) and by the fact that \( \varphi(t) < t \) for all \( t > 0 \), we have

\[
\max \left\{ \begin{array}{ll}
& d(g_{x^{n-1}}, g_{x^n}), d(g_{y^{n-1}}, g_{y^n}), \\
& d(g_{z^{n-1}}, g_{z^n}), d(g_{w^{n-1}}, g_{w^n})
\end{array} \right\} \leq \varphi \left( \max \left\{ \begin{array}{ll}
& d(g_{x^n}, g_{x^{n+1}}), d(g_{y^n}, g_{y^{n+1}}), \\
& d(g_{z^n}, g_{z^{n+1}}), d(g_{w^n}, g_{w^{n+1}})
\end{array} \right\} \right)
\]

\[
< \max \left\{ d(g_{x^n}, g_{x^{n+1}}), d(g_{y^n}, g_{y^{n+1}}), d(g_{z^n}, g_{z^{n+1}}), d(g_{w^n}, g_{w^{n+1}}) \right\} ,
\]

which is a contradiction. Thus we must have

\[
\max \left\{ \begin{array}{ll}
& d(g_{x^{n-1}}, g_{x^n}), d(g_{y^{n-1}}, g_{y^n}), d(g_{z^{n-1}}, g_{z^n}), \\
& d(g_{w^{n-1}}, g_{w^n}), d(g_{x^n}, g_{x^{n+1}}), d(g_{y^n}, g_{y^{n+1}}), \\
& d(g_{z^n}, g_{z^{n+1}}), d(g_{w^n}, g_{w^{n+1}})
\end{array} \right\} = \max \left\{ d(g_{x^{n-1}}, g_{x^n}), d(g_{y^{n-1}}, g_{y^n}), d(g_{z^{n-1}}, g_{z^n}), d(g_{w^{n-1}}, g_{w^n}) \right\} .
\]
Hence by (2), we have for all \( n \in \mathbb{N} \),

\[
\max \left\{ \begin{array}{c}
d(gx_n, gx_{n+1}), \
d(gy_n, gy_{n+1}), \\
d(gz_n, gz_{n+1}), \
d(gw_n, gw_{n+1})
\end{array} \right\} \leq \varphi \left[ \max \left\{ \begin{array}{c}
d(gx_{n-1}, gx_n), \
d(gy_{n-1}, gy_n), \\
d(gz_{n-1}, gz_n), \
d(gw_{n-1}, gw_n)
\end{array} \right\} \right] \\
\leq \varphi^n \left[ \max \left\{ \begin{array}{c}
d(gx_0, gx_1), \
d(gy_0, gy_1), \\
d(gz_0, gz_1), \
d(gw_0, gw_1)
\end{array} \right\} \right].
\]

Thus

\[
\max \left\{ \begin{array}{c}
d(gx_n, gx_{n+1}), \\
d(gy_n, gy_{n+1}), \\
d(gz_n, gz_{n+1}), \\
d(gw_n, gw_{n+1})
\end{array} \right\} \leq \varphi^n(\delta),
\]

where

\[ \delta = \max \left\{ \begin{array}{c}
d(gx_0, gx_1), \\
d(gy_0, gy_1), \\
d(gz_0, gz_1), \\
d(gw_0, gw_1)
\end{array} \right\}. \]

Without loss of generality, one can assume that \( \max\{d(gx_0, gx_1), d(gy_0, gy_1), d(gz_0, gz_1), d(gw_0, gw_1)\} \neq 0 \). In fact, if this is not true, then \( gx_0 = gx_1 \in F(x_0, y_0, z_0, w_0), gy_0 = gy_1 \in F(y_0, z_0, w_0, x_0), gz_0 = gz_1 \in F(z_0, w_0, x_0, y_0) \) and \( gw_0 = gw_1 \in F(w_0, x_0, y_0, z_0) \), that is, \( (x_0, y_0, z_0, w_0) \) is a quadruple coincidence point of \( F \) and \( g \). Thus, for \( m, n \in \mathbb{N} \) with \( m > n \), we have by (3) that

\[
d(gx_n, gx_{m+n}) \\
\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \ldots + d(gx_{n+m-1}, gx_{m+n}) \\
\leq \max \left\{ \begin{array}{c}
d(gx_n, gx_{n+1}), \\
d(gy_n, gy_{n+1}), \\
d(gz_n, gz_{n+1}), \\
d(gw_n, gw_{n+1})
\end{array} \right\} \\
+ \max \left\{ \begin{array}{c}
d(gx_{n+1}, gx_{n+2}), \\
d(gy_{n+1}, gy_{n+2}), \\
d(gz_{n+1}, gz_{n+2}), \\
d(gw_{n+1}, gw_{n+2})
\end{array} \right\} \\
+ \ldots + \max \left\{ \begin{array}{c}
d(gx_{n+m-1}, gx_{n+m}), \\
d(gy_{n+m-1}, gy_{n+m}), \\
d(gz_{n+m-1}, gz_{n+m}), \\
d(gw_{n+m-1}, gw_{n+m})
\end{array} \right\} \\
\leq \varphi^n(\delta) + \varphi^{n+1}(\delta) + \ldots + \varphi^{n+m-1}(\delta) \\
\leq \sum_{i=n}^{n+m-1} \varphi^i(\delta),
\]

which implies, by \( (ii)_\varphi \), that \( \{gx_n\} \) is a Cauchy sequence in \( g(X) \). Similarly we obtain that \( \{gy_n\}, \{gz_n\} \) and \( \{gw_n\} \) are Cauchy sequences in \( g(X) \). Since \( g(X) \) is complete, therefore there exist \( x, y, z, w \in X \) such that

\[
lim_{n \to \infty} gx_n = gx, \quad \lim_{n \to \infty} gy_n = gy, \quad \lim_{n \to \infty} gz_n = gz \quad \text{and} \quad \lim_{n \to \infty} gw_n = gw. \tag{4}
\]
Now, since $g x_{n+1} \in F(x_n, y_n, z_n, w_n)$, $g y_{n+1} \in F(y_n, z_n, w_n, x_n)$, $g z_{n+1} \in F(z_n, w_n, x_n, y_n)$ and $g w_{n+1} \in F(w_n, x_n, y_n, z_n)$, therefore

$$D(g x_{n+1}, F(x, y, z, w)) \leq H(F(x_n, y_n, z_n, w_n), F(x, y, z, w)),$$
$$D(g y_{n+1}, F(y, z, w, x)) \leq H(F(y_n, z_n, w_n, x_n), F(y, z, w, x)),$$
$$D(g z_{n+1}, F(z, w, x, y)) \leq H(F(z_n, w_n, x_n, y_n), F(z, w, x, y)),$$
$$D(g w_{n+1}, F(w, x, y, z)) \leq H(F(w_n, x_n, y_n, z_n), F(w, x, y, z)),$$

by using condition (1), we get

$$D(g x_{n+1}, F(x, y, z, w)) \leq \varphi[\Delta_n],$$
$$D(g y_{n+1}, F(y, z, w, x)) \leq \varphi[\Delta_n],$$
$$D(g z_{n+1}, F(z, w, x, y)) \leq \varphi[\Delta_n],$$
$$D(g w_{n+1}, F(w, x, y, z)) \leq \varphi[\Delta_n].$$

Thus

$$\max \left\{ \begin{array}{l}
D(g x_{n+1}, F(x, y, z, w)), \\
D(g y_{n+1}, F(y, z, w, x)), \\
D(g z_{n+1}, F(z, w, x, y)), \\
D(g w_{n+1}, F(w, x, y, z))
\end{array} \right\} \leq \varphi[\Delta_n],$$

where

$$\Delta_n = \max \left\{ \begin{array}{l}
d(g x_n, g x), d(g x_n, g x_{n+1}), D(g x, F(x, y, z, w)), \\
d(g y_n, g y), d(g y_n, g y_{n+1}), D(g y, F(y, z, w, x)), \\
d(g z_n, g z), d(g z_n, g z_{n+1}), D(g z, F(z, w, x, y)), \\
d(g w_n, g w), d(g w_n, g w_{n+1}), D(g w, F(w, x, y, z)), \\
\frac{D(g x_n, F(x, y, z, w)) + d(g x, g x_{n+1})}{2}, \\
\frac{D(g y_n, F(y, z, w, x)) + d(g y, g y_{n+1})}{2}, \\
\frac{D(g z_n, F(z, w, x, y)) + d(g z, g z_{n+1})}{2}, \\
\frac{D(g w_n, F(w, x, y, z)) + d(g w, g w_{n+1})}{2}
\end{array} \right\}.$$

Now, by (4), there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\Delta_n = \max \left\{ \begin{array}{l}
D(g x, F(x, y, z, w)), \\
D(g y, F(y, z, w, x)), \\
D(g z, F(z, w, x, y)), \\
D(g w, F(w, x, y, z))
\end{array} \right\}.$$
Combining this with (5), we get for all \( n > n_0 \),

\[
\max \left\{ \begin{array}{l}
D(g_{x_{n+1}}, F(x, y, z, w)), \\
D(g_{y_{n+1}}, F(y, z, w, x)), \\
D(g_{z_{n+1}}, F(z, w, x, y)), \\
D(g_{w_{n+1}}, F(w, x, y, z))
\end{array} \right\} \leq \varphi \max \left\{ \begin{array}{l}
D(gx, F(x, y, z, w)), \\
D(gy, F(y, z, w, x)), \\
D(gz, F(z, w, x, y)), \\
D(gw, F(w, x, y, z))
\end{array} \right\}.
\]

(6)

Now, we claim that

\[
\max \left\{ \begin{array}{l}
D(gx, F(x, y, z, w)), \\
D(gy, F(y, z, w, x)), \\
D(gz, F(z, w, x, y)), \\
D(gw, F(w, x, y, z))
\end{array} \right\} = 0. \tag{7}
\]

If this is not true, then

\[
\max \left\{ \begin{array}{l}
D(gx, F(x, y, z, w)), \\
D(gy, F(y, z, w, x)), \\
D(gz, F(z, w, x, y)), \\
D(gw, F(w, x, y, z))
\end{array} \right\} > 0.
\]

Then, by (6) and by the fact that \( \varphi(t) < t \) for all \( t > 0 \), we get

\[
\max \left\{ \begin{array}{l}
D(g_{x_{n+1}}, F(x, y, z, w)), \\
D(g_{y_{n+1}}, F(y, z, w, x)), \\
D(g_{z_{n+1}}, F(z, w, x, y)), \\
D(g_{w_{n+1}}, F(w, x, y, z))
\end{array} \right\} < \max \left\{ \begin{array}{l}
D(gx, F(x, y, z, w)), \\
D(gy, F(y, z, w, x)), \\
D(gz, F(z, w, x, y)), \\
D(gw, F(w, x, y, z))
\end{array} \right\}.
\]

Letting \( n \to \infty \) in the above inequality, by using (4), we obtain

\[
\max \left\{ \begin{array}{l}
D(gx, F(x, y, z, w)), \\
D(gy, F(y, z, w, x)), \\
D(gz, F(z, w, x, y)), \\
D(gw, F(w, x, y, z))
\end{array} \right\} \leq \max \left\{ \begin{array}{l}
D(gx, F(x, y, z, w)), \\
D(gy, F(y, z, w, x)), \\
D(gz, F(z, w, x, y)), \\
D(gw, F(w, x, y, z))
\end{array} \right\},
\]

which is a contradiction. So (7) holds. Thus, it follows that

\[
gx \in F(x, y, z, w), \ gy \in F(y, z, w, x), \\
gz \in F(z, w, x, y), \ gw \in F(w, x, y, z),
\]
that is, \((x, y, z, w)\) is a quadruple coincidence point of \(F\) and \(g\). Hence \(C\{F, g\}\) is non-empty.

Suppose now that \((a)\) holds. Assume that for some \((x, y, z, w) \in C\{F, g\}\),
\[
\lim_{n \to \infty} g^n x = p, \quad \lim_{n \to \infty} g^n y = q, \quad \lim_{n \to \infty} g^n z = r \quad \text{and} \quad \lim_{n \to \infty} g^n w = s,
\]
where \(p, q, r, s \in X\). Since \(g\) is continuous at \(p, q, r\) and \(s\). We have, by (8), that \(p, q, r\) and \(s\) are fixed points of \(g\), that is,
\[
gp = p, \quad gq = q, \quad gr = r \quad \text{and} \quad gs = s.
\]
As \(F\) and \(g\) are \(w\)-compatible, so, for all \(n \geq 1\),
\[
g^n x \in F(g^{n-1} x, g^{n-1} y, g^{n-1} z, g^{n-1} w),
g^n y \in F(g^{n-1} y, g^{n-1} z, g^{n-1} w, g^{n-1} x),
g^n z \in F(g^{n-1} z, g^{n-1} w, g^{n-1} x, g^{n-1} y),
g^n w \in F(g^{n-1} w, g^{n-1} x, g^{n-1} y, g^{n-1} z).
\]
Now, by using (10), we obtain
\[
D(g^n x, F(p,q,r,s)) \leq H(F(g^{n-1} x, g^{n-1} y, g^{n-1} z, g^{n-1} w), F(p,q,r,s)),
\]
\[
D(g^n y, F(q,r,s,p)) \leq H(F(g^{n-1} y, g^{n-1} z, g^{n-1} w, g^{n-1} x), F(q,r,s,p)),
\]
\[
D(g^n z, F(r,s,p,q)) \leq H(F(g^{n-1} z, g^{n-1} w, g^{n-1} x, g^{n-1} y), F(r,s,p,q)),
\]
\[
D(g^n w, F(s,p,q,r)) \leq H(F(g^{n-1} w, g^{n-1} x, g^{n-1} y, g^{n-1} z), F(s,p,q,r)),
\]
which, by (1), implies that
\[
D(g^n x, F(p, q, r, s)) \leq \varphi[\nabla_n],
\]
\[
D(g^n y, F(q, r, s, p)) \leq \varphi[\nabla_n],
\]
\[
D(g^n z, F(r, s, p, q)) \leq \varphi[\nabla_n],
\]
\[
D(g^n w, F(s, p, q, r)) \leq \varphi[\nabla_n],
\]
where
\[
\nabla_n = \max \left\{ \begin{array}{c}
d(g^n x, gp), D(gp, F(p,q,r,s)), D(g^n x, F(p,q,r,s)) + D(gp, g^n x) \\
d(g^n y, gq), D(gq, F(q,r,s,p)), D(g^n y, F(q,r,s,p)) + D(gq, g^n y) \\
d(g^n z, gr), D(gr, F(r,s,p,q)), D(g^n z, F(r,s,p,q)) + D(gr, g^n z) \\
d(g^n w, gs), D(gs, F(s,p,q,r)), D(g^n w, F(s,p,q,r)) + D(gs, g^n w)
\end{array} \right\}.
\]
By (8) and (9), there exists \(n_0 \in \mathbb{N}\) such that for all \(n > n_0\),
\[
\nabla_n = \max \left\{ \begin{array}{c}
D(gp, F(p, q, r, s)), \\
D(gq, F(q, r, s, p)), \\
D(gr, F(r, s, p, q)), \\
D(gs, F(s, p, q, r))
\end{array} \right\}.
\]
Combining this with (11), we get for all $n > n_0$,

\[
\max \left\{ \begin{array}{ll}
D(g^n x, F(p, q, r, s)), \\
D(g^n y, F(q, r, s, p)), \\
D(g^n z, F(r, s, p, q)), \\
D(g^n w, F(s, p, q, r))
\end{array} \right\}
\leq \nu \left( \max \left\{ \begin{array}{ll}
D(gp, F(p, q, r, s)), \\
D(gq, F(q, r, s, p)), \\
D(gr, F(r, s, p, q)), \\
D(gs, F(s, p, q, r))
\end{array} \right\} \right).
\]

(12)

Now, we claim that

\[
\max \left\{ \begin{array}{ll}
D(gp, F(p, q, r, s)), \\
D(gq, F(q, r, s, p)), \\
D(gr, F(r, s, p, q)), \\
D(gs, F(s, p, q, r))
\end{array} \right\} = 0.
\]

(13)

If this is not true, then

\[
\max \left\{ \begin{array}{ll}
D(gp, F(p, q, r, s)), \\
D(gq, F(q, r, s, p)), \\
D(gr, F(r, s, p, q)), \\
D(gs, F(s, p, q, r))
\end{array} \right\} > 0.
\]

Then, by (12) and by the fact that $\nu(t) < t$ for all $t > 0$, we get for all $n > n_0$,

\[
\max \left\{ \begin{array}{ll}
D(g^n x, F(p, q, r, s)), \\
D(g^n y, F(q, r, s, p)), \\
D(g^n z, F(r, s, p, q)), \\
D(g^n w, F(s, p, q, r))
\end{array} \right\} < \max \left\{ \begin{array}{ll}
D(gp, F(p, q, r, s)), \\
D(gq, F(q, r, s, p)), \\
D(gr, F(r, s, p, q)), \\
D(gs, F(s, p, q, r))
\end{array} \right\}.
\]

On taking limit as $n \to \infty$ in the above inequality, by using (8) and (9), we get

\[
\max \left\{ \begin{array}{ll}
D(gp, F(p, q, r, s)), \\
D(gq, F(q, r, s, p)), \\
D(gr, F(r, s, p, q)), \\
D(gs, F(s, p, q, r))
\end{array} \right\} < \max \left\{ \begin{array}{ll}
D(gp, F(p, q, r, s)), \\
D(gq, F(q, r, s, p)), \\
D(gr, F(r, s, p, q)), \\
D(gs, F(s, p, q, r))
\end{array} \right\},
\]

which is a contradiction. So (13) holds. Thus, it follows that

\[
gp \in F(p, q, r, s), \quad gq \in F(q, r, s, p), \\
gr \in F(r, s, p, q), \quad gs \in F(s, p, q, r).
\]

(14)
Thus, by (9) and (14), we have

\[ p = gp \in F(p, q, r, s), \quad q = gq \in F(q, r, s, p), \]
\[ r = gr \in F(r, s, p, q), \quad s = gs \in F(s, p, q, r), \]

that is, \((p, q, r, s)\) is a common quadruple fixed point of \(F\) and \(g\).

Suppose now that (b) holds. Assume that for some \((x, y, z, w) \in C\{F, g\}\), \(g\) is \(F\)-weakly commuting, that is, \(g^2x \in F(gx, gy, gz, gw), g^2y \in F(gy, gz, gw, gx), g^2z \in F(gz, gw, gx, gy), g^2w \in F(gw, gx, gy, gz)\) and \(g^2x = gx, g^2y = gy, g^2z = gz, g^2w = gw\). Thus \(gx = g^2x \in F(gx, gy, gz, gw), gy = g^2y \in F(gy, gz, gw, gx), gz = g^2z \in F(gz, gw, gx, gy)\) and \(gw = g^2w \in F(gw, gx, gy, gz)\), that is, \((gx, gy, gz, gw)\) is a common quadruple fixed point of \(F\) and \(g\).

Suppose now that (c) holds. Assume that for some \((x, y, z, w) \in C\{F, g\}\) and for some \(p, q, r, s \in X\),

\[ \lim_{n \to \infty} g^n p = x, \quad \lim_{n \to \infty} g^n q = y, \quad \lim_{n \to \infty} g^n r = z \quad \text{and} \quad \lim_{n \to \infty} g^n s = w. \tag{15} \]

Since \(g\) is continuous at \(x, y, z\) and \(w\). Therefore, by (15), we get that \(x, y, z\) and \(w\) are fixed points of \(g\), that is,

\[ gx = x, \quad gy = y, \quad gz = z \quad \text{and} \quad gw = w. \tag{16} \]

Since \((x, y, z, w) \in C\{F, g\}\). Therefore, by (16), we get

\[ x = gx \in F(x, y, z, w), \quad y = gy \in F(y, z, w, x), \]
\[ z = gz \in F(z, w, x, y), \quad w = gw \in F(w, x, y, z), \]

that is, \((x, y, z, w)\) is a common quadruple fixed point of \(F\) and \(g\).

Finally, suppose that (d) holds. Let \(g(C\{F, g\}) = \{(x, x, x, x)\}\). Then \(\{x\} = \{gx\} = F(x, x, x, x)\). Hence \((x, x, x, x)\) is quadruple fixed point of \(F\) and \(g\). \(\square\)

**Example 2.2.** Suppose that \(X = [0, 1]\), equipped with the metric \(d : X \times X \to [0, +\infty)\) defined by \(d(x, y) = \max\{x, y\}\) and \(d(x, x) = 0\) for all \(x, y \in X\). Let \(F : X^4 \to CB(X)\) be defined as

\[
F(x, y, z, w) = \begin{cases} 
\{0\}, & \text{for } x, y, z, w = 1, \\
0, \frac{x^2 + y^2 + z^2 + w^2}{8}, & \text{for } x, y, z, w \in [0, 1),
\end{cases}
\]

and \(g : X \to X\) be defined as

\[ g(x) = x^2, \quad \text{for all } x \in X. \]
Define \( \varphi : [0, +\infty) \to [0, +\infty) \) by
\[
\varphi(t) = \begin{cases} 
\frac{t}{4}, & \text{for } t \neq 1 \\
\frac{t}{4}, & \text{for } t = 1.
\end{cases}
\]

Now, for all \( x, y, z, w, p, q, r, s \in X \) with \( x, y, z, w, p, q, r, s \in [0, 1) \), we have

**Case (a)** If \( x^2 + y^2 + z^2 + w^2 = p^2 + q^2 + r^2 + s^2 \), then
\[
H(F(x, y, z, w), F(p, q, r, s)) = \frac{p^2 + q^2 + r^2 + s^2}{8}
\]
\[
\leq \frac{1}{8} \max \{x^2, p^2\} + \frac{1}{8} \max \{y^2, q^2\} + \frac{1}{8} \max \{z^2, r^2\} + \frac{1}{8} \max \{w^2, s^2\}
\]
\[
\leq \frac{1}{8} \left( d(gx, gp) + \frac{1}{8} d(gy, gq) + \frac{1}{8} d(gz, gr) + \frac{1}{8} d(gw, gs) \right)
\]
\[
\leq \frac{1}{2} \max \left\{ \begin{array}{l}
d(gx, gp), D(gx, F(x, y, z, w)), D(gp, F(p, q, r, s)),
d(gy, gq), D(gy, F(y, z, w, x)), D(gq, F(q, r, s, p)),
d(gz, gr), D(gz, F(z, w, x, y)), D(gr, F(r, s, p, q)),
d(gw, gs), D(gw, F(w, x, y, z)), D(gs, F(s, p, q, r)),
\end{array} \right\}
\]
\[
\leq \varphi \max \left\{ \begin{array}{l}
d(gx, gp), D(gx, F(x, y, z, w)), D(gp, F(p, q, r, s)),
d(gy, gq), D(gy, F(y, z, w, x)), D(gq, F(q, r, s, p)),
d(gz, gr), D(gz, F(z, w, x, y)), D(gr, F(r, s, p, q)),
d(gw, gs), D(gw, F(w, x, y, z)), D(gs, F(s, p, q, r)),
\end{array} \right\}
\]

**Case (b)** If \( x^2 + y^2 + z^2 + w^2 \neq p^2 + q^2 + r^2 + s^2 \) with \( x^2 + y^2 + z^2 + w^2 < p^2 + q^2 + r^2 + s^2 \), then
\[
H(F(x, y, z, w), F(p, q, r, s)) = \frac{p^2 + q^2 + r^2 + s^2}{8}
\]
\[
\leq \frac{1}{8} \max \{x^2, p^2\} + \frac{1}{8} \max \{y^2, q^2\} + \frac{1}{8} \max \{z^2, r^2\} + \frac{1}{8} \max \{w^2, s^2\}
\]
\[
\leq \frac{1}{8} \left( d(gx, gp) + \frac{1}{8} d(gy, gq) + \frac{1}{8} d(gz, gr) + \frac{1}{8} d(gw, gs) \right) \leq
\]

Similarly, we obtain the same result for \( p^2 + q^2 + r^2 + s^2 < x^2 + y^2 + z^2 + w^2 \). Thus the contractive condition (1) is satisfied for all \( x, y, z, w, p, q, r, s \in X \) with \( x, y, z, w, p, q, r, s \in [0, 1] \). Again, for all \( x, y, z, w, p, q, r, s \in X \) with \( x, y, z, \) \( w \in [0, 1] \) and \( p, q, r, s = 1 \), we have

\[
H(F(x, y, z, w), F(p, q, r, s)) = \frac{x^2 + y^2 + z^2 + w^2}{8} \leq \frac{1}{8} \max\{x^2, p^2\} + \frac{1}{8} \max\{y^2, q^2\} + \frac{1}{8} \max\{z^2, r^2\} + \frac{1}{8} \max\{w^2, s^2\} \leq \frac{1}{8} d(gx, gp) + \frac{1}{8} d(gy, gq) + \frac{1}{8} d(gz, gr) + \frac{1}{8} d(gw, gs)
\]
Corollary 2.4. Let \((X, d)\) be a complete metric space, \(F : X^4 \to CB(X)\) be a
mapping. Suppose that there exists $\varphi \in \Phi$ such that

$$H(F(x, y, z, w), F(p, q, r, s))$$

$$\leq \varphi \max \left\{ \frac{d(x, p), D(x, F(x, y, z, w)), D(p, F(p, q, r, s))}{2}, \frac{d(y, q), D(y, F(y, z, w, x)), D(q, F(q, r, s, p))}{2}, \frac{d(z, r), D(z, F(z, w, x, y)), D(r, F(r, s, p, q))}{2}, \frac{d(w, s), D(w, F(w, x, y, z)), D(s, F(s, p, q, r))}{2} \right\},$$

for all $x, y, z, w, p, q, r, s \in X$. Then $F$ has a quadruple fixed point.

If we put $\varphi(t) = kt$ where $0 < k < 1$ in Theorem 2.1, then we have the following result:

**Corollary 2.5.** Let $(X, d)$ be a metric space. Assume $F : X^4 \to CB(X)$ and $g : X \to X$ be two mappings satisfying

$$H(F(x, y, z, w), F(p, q, r, s))$$

$$\leq k \max \left\{ \frac{d(gx, gp), D(gx, F(x, y, z, w)), D(gp, F(p, q, r, s))}{2}, \frac{d(gy, gq), D(gy, F(y, z, w, x)), D(gq, F(q, r, s, p))}{2}, \frac{d(gz, gr), D(gz, F(z, w, x, y)), D(gr, F(r, s, p, q))}{2}, \frac{d(gw, gs), D(gw, F(w, x, y, z)), D(gs, F(s, p, q, r))}{2} \right\},$$

for all $x, y, z, w, p, q, r, s \in X$, where $0 < k < 1$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of $X$. Then $F$ and $g$ have a quadruple coincidence point. Moreover, $F$ and $g$ have a common quadruple fixed point, if one of the following conditions holds:

(a) $F$ and $g$ are $w$-compatible.

$$\lim_{n \to \infty} g^n x = p, \lim_{n \to \infty} g^n y = q, \lim_{n \to \infty} g^n z = r, \lim_{n \to \infty} g^n w = s$$

for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and $g$ is continuous at $p, q, r$ and $s$.

(b) $g$ is $F$-weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and $gx, gy, gz$ and $gw$ are fixed points of $g$, that is, $g^2 x = gx, g^2 y = gy, g^2 z = gz$ and $g^2 w = gw.$
(c) $g$ is continuous at $x, y, z$ and $w$.

$$\lim_{n \to \infty} g^n p = x, \lim_{n \to \infty} g^n q = y, \lim_{n \to \infty} g^n r = z, \lim_{n \to \infty} g^n s = w$$

for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.

(d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

If we put $g = I$ ($I$ is the identity mapping) in Corollary 2.5, then we have the following result:

**Corollary 2.6.** Let $(X, d)$ be a complete metric space, $F : X^4 \to CB(X)$ be a mapping satisfying

$$H(F(x, y, z, w), F(p, q, r, s)) \leq k \max\left\{ \begin{array}{l}
\frac{d(x, p) + D(x, F(x, y, z, w)) + D(p, F(p, q, r, s))}{d(y, q) + D(y, F(y, z, w, x)) + D(q, F(q, r, s, p))}, \\
\frac{d(z, r) + D(z, F(z, w, x, y)) + D(r, F(r, s, p, q))}{d(w, s) + D(w, F(w, x, y, z)) + D(s, F(s, p, q, r))}, \\
\frac{D(x, F(p, q, r, s)) + D(p, F(x, y, z, w))}{D(y, F(q, r, s, p)) + D(q, F(y, z, w, x))}, \\
\frac{D(z, F(r, s, p, q)) + D(r, F(z, w, x, y))}{D(w, F(s, p, q, r)) + D(s, F(w, x, y, z))} \end{array} \right\},$$

for all $x, y, z, w, p, q, r, s \in X$, where $0 < k < 1$. Then $F$ has a quadruple fixed point.

**REFERENCES**


[38] Z. Mustafa - W. A. Shatanawi - E. Karapinar, Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces, J. Appl. Math. 2012, Article Id: 951912.


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