LE MATEMATICHE
Vol. LXX (2015) – Fasc. I, pp. 157–177
doi: 10.4418/2015.70.1.12

QUADRUPLE FIXED POINT THEOREM FOR HYBRID PAIR OF MAPPINGS UNDER GENERALIZED NONLINEAR CONTRACTION

BHAVANA DESHPANDE - AMRISH HANDA

We establish a quadruple coincidence and common quadruple fixed point theorem for hybrid pair of mappings under generalized nonlinear contraction on a non-complete metric space, which is not partially ordered. It is to be noted that to find quadruple coincidence point, we do not employ the condition of continuity of any mapping involved therein. We also give an example to validate our result. We improve, extend and generalize various known results.

1. Introduction and Preliminaries

Let $(X,\ d)$ be a metric space and $CB(X)$ be the set of all non-empty closed bounded subsets of X. Let $D(x,A)$ denote the distance from x to $A \subset X$ and H denote the Hausdorff metric induced by d, that is,

$$D(x,\ A) = \inf_{a \in A} d(x,\ a)$$
$$\text{and } H(A,\ B) = \max\left\{\sup_{a \in A} D(a,\ B),\ \sup_{b \in B} D(b,\ A)\right\},\ \text{for all } A,\ B \in CB(X).$$

Entrato in redazione: 23 maggio 2014

AMS 2010 Subject Classification: 47H10, 54H25.

Keywords: Quadruple fixed point, quadruple coincidence point, generalized nonlinear contraction, w-compatibility, F-weakly commutativity.

The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was initiated by Markin [31]. The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to ([4], [14], [15], [26], [35], [43]) and the reference therein. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics. Nadler [39] extended the famous Banach Contraction Principle [6] from single-valued mapping to multivalued mapping.

Bhaskar and Lakshmikantham [10] established some coupled fixed point theorems and applied these results to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [27] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results of Bhaskar and Lakshmikantham [10].

Berinde and Borcut [8] introduced the concept of tripled fixed point for single-valued mappings in partially ordered metric spaces and established the existence of tripled fixed point for single-valued mappings in partially ordered metric spaces.

Karapinar [25] introduced the concept of quadruple fixed point for single valued mappings in partially ordered metric spaces and established the existence of quadruple fixed point for single-valued mappings in partially ordered metric spaces.

Many researchers have studied coupled, tripled and quadruple fixed point theorems for single valued mappings including([3], [5], [7], [9], [11], [16], [17], [18], [19], [23], [24], [28], [30], [36], [37], [38], [40], [42], [45]).

Recently Samet et al. [41] claimed that most of the coupled fixed point theorems for single valued mappings on ordered metric spaces are consequences of well-known fixed point theorems. Some of our basic references are ([12], [13], [32], [33], [34]).

Coupled fixed point theory were extended by Abbas et al. [2] to multivalued mappings and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces.

The concepts of tripled fixed point theory were extended by Deshpande et al. [20] in the settings of multivalued mappings and obtained tripled coincidence point and common tripled fixed point theorems involving hybrid pair of mappings under generalized nonlinear contraction.

Further quadruple fixed point theory were extended to multivalued mappings by Deshpande and Handa [21]: they obtained quadruple coincidence point and common quadruple fixed point theorems involving hybrid pair of mappings under $\varphi - \psi$ contraction.

In [21], Deshpande and Handa introduced the following for multivalued mappings:

Definition 1.1. Let X be a non-empty set, $F : X^4 \to 2^X$ (a collection of all non-empty subsets of X) be a multivalued mapping and g be a self-mapping on X. An element $(x, y, z, w) \in X^4$ is called

(1) a quadruple fixed point of F if $x \in F(x, y, z, w)$, $y \in F(y, z, w, x)$, $z \in F(z, w, x, y)$ and $w \in F(w, x, y, z)$.

(2) a quadruple coincidence point of hybrid pair $\{F, g\}$ if $g(x) \in F(x, y, z, w)$, $g(y) \in F(y, z, w, x)$, $g(z) \in F(z, w, x, y)$ and $g(w) \in F(w, x, y, z)$.

(3) a common quadruple fixed point of hybrid pair $\{F, g\}$ if $x = g(x) \in F(x, y, z, w)$, $y = g(y) \in F(y, z, w, x)$, $z = g(z) \in F(z, w, x, y)$ and $w = g(w) \in F(w, x, y, z)$.

We denote the set of quadruple coincidence points of mappings F and g by $C\{F, g\}$. Note that if $(x, y, z, w) \in C\{F, g\}$, then (y, z, w, x), (z, w, x, y) and (w, x, y, z) are also in $C\{F, g\}$.

Definition 1.2. Let $F : X^4 \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called w-compatible if $g(F(x, y, z, w)) \subseteq F(gx, gy, gz, gw)$ whenever $(x, y, z, w) \in C\{F, g\}$.

Definition 1.3. Let $F : X^4 \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called F-weakly commuting at some point $(x, y, z, w) \in X^4$ if $g^2x \in F(gx, gy, gz, gw)$, $g^2y \in F(gy, gz, gw, gx)$, $g^2z \in F(gz, gw, gx, gy)$ and $g^2w \in F(gw, gx, gy, gz)$.

Lemma 1.4. *Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in CB(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b \in B} d(a, b)$.*

Very few papers were devoted to coupled, tripled and quadruple fixed point problems for hybrid pair of mappings including ([1], [2], [20], [21], [22], [29], [44]).

In this paper, we prove a quadruple coincidence and common quadruple fixed point for hybrid pair of mappings under generalized nonlinear contraction on a non-complete metric space, which is not partially ordered. It is to be noted that to find quadruple coincidence point, we do not employ the condition of continuity of any mapping involved therein. We improve, extend and generalize the results of Abbas et al. [2], Bhaskar and Lakshmikantham [10], Ding et al. [23] and Lakshmikantham and Ciric [27]. An example is also given to demonstrate the effectiveness of our result.

2. Main results

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_φ) φ is non-decreasing,

(ii_φ) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, where $\varphi^{n+1}(t) = \varphi^n(\varphi(t))$.

It is clear that $\varphi(t) < t$ for each $t > 0$. In fact, if $\varphi(t_0) \geq t_0$ for some $t_0 > 0$, then, since φ is non-decreasing, $\varphi^n(t_0) \geq t_0$ for all $n \in \mathbb{N}$, which contradicts with $\lim_{n\to\infty} \varphi^n(t_0) = 0$. In addition, it is easy to see that $\varphi(0) = 0$.

Theorem 2.1. *Let (X, d) be a metric space, $F : X^4 \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exists $\varphi \in \Phi$ such that*

$$H(F(x,y,z,w),F(p,q,r,s)) \tag{1}$$
$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx,\, gp),\, D(gx,\, F(x,\, y,\, z,\, w)),\, D(gp,\, F(p,\, q,\, r,\, s)),\\ d(gy,\, gq),\, D(gy,\, F(y,\, z,\, w,\, x)),\, D(gq,\, F(q,\, r,\, s,\, p)),\\ d(gz,\, gr),\, D(gz,\, F(z,\, w,\, x,\, y)),\, D(gr,\, F(r,\, s,\, p,\, q)),\\ d(gw,\, gs),\, D(gw,\, F(w,\, x,\, y,\, z)),\, D(gs,\, F(s,\, p,\, q,\, r)),\\ \frac{D(gx,\, F(p,\, q,\, r,\, s))+D(gp,\, F(x,\, y,\, z,\, w))}{2},\\ \frac{D(gy,\, F(q,\, r,\, s,\, p))+D(gq,\, F(y,\, z,\, w,\, x))}{2},\\ \frac{D(gz,\, F(r,\, s,\, p,\, q))+D(gr,\, F(z,\, w,\, x,\, y))}{2},\\ \frac{D(gw,\, F(s,\, p,\, q,\, r))+D(gs,\, F(w,\, x,\, y,\, z))}{2} \end{array}\right\}\right],$$

for all $x, y, z, w, p, q, r, s \in X$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a quadruple coincidence point. Moreover, F and g have a common quadruple fixed point if one of the following conditions holds:

(a) F and g are w-compatible.

$$\lim_{n\to\infty} g^n x = p,\ \lim_{n\to\infty} g^n y = q,\ \lim_{n\to\infty} g^n z = r,\ \lim_{n\to\infty} g^n w = s$$

for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and g is continuous at p, q, r and s.

(b) g is F-weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and gx, gy, gz and gw are fixed points of g, that is, $g^2x = gx$, $g^2y = gy$, $g^2z = gz$ and $g^2w = gw$.

(c) g is continuous at x, y, z and w.

$$\lim_{n\to\infty} g^n p = x,\ \lim_{n\to\infty} g^n q = y,\ \lim_{n\to\infty} g^n r = z,\ \lim_{n\to\infty} g^n s = w$$

for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.

(d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

Proof. Let $x_0, y_0, z_0, w_0 \in X$ be arbitrary. Then $F(x_0, y_0, z_0, w_0)$, $F(y_0, z_0, w_0, x_0)$, $F(z_0, w_0, x_0, y_0)$ and $F(w_0, x_0, y_0, z_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0, z_0, w_0)$, $gy_1 \in F(y_0, z_0, w_0, x_0)$, $gz_1 \in F(z_0, w_0, x_0, y_0)$ and $gw_1 \in F(w_0, x_0, y_0, z_0)$, because $F(X^4) \subseteq g(X)$. Since $F : X^4 \to CB(X)$, therefore by Lemma 1.4, there exist $u_1 \in F(x_1, y_1, z_1, w_1)$, $u_2 \in F(y_1, z_1, w_1, x_1)$, $u_3 \in F(z_1, w_1, x_1, y_1)$ and $u_4 \in F(w_1, x_1, y_1, z_1)$ such that

$$\begin{aligned}
d(gx_1, u_1) &\leq H(F(x_0, y_0, z_0, w_0), F(x_1, y_1, z_1, w_1)),\\
d(gy_1, u_2) &\leq H(F(y_0, z_0, w_0, x_0), F(y_1, z_1, w_1, x_1)),\\
d(gz_1, u_3) &\leq H(F(z_0, w_0, x_0, y_0), F(z_1, w_1, x_1, y_1)),\\
d(gw_1, u_4) &\leq H(F(w_0, x_0, y_0, z_0), F(w_1, x_1, y_1, z_1)).
\end{aligned}$$

Since $F(X^4) \subseteq g(X)$, therefore there exist $x_2, y_2, z_2, w_2 \in X$ such that $u_1 = gx_2$, $u_2 = gy_2$, $u_3 = gz_2$ and $u_4 = gw_2$. Thus

$$\begin{aligned}
d(gx_1, gx_2) &\leq H(F(x_0, y_0, z_0, w_0), F(x_1, y_1, z_1, w_1)),\\
d(gy_1, gy_2) &\leq H(F(y_0, z_0, w_0, x_0), F(y_1, z_1, w_1, x_1)),\\
d(gz_1, gz_2) &\leq H(F(z_0, w_0, x_0, y_0), F(z_1, w_1, x_1, y_1)),\\
d(gw_1, gw_2) &\leq H(F(w_0, x_0, y_0, z_0), F(w_1, x_1, y_1, z_1)).
\end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in X such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n, y_n, z_n, w_n)$, $gy_{n+1} \in F(y_n, z_n, w_n, x_n)$, $gz_{n+1} \in F(z_n, w_n, x_n, y_n)$ and $gw_{n+1} \in F(w_n, x_n, y_n, z_n)$ such that

$$\begin{aligned}
&d(gx_n, gx_{n+1})\\
&\leq H(F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), F(x_n, y_n, z_n, w_n))\\
&\leq \varphi\left[\max\left\{\begin{array}{c}
d(gx_{n-1}, gx_n), D(gx_n, F(x_n, y_n, z_n, w_n)),\\
D(gx_{n-1}, F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1})),\\
d(gy_{n-1}, gy_n), D(gy_n, F(y_n, z_n, w_n, x_n)),\\
D(gy_{n-1}, F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1})),\\
d(gz_{n-1}, gz_n), D(gz_n, F(z_n, w_n, x_n, y_n)),\\
D(gz_{n-1}, F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1})),\\
d(gw_{n-1}, gw_n), D(gw_n, F(w_n, x_n, y_n, z_n)),\\
D(gw_{n-1}, F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1})),\\
\frac{D(gx_{n-1}, F(x_n, y_n, z_n, w_n)) + D(gx_n, F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}))}{2},\\
\frac{D(gy_{n-1}, F(y_n, z_n, w_n, x_n)) + D(gy_n, F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}))}{2},\\
\frac{D(gz_{n-1}, F(z_n, w_n, x_n, y_n)) + D(gz_n, F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}))}{2},\\
\frac{D(gw_{n-1}, F(w_n, x_n, y_n, z_n)) + D(gw_n, F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}))}{2}
\end{array}\right\}\right] \leq
\end{aligned}$$

$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),d(gy_{n-1},gy_n), \\ d(gy_n,gy_{n+1}),d(gz_{n-1},gz_n),d(gz_n,gz_{n+1}), \\ d(gw_{n-1},gw_n),d(gw_n,gw_{n+1}),\frac{d(gx_{n-1},gx_{n+1})}{2}, \\ \frac{d(gy_{n-1},gy_{n+1})}{2},\frac{d(gz_{n-1},gz_{n+1})}{2},\frac{d(gw_{n-1},gw_{n+1})}{2} \end{array}\right\}\right].$$

Thus

$$d(gx_n,gx_{n+1})$$
$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),d(gy_{n-1},gy_n), \\ d(gy_n,gy_{n+1}),d(gz_{n-1},gz_n),d(gz_n,gz_{n+1}), \\ d(gw_{n-1},gw_n),d(gw_n,gw_{n+1}),\frac{d(gx_{n-1},gx_{n+1})}{2}, \\ \frac{d(gy_{n-1},gy_{n+1})}{2},\frac{d(gz_{n-1},gz_{n+1})}{2},\frac{d(gw_{n-1},gw_{n+1})}{2} \end{array}\right\}\right].$$

Similarly

$$d(gy_n,gy_{n+1})$$
$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),d(gy_{n-1},gy_n), \\ d(gy_n,gy_{n+1}),d(gz_{n-1},gz_n),d(gz_n,gz_{n+1}), \\ d(gw_{n-1},gw_n),d(gw_n,gw_{n+1}),\frac{d(gx_{n-1},gx_{n+1})}{2}, \\ \frac{d(gy_{n-1},gy_{n+1})}{2},\frac{d(gz_{n-1},gz_{n+1})}{2},\frac{d(gw_{n-1},gw_{n+1})}{2} \end{array}\right\}\right],$$

$$d(gz_n,gz_{n+1})$$
$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),d(gy_{n-1},gy_n), \\ d(gy_n,gy_{n+1}),d(gz_{n-1},gz_n),d(gz_n,gz_{n+1}), \\ d(gw_{n-1},gw_n),d(gw_n,gw_{n+1}),\frac{d(gx_{n-1},gx_{n+1})}{2}, \\ \frac{d(gy_{n-1},gy_{n+1})}{2},\frac{d(gz_{n-1},gz_{n+1})}{2},\frac{d(gw_{n-1},gw_{n+1})}{2} \end{array}\right\}\right],$$

$$d(gw_n,gw_{n+1})$$
$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),d(gy_{n-1},gy_n), \\ d(gy_n,gy_{n+1}),d(gz_{n-1},gz_n),d(gz_n,gz_{n+1}), \\ d(gw_{n-1},gw_n),d(gw_n,gw_{n+1}),\frac{d(gx_{n-1},gx_{n+1})}{2}, \\ \frac{d(gy_{n-1},gy_{n+1})}{2},\frac{d(gz_{n-1},gz_{n+1})}{2},\frac{d(gw_{n-1},gw_{n+1})}{2} \end{array}\right\}\right].$$

Combining them, we get

$$\max\left\{\begin{array}{c} d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}), \\ d(gz_n,gz_{n+1}),d(gw_n,gw_{n+1}) \end{array}\right\}$$
$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),d(gy_{n-1},gy_n), \\ d(gy_n,gy_{n+1}),d(gz_{n-1},gz_n),d(gz_n,gz_{n+1}), \\ d(gw_{n-1},gw_n),d(gw_n,gw_{n+1}),\frac{d(gx_{n-1},gx_{n+1})}{2}, \\ \frac{d(gy_{n-1},gy_{n+1})}{2},\frac{d(gz_{n-1},gz_{n+1})}{2},\frac{d(gw_{n-1},gw_{n+1})}{2} \end{array}\right\}\right] \leq$$

$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),d(gy_{n-1},gy_n),\\ d(gy_n,gy_{n+1}),d(gz_{n-1},gz_n),d(gz_n,gz_{n+1}),\\ d(gw_{n-1},gw_n),d(gw_n,gw_{n+1}),\\ \frac{d(gx_{n-1},gx_n)+d(gx_n,gx_{n+1})}{2},\frac{d(gy_{n-1},gy_n)+d(gy_n,gy_{n+1})}{2},\\ \frac{d(gz_{n-1},gz_n)+d(gz_n,gz_{n+1})}{2},\frac{d(gw_{n-1},gw_n)+d(gw_n,gw_{n+1})}{2} \end{array}\right\}\right]$$

$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},gx_n),d(gy_{n-1},gy_n),d(gz_{n-1},gz_n),\\ d(gw_{n-1},gw_n),d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),\\ d(gz_n,gz_{n+1}),d(gw_n,gw_{n+1}) \end{array}\right\}\right].$$

Thus

$$\max\left\{\begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}),\\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\} \tag{2}$$

$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n),\ d(gz_{n-1},\ gz_n),\\ d(gw_{n-1},\ gw_n),\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}),\\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\}\right].$$

If we suppose that

$$\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n),\ d(gz_{n-1},\ gz_n),\\ d(gw_{n-1},\ gw_n),\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}),\\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}),\\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\}.$$

Then, by (2) and by the fact that $\varphi(t) < t$ for all $t > 0$, we have

$$\max\left\{\begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}),\\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\}$$
$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}),\\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\}\right]$$
$$< \max\left\{\begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}),\\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\},$$

which is a contradiction. Thus we must have

$$\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n),\ d(gz_{n-1},\ gz_n),\\ d(gw_{n-1},\ gw_n),\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}),\\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n),\\ d(gz_{n-1},\ gz_n),\ d(gw_{n-1},\ gw_n) \end{array}\right\}.$$

Hence by (2), we have for all $n \in \mathbb{N}$,

$$
\begin{aligned}
&\max\left\{\begin{array}{l} d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}), \\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\} \\
&\leq \varphi\left[\max\left\{\begin{array}{l} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gz_{n-1},\ gz_n),\ d(gw_{n-1},\ gw_n) \end{array}\right\}\right] \\
&\leq\ \varphi^n\left[\max\left\{\begin{array}{l} d(gx_0,\ gx_1),\ d(gy_0,\ gy_1), \\ d(gz_0,\ gz_1),\ d(gw_0,\ gw_1) \end{array}\right\}\right].
\end{aligned}
$$

Thus

$$
\max\left\{\begin{array}{l} d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}), \\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\} \leq \varphi^n(\delta), \tag{3}
$$

where

$$
\delta = \max\left\{\begin{array}{l} d(gx_0,\ gx_1),\ d(gy_0,\ gy_1), \\ d(gz_0,\ gz_1),\ d(gw_0,\ gw_1) \end{array}\right\}.
$$

Without loss of generality, one can assume that $\max\{d(gx_0,\ gx_1),\ d(gy_0,\ gy_1),\ d(gz_0,\ gz_1),\ d(gw_0,\ gw_1)\} \neq 0$. In fact, if this is not true, then $gx_0 = gx_1 \in F(x_0,\ y_0,\ z_0,\ w_0)$, $gy_0 = gy_1 \in F(y_0,\ z_0,\ w_0,\ x_0)$, $gz_0 = gz_1 \in F(z_0,\ w_0,\ x_0,\ y_0)$ and $gw_0 = gw_1 \in F(w_0, x_0, y_0, z_0)$, that is, (x_0, y_0, z_0, w_0) is a quadruple coincidence point of F and g. Thus, for $m,\ n \in \mathbb{N}$ with $m > n$, we have by (3) that

$$
\begin{aligned}
&d(gx_n,\ gx_{m+n}) \\
&\leq d(gx_n,\ gx_{n+1}) + d(gx_{n+1},\ gx_{n+2}) + \ldots + d(gx_{n+m-1},\ gx_{m+n}) \\
&\leq \max\left\{\begin{array}{l} d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}), \\ d(gz_n,\ gz_{n+1}),\ d(gw_n,\ gw_{n+1}) \end{array}\right\} \\
&+ \max\left\{\begin{array}{l} d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2}), \\ d(gz_{n+1},\ gz_{n+2}),\ d(gw_{n+1},\ gw_{n+2}) \end{array}\right\} \\
&+ \ldots + \max\left\{\begin{array}{l} d(gx_{n+m-1},\ gx_{n+m}),\ d(gy_{n+m-1},\ gy_{n+m}), \\ d(gz_{n+m-1},\ gz_{n+m}),\ d(gw_{n+m-1},\ gw_{n+m}) \end{array}\right\} \\
&\leq \varphi^n(\delta) + \varphi^{n+1}(\delta) + \ldots + \varphi^{n+m-1}(\delta) \\
&\leq \sum_{i=n}^{n+m-1} \varphi^i(\delta),
\end{aligned}
$$

which implies, by (ii_φ), that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Similarly we obtain that $\{gy_n\}$, $\{gz_n\}$ and $\{gw_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, therefore there exist $x, y, z, w \in X$ such that

$$
\lim_{n\to\infty} gx_n = gx,\ \lim_{n\to\infty} gy_n = gy,\ \lim_{n\to\infty} gz_n = gz \text{ and } \lim_{n\to\infty} gw_n = gw. \tag{4}
$$

Now, since $gx_{n+1} \in F(x_n, y_n, z_n, w_n)$, $gy_{n+1} \in F(y_n, z_n, w_n, x_n)$, $gz_{n+1} \in F(z_n, w_n, x_n, y_n)$ and $gw_{n+1} \in F(w_n, x_n, y_n, z_n)$, therefore

$$\begin{aligned} D(gx_{n+1}, F(x, y, z, w)) &\leq H(F(x_n, y_n, z_n, w_n), F(x, y, z, w)), \\ D(gy_{n+1}, F(y, z, w, x)) &\leq H(F(y_n, z_n, w_n, x_n), F(y, z, w, x)), \\ D(gz_{n+1}, F(z, w, x, y)) &\leq H(F(z_n, w_n, x_n, y_n), F(z, w, x, y)), \\ D(gw_{n+1}, F(w, x, y, z)) &\leq H(F(w_n, x_n, y_n, z_n), F(w, x, y, z)), \end{aligned}$$

by using condition (1), we get

$$\begin{aligned} D(gx_{n+1}, F(x, y, z, w)) &\leq \varphi[\Delta_n], \\ D(gy_{n+1}, F(y, z, w, x)) &\leq \varphi[\Delta_n], \\ D(gz_{n+1}, F(z, w, x, y)) &\leq \varphi[\Delta_n], \\ D(gw_{n+1}, F(w, x, y, z)) &\leq \varphi[\Delta_n]. \end{aligned}$$

Thus

$$\max\left\{\begin{array}{c} D(gx_{n+1}, F(x, y, z, w)), \\ D(gy_{n+1}, F(y, z, w, x)), \\ D(gz_{n+1}, F(z, w, x, y)), \\ D(gw_{n+1}, F(w, x, y, z)) \end{array}\right\} \leq \varphi[\Delta_n], \tag{5}$$

where

$$\Delta_n = \max\left\{\begin{array}{c} d(gx_n, gx), d(gx_n, gx_{n+1}), D(gx, F(x,y,z,w)), \\ d(gy_n, gy), d(gy_n, gy_{n+1}), D(gy, F(y,z,w,x)), \\ d(gz_n, gz), d(gz_n, gz_{n+1}), D(gz, F(z,w,x,y)), \\ d(gw_n, gw), d(gw_n, gw_{n+1}), D(gw, F(w,x,y,z)), \\ \frac{D(gx_n, F(x,y,z,w)) + d(gx, gx_{n+1})}{2}, \\ \frac{D(gy_n, F(y,z,w,x)) + d(gy, gy_{n+1})}{2}, \\ \frac{D(gz_n, F(z,w,x,y)) + d(gz, gz_{n+1})}{2}, \\ \frac{D(gw_n, F(w,x,y,z)) + d(gw, gw_{n+1})}{2} \end{array}\right\}.$$

Now, by (4), there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\Delta_n = \max\left\{\begin{array}{c} D(gx, F(x, y, z, w)), \\ D(gy, F(y, z, w, x)), \\ D(gz, F(z, w, x, y)), \\ D(gw, F(w, x, y, z)) \end{array}\right\}.$$

Combining this with (5), we get for all $n > n_0$,

$$\max\left\{\begin{array}{c} D(gx_{n+1},\ F(x,\ y,\ z,\ w)), \\ D(gy_{n+1},\ F(y,\ z,\ w,\ x)), \\ D(gz_{n+1},\ F(z,\ w,\ x,\ y)), \\ D(gw_{n+1},\ F(w,\ x,\ y,\ z)) \end{array}\right\} \tag{6}$$

$$\leq \varphi\left[\max\left\{\begin{array}{c} D(gx,\ F(x,\ y,\ z,\ w)), \\ D(gy,\ F(y,\ z,\ w,\ x)), \\ D(gz,\ F(z,\ w,\ x,\ y)), \\ D(gw,\ F(w,\ x,\ y,\ z)) \end{array}\right\}\right].$$

Now, we claim that

$$\max\left\{\begin{array}{c} D(gx,\ F(x,\ y,\ z,\ w)), \\ D(gy,\ F(y,\ z,\ w,\ x)), \\ D(gz,\ F(z,\ w,\ x,\ y)), \\ D(gw,\ F(w,\ x,\ y,\ z)) \end{array}\right\} = 0. \tag{7}$$

If this is not true, then

$$\max\left\{\begin{array}{c} D(gx,\ F(x,\ y,\ z,\ w)), \\ D(gy,\ F(y,\ z,\ w,\ x)), \\ D(gz,\ F(z,\ w,\ x,\ y)), \\ D(gw,\ F(w,\ x,\ y,\ z)) \end{array}\right\} > 0.$$

Then, by (6) and by the fact that $\varphi(t) < t$ for all $t > 0$, we get

$$\max\left\{\begin{array}{c} D(gx_{n+1},\ F(x,\ y,\ z,\ w)), \\ D(gy_{n+1},\ F(y,\ z,\ w,\ x)), \\ D(gz_{n+1},\ F(z,\ w,\ x,\ y)), \\ D(gw_{n+1},\ F(w,\ x,\ y,\ z)) \end{array}\right\} < \max\left\{\begin{array}{c} D(gx,\ F(x,\ y,\ z,\ w)), \\ D(gy,\ F(y,\ z,\ w,\ x)), \\ D(gz,\ F(z,\ w,\ x,\ y)), \\ D(gw,\ F(w,\ x,\ y,\ z)) \end{array}\right\}.$$

Letting $n \to \infty$ in the above inequality, by using (4), we obtain

$$\max\left\{\begin{array}{c} D(gx,\ F(x,\ y,\ z,\ w)), \\ D(gy,\ F(y,\ z,\ w,\ x)), \\ D(gz,\ F(z,\ w,\ x,\ y)), \\ D(gw,\ F(w,\ x,\ y,\ z)) \end{array}\right\} < \max\left\{\begin{array}{c} D(gx,\ F(x,\ y,\ z,\ w)), \\ D(gy,\ F(y,\ z,\ w,\ x)), \\ D(gz,\ F(z,\ w,\ x,\ y)), \\ D(gw,\ F(w,\ x,\ y,\ z)) \end{array}\right\},$$

which is a contradiction. So (7) holds. Thus, it follows that

$$gx \in F(x,\ y,\ z,\ w),\ gy \in F(y,\ z,\ w,\ x),$$
$$gz \in F(z,\ w,\ x,\ y),\ gw \in F(w,\ x,\ y,\ z),$$

that is, (x, y, z, w) is a quadruple coincidence point of F and g. Hence $C\{F, g\}$ is non-empty.

Suppose now that (a) holds. Assume that for some $(x, y, z, w) \in C\{F, g\}$,

$$\lim_{n\to\infty} g^n x = p,\ \lim_{n\to\infty} g^n y = q,\ \lim_{n\to\infty} g^n z = r \text{ and } \lim_{n\to\infty} g^n w = s, \tag{8}$$

where $p, q, r, s \in X$. Since g is continuous at p, q, r and s. We have, by (8), that p, q, r and s are fixed points of g, that is,

$$gp = p,\ gq = q,\ gr = r \text{ and } gs = s. \tag{9}$$

As F and g are w-compatible, so, for all $n \geq 1$,

$$\begin{aligned} g^n x &\in F(g^{n-1}x,\ g^{n-1}y,\ g^{n-1}z,\ g^{n-1}w), \\ g^n y &\in F(g^{n-1}y,\ g^{n-1}z,\ g^{n-1}w,\ g^{n-1}x), \\ g^n z &\in F(g^{n-1}z,\ g^{n-1}w,\ g^{n-1}x,\ g^{n-1}y), \\ g^n w &\in F(g^{n-1}w,\ g^{n-1}x,\ g^{n-1}y,\ g^{n-1}z). \end{aligned} \tag{10}$$

Now, by using (10), we obtain

$$\begin{aligned} D(g^n x, F(p,q,r,s)) &\leq H(F(g^{n-1}x, g^{n-1}y, g^{n-1}z, g^{n-1}w), F(p,q,r,s)), \\ D(g^n y, F(q,r,s,p)) &\leq H(F(g^{n-1}y, g^{n-1}z, g^{n-1}w, g^{n-1}x), F(q,r,s,p)), \\ D(g^n z, F(r,s,p,q)) &\leq H(F(g^{n-1}z, g^{n-1}w, g^{n-1}x, g^{n-1}y), F(r,s,p,q)), \\ D(g^n w, F(s,p,q,r)) &\leq H(F(g^{n-1}w, g^{n-1}x, g^{n-1}y, g^{n-1}z), F(s,p,q,r)), \end{aligned}$$

which, by (1), implies that

$$\begin{aligned} D(g^n x,\ F(p,\ q,\ r,\ s)) &\leq \varphi[\nabla_n], \\ D(g^n y,\ F(q,\ r,\ s,\ p)) &\leq \varphi[\nabla_n], \\ D(g^n z,\ F(r,\ s,\ p,\ q)) &\leq \varphi[\nabla_n], \\ D(g^n w,\ F(s,\ p,\ q,\ r)) &\leq \varphi[\nabla_n], \end{aligned} \tag{11}$$

where

$$\nabla_n = \max \left\{ \begin{array}{c} d(g^n x, gp), D(gp, F(p,q,r,s)), \frac{D(g^n x, F(p,q,r,s)) + d(gp, g^n x)}{2} \\ d(g^n y, gq), D(gq, F(q,r,s,p)), \frac{D(g^n y, F(q,r,s,p)) + d(gq, g^n y)}{2} \\ d(g^n z, gr), D(gr, F(r,s,p,q)), \frac{D(g^n z, F(r,s,p,q)) + d(gr, g^n z)}{2} \\ d(g^n w, gs), D(gs, F(s,p,q,r)), \frac{D(g^n w, F(s,p,q,r)) + d(gs, g^n w)}{2} \end{array} \right\}.$$

By (8) and (9), there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\nabla_n = \max \left\{ \begin{array}{c} D(gp,\ F(p,\ q,\ r,\ s)), \\ D(gq,\ F(q,\ r,\ s,\ p)), \\ D(gr,\ F(r,\ s,\ p,\ q)), \\ D(gs,\ F(s,\ p,\ q,\ r)) \end{array} \right\}.$$

Combining this with (11), we get for all $n > n_0$,

$$\max\left\{\begin{array}{c} D(g^n x,\ F(p,\ q,\ r,\ s)), \\ D(g^n y,\ F(q,\ r,\ s,\ p)), \\ D(g^n z,\ F(r,\ s,\ p,\ q)), \\ D(g^n w,\ F(s,\ p,\ q,\ r)) \end{array}\right\} \leq \varphi\left[\max\left\{\begin{array}{c} D(gp,\ F(p,\ q,\ r,\ s)), \\ D(gq,\ F(q,\ r,\ s,\ p)), \\ D(gr,\ F(r,\ s,\ p,\ q)), \\ D(gs,\ F(s,\ p,\ q,\ r)) \end{array}\right\}\right]. \tag{12}$$

Now, we claim that

$$\max\left\{\begin{array}{c} D(gp,\ F(p,\ q,\ r,\ s)), \\ D(gq,\ F(q,\ r,\ s,\ p)), \\ D(gr,\ F(r,\ s,\ p,\ q)), \\ D(gs,\ F(s,\ p,\ q,\ r)) \end{array}\right\} = 0. \tag{13}$$

If this is not true, then

$$\max\left\{\begin{array}{c} D(gp,\ F(p,\ q,\ r,\ s)), \\ D(gq,\ F(q,\ r,\ s,\ p)), \\ D(gr,\ F(r,\ s,\ p,\ q)), \\ D(gs,\ F(s,\ p,\ q,\ r)) \end{array}\right\} > 0.$$

Then, by (12) and by the fact that $\varphi(t) < t$ for all $t > 0$, we get for all $n > n_0$,

$$\max\left\{\begin{array}{c} D(g^n x,\ F(p,\ q,\ r,\ s)), \\ D(g^n y,\ F(q,\ r,\ s,\ p)), \\ D(g^n z,\ F(r,\ s,\ p,\ q)), \\ D(g^n w,\ F(s,\ p,\ q,\ r)) \end{array}\right\} < \max\left\{\begin{array}{c} D(gp,\ F(p,\ q,\ r,\ s)), \\ D(gq,\ F(q,\ r,\ s,\ p)), \\ D(gr,\ F(r,\ s,\ p,\ q)), \\ D(gs,\ F(s,\ p,\ q,\ r)) \end{array}\right\}.$$

On taking limit as $n \to \infty$ in the above inequality, by using (8) and (9), we get

$$\max\left\{\begin{array}{c} D(gp,\ F(p,\ q,\ r,\ s)), \\ D(gq,\ F(q,\ r,\ s,\ p)), \\ D(gr,\ F(r,\ s,\ p,\ q)), \\ D(gs,\ F(s,\ p,\ q,\ r)) \end{array}\right\} < \max\left\{\begin{array}{c} D(gp,\ F(p,\ q,\ r,\ s)), \\ D(gq,\ F(q,\ r,\ s,\ p)), \\ D(gr,\ F(r,\ s,\ p,\ q)), \\ D(gs,\ F(s,\ p,\ q,\ r)) \end{array}\right\},$$

which is a contradiction. So (13) holds. Thus, it follows that

$$\begin{gathered} gp \in F(p,\ q,\ r,\ s),\ gq \in F(q,\ r,\ s,\ p), \\ gr \in F(r,\ s,\ p,\ q),\ gs \in F(s,\ p,\ q,\ r). \end{gathered} \tag{14}$$

Thus, by (9) and (14), we have

$$p = gp \in F(p,\ q,\ r,\ s),\ q = gq \in F(q,\ r,\ s,\ p),$$
$$r = gr \in F(r,\ s,\ p,\ q),\ s = gs \in F(s,\ p,\ q,\ r),$$

that is, (p, q, r, s) is a common quadruple fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x, y, z, w) \in C\{F, g\}$, g is F-weakly commuting, that is, $g^2x \in F(gx, gy, gz, gw)$, $g^2y \in F(gy, gz, gw, gx)$, $g^2z \in F(gz, gw, gx, gy)$, $g^2w \in F(gw, gx, gy, gz)$ and $g^2x = gx$, $g^2y = gy$, $g^2z = gz$, $g^2w = gw$. Thus $gx = g^2x \in F(gx, gy, gz, gw)$, $gy = g^2y \in F(gy, gz, gw, gx)$, $gz = g^2z \in F(gz, gw, gx, gy)$ and $gw = g^2w \in F(gw, gx, gy, gz)$, that is, (gx, gy, gz, gw) is a common quadruple fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$,

$$\lim_{n\to\infty} g^n p = x,\ \lim_{n\to\infty} g^n q = y,\ \lim_{n\to\infty} g^n r = z \text{ and } \lim_{n\to\infty} g^n s = w. \tag{15}$$

Since g is continuous at x, y, z and w. Therefore, by (15), we get that x, y, z and w are fixed points of g, that is,

$$gx = x,\ gy = y,\ gz = z \text{ and } gw = w. \tag{16}$$

Since $(x, y, z, w) \in C\{F, g\}$. Therefore, by (16), we get

$$x = gx \in F(x,\ y,\ z,\ w),\ y = gy \in F(y,\ z,\ w,\ x),$$
$$z = gz \in F(z,\ w,\ x,\ y),\ w = gw \in F(w,\ x,\ y,\ z),$$

that is, (x, y, z, w) is a common quadruple fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C\{F, g\}) = \{(x, x, x, x)\}$. Then $\{x\} = \{gx\} = F(x, x, x, x)$. Hence (x, x, x, x) is quadruple fixed point of F and g. $\square$

Example 2.2. Suppose that $X = [0,\ 1]$, equipped with the metric $d : X \times X \to [0,\ +\infty)$ defined by $d(x,\ y) = \max\{x,\ y\}$ and $d(x,\ x) = 0$ for all $x,\ y \in X$. Let $F : X^4 \to CB(X)$ be defined as

$$F(x,\ y,\ z,\ w) = \begin{cases} \{0\}, \text{ for } x,\ y,\ z,\ w = 1, \\ \left[0,\ \frac{x^2+y^2+z^2+w^2}{8}\right], \text{ for } x,\ y,\ z,\ w \in [0,\ 1), \end{cases}$$

and $g : X \to X$ be defined as

$$g(x) = x^2, \text{ for all } x \in X.$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{2}, & \text{for } t \neq 1 \\ \frac{3}{4}, & \text{for } t = 1. \end{cases}$$

Now, for all $x, y, z, w, p, q, r, s \in X$ with $x, y, z, w, p, q, r, s \in [0, 1)$, we have

Case (a) If $x^2+y^2+z^2+w^2 = p^2+q^2+r^2+s^2$, then

$$\begin{aligned}
&H(F(x, y, z, w), F(p, q, r, s)) \\
&= \frac{p^2+q^2+r^2+s^2}{8} \\
&\leq \frac{1}{8}\max\{x^2, p^2\} + \frac{1}{8}\max\{y^2, q^2\} + \frac{1}{8}\max\{z^2, r^2\} + \frac{1}{8}\max\{w^2, s^2\} \\
&\leq \frac{1}{8}d(gx, gp) + \frac{1}{8}d(gy, gq) + \frac{1}{8}d(gz, gr) + \frac{1}{8}d(gw, gs) \\
&\leq \frac{1}{2}\max\left\{\begin{array}{c} d(gx, gp), D(gx, F(x, y, z, w)), D(gp, F(p, q, r, s)), \\ d(gy, gq), D(gy, F(y, z, w, x)), D(gq, F(q, r, s, p)), \\ d(gz, gr), D(gz, F(z, w, x, y)), D(gr, F(r, s, p, q)), \\ d(gw, gs), D(gw, F(w, x, y, z)), D(gs, F(s, p, q, r)), \\ \frac{D(gx, F(p, q, r, s))+D(gp, F(x, y, z, w))}{2}, \\ \frac{D(gy, F(q, r, s, p))+D(gq, F(y, z, w, x))}{2}, \\ \frac{D(gz, F(r, s, p, q))+D(gr, F(z, w, x, y))}{2}, \\ \frac{D(gw, F(s, p, q, r))+D(gs, F(w, x, y, z))}{2} \end{array}\right\} \\
&\leq \varphi\left[\max\left\{\begin{array}{c} d(gx, gp), D(gx, F(x, y, z, w)), D(gp, F(p, q, r, s)), \\ d(gy, gq), D(gy, F(y, z, w, x)), D(gq, F(q, r, s, p)), \\ d(gz, gr), D(gz, F(z, w, x, y)), D(gr, F(r, s, p, q)), \\ d(gw, gs), D(gw, F(w, x, y, z)), D(gs, F(s, p, q, r)), \\ \frac{D(gx, F(p, q, r, s))+D(gp, F(x, y, z, w))}{2}, \\ \frac{D(gy, F(q, r, s, p))+D(gq, F(y, z, w, x))}{2}, \\ \frac{D(gz, F(r, s, p, q))+D(gr, F(z, w, x, y))}{2}, \\ \frac{D(gw, F(s, p, q, r))+D(gs, F(w, x, y, z))}{2} \end{array}\right\}\right].
\end{aligned}$$

Case (b) If $x^2+y^2+z^2+w^2 \neq p^2+q^2+r^2+s^2$ with $x^2+y^2+z^2+w^2 < p^2+q^2+r^2+s^2$, then

$$\begin{aligned}
&H(F(x, y, z, w), F(p, q, r, s)) \\
&= \frac{p^2+q^2+r^2+s^2}{8} \\
&\leq \frac{1}{8}\max\{x^2, p^2\} + \frac{1}{8}\max\{y^2, q^2\} + \frac{1}{8}\max\{z^2, r^2\} + \frac{1}{8}\max\{w^2, s^2\} \\
&\leq \frac{1}{8}d(gx, gp) + \frac{1}{8}d(gy, gq) + \frac{1}{8}d(gz, gr) + \frac{1}{8}d(gw, gs) \leq
\end{aligned}$$

$$\leq \frac{1}{2}\max\left\{\begin{array}{c} d(gx,\ gp),\ D(gx,\ F(x,\ y,\ z,\ w)),\ D(gp,\ F(p,\ q,\ r,\ s)), \\ d(gy,\ gq),\ D(gy,\ F(y,\ z,\ w,\ x)),\ D(gq,\ F(q,\ r,\ s,\ p)), \\ d(gz,\ gr),\ D(gz,\ F(z,\ w,\ x,\ y)),\ D(gr,\ F(r,\ s,\ p,\ q)), \\ d(gw,\ gs),\ D(gw,\ F(w,\ x,\ y,\ z)),\ D(gs,\ F(s,\ p,\ q,\ r)), \\ \frac{D(gx,\ F(p,\ q,\ r,\ s))+D(gp,\ F(x,\ y,\ z,\ w))}{2}, \\ \frac{D(gy,\ F(q,\ r,\ s,\ p))+D(gq,\ F(y,\ z,\ w,\ x))}{2}, \\ \frac{D(gz,\ F(r,\ s,\ p,\ q))+D(gr,\ F(z,\ w,\ x,\ y))}{2}, \\ \frac{D(gw,\ F(s,\ p,\ q,\ r))+D(gs,\ F(w,\ x,\ y,\ z))}{2} \end{array}\right\}$$

$$\leq \varphi\left[\max\left\{\begin{array}{c} d(gx,\ gp),\ D(gx,\ F(x,\ y,\ z,\ w)),\ D(gp,\ F(p,\ q,\ r,\ s)), \\ d(gy,\ gq),\ D(gy,\ F(y,\ z,\ w,\ x)),\ D(gq,\ F(q,\ r,\ s,\ p)), \\ d(gz,\ gr),\ D(gz,\ F(z,\ w,\ x,\ y)),\ D(gr,\ F(r,\ s,\ p,\ q)), \\ d(gw,\ gs),\ D(gw,\ F(w,\ x,\ y,\ z)),\ D(gs,\ F(s,\ p,\ q,\ r)), \\ \frac{D(gx,\ F(p,\ q,\ r,\ s))+D(gp,\ F(x,\ y,\ z,\ w))}{2}, \\ \frac{D(gy,\ F(q,\ r,\ s,\ p))+D(gq,\ F(y,\ z,\ w,\ x))}{2}, \\ \frac{D(gz,\ F(r,\ s,\ p,\ q))+D(gr,\ F(z,\ w,\ x,\ y))}{2}, \\ \frac{D(gw,\ F(s,\ p,\ q,\ r))+D(gs,\ F(w,\ x,\ y,\ z))}{2} \end{array}\right\}\right].$$

Similarly, we obtain the same result for $p^2+q^2+r^2+s^2 < x^2+y^2+z^2+w^2$. Thus the contractive condition (1) is satisfied for all $x,\ y,\ z,\ w,\ p,\ q,\ r,\ s \in X$ with $x,\ y,\ z,\ w,\ p,\ q,\ r,\ s \in [0,\ 1)$. Again, for all $x,\ y,\ z,\ w,\ p,\ q,\ r,\ s \in X$ with $x,\ y,\ z,\ w \in [0,\ 1)$ and $p,\ q,\ r,\ s = 1$, we have

$$\begin{aligned} & H(F(x,\ y,\ z,\ w),\ F(p,\ q,\ r,\ s)) \\ & = \frac{x^2+y^2+z^2+w^2}{8} \\ & \leq \frac{1}{8}\max\{x^2,\ p^2\}+\frac{1}{8}\max\{y^2,\ q^2\}+\frac{1}{8}\max\{z^2,\ r^2\}+\frac{1}{8}\max\{w^2,\ s^2\} \\ & \leq \frac{1}{8}d(gx,\ gp)+\frac{1}{8}d(gy,\ gq)+\frac{1}{8}d(gz,\ gr)+\frac{1}{8}d(gw,\ gs) \end{aligned}$$

$$\leq \frac{1}{2}\max\left\{\begin{array}{c} d(gx,\ gp),\ D(gx,\ F(x,\ y,\ z,\ w)),\ D(gp,\ F(p,\ q,\ r,\ s)), \\ d(gy,\ gq),\ D(gy,\ F(y,\ z,\ w,\ x)),\ D(gq,\ F(q,\ r,\ s,\ p)), \\ d(gz,\ gr),\ D(gz,\ F(z,\ w,\ x,\ y)),\ D(gr,\ F(r,\ s,\ p,\ q)), \\ d(gw,\ gs),\ D(gw,\ F(w,\ x,\ y,\ z)),\ D(gs,\ F(s,\ p,\ q,\ r)), \\ \frac{D(gx,\ F(p,\ q,\ r,\ s))+D(gp,\ F(x,\ y,\ z,\ w))}{2}, \\ \frac{D(gy,\ F(q,\ r,\ s,\ p))+D(gq,\ F(y,\ z,\ w,\ x))}{2}, \\ \frac{D(gz,\ F(r,\ s,\ p,\ q))+D(gr,\ F(z,\ w,\ x,\ y))}{2}, \\ \frac{D(gw,\ F(s,\ p,\ q,\ r))+D(gs,\ F(w,\ x,\ y,\ z))}{2} \end{array}\right\} \leq$$

$$\leq \varphi \left[\max \left\{ \begin{array}{c} d(gx,\ gp),\ D(gx,\ F(x,\ y,\ z,\ w)),\ D(gp,\ F(p,\ q,\ r,\ s)), \\ d(gy,\ gq),\ D(gy,\ F(y,\ z,\ w,\ x)),\ D(gq,\ F(q,\ r,\ s,\ p)), \\ d(gz,\ gr),\ D(gz,\ F(z,\ w,\ x,\ y)),\ D(gr,\ F(r,\ s,\ p,\ q)), \\ d(gw,\ gs),\ D(gw,\ F(w,\ x,\ y,\ z)),\ D(gs,\ F(s,\ p,\ q,\ r)), \\ \frac{D(gx,\ F(p,\ q,\ r,\ s))+D(gp,\ F(x,\ y,\ z,\ w))}{2}, \\ \frac{D(gy,\ F(q,\ r,\ s,\ p))+D(gq,\ F(y,\ z,\ w,\ x))}{2}, \\ \frac{D(gz,\ F(r,\ s,\ p,\ q))+D(gr,\ F(z,\ w,\ x,\ y))}{2}, \\ \frac{D(gw,\ F(s,\ p,\ q,\ r))+D(gs,\ F(w,\ x,\ y,\ z))}{2} \end{array} \right\} \right].$$

Thus the contractive condition (1) is satisfied for all $x,\ y,\ z,\ w,\ p,\ q,\ r,\ s \in X$ with $x,\ y,\ z,\ w \in [0,\ 1)$ and $p,\ q,\ r,\ s = 1$. Similarly, we can see that the contractive condition (1) is satisfied for all $x,\ y,\ z,\ w,\ p,\ q,\ r,\ s \in X$ with $x,\ y,\ z,\ w,\ p,\ q,\ r,\ s = 1$. Hence, the hybrid pair $\{F,\ g\}$ satisfies the contractive condition (1), for all $x,\ y,\ z,\ w,\ p,\ q,\ r,\ s \in X$. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = (0,\ 0,\ 0,\ 0)$ is a common quadruple fixed point of hybrid pair $\{F,\ g\}$. The function $F : X^4 \to CB(X)$ involved in this example is not continuous at the point $(1,\ 1,\ 1,\ 1) \in X^4$.

Remark 2.3. We improve, extend and generalize the result of Ding et al. [23] in the following sense:

(i) We prove quadruple coincidence and common quadruple fixed point theorem while Ding et al. [23] proved coupled coincidence and common coupled fixed point theorems.

(ii) We prove our result in the settings of hybrid pair of mappings.

(iii) To prove our result we consider non complete metric space and the space is also not partially ordered.

(iv) The mapping $F : X^4 \to CB(X)$ is discontinuous and not satisfying mixed g-monotone property.

(v) The function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ involved in our theorem and example is discontinuous.

(vi) Our proof is simple and different from the other results in the existing literature.

If we put $g = I$ (I is the identity mapping) in Theorem 2.1, then we have the following result:

Corollary 2.4. *Let $(X,\ d)$ be a complete metric space, $F : X^4 \to CB(X)$ be a* *mapping. Suppose that there exists* $\varphi \in \Phi$ *such that*

$$H(F(x,\,y,\,z,\,w),\,F(p,\,q,\,r,\,s)) \leq \varphi\left[\max\left\{\begin{array}{c} d(x,\,p),\,D(x,\,F(x,\,y,\,z,\,w)),\,D(p,\,F(p,\,q,\,r,\,s)),\\ d(y,\,q),\,D(y,\,F(y,\,z,\,w,\,x)),\,D(q,\,F(q,\,r,\,s,\,p)),\\ d(z,\,r),\,D(z,\,F(z,\,w,\,x,\,y)),\,D(r,\,F(r,\,s,\,p,\,q)),\\ d(w,\,s),\,D(w,\,F(w,\,x,\,y,\,z)),\,D(s,\,F(s,\,p,\,q,\,r)),\\ \frac{D(x,\,F(p,\,q,\,r,\,s))+D(p,\,F(x,\,y,\,z,\,w))}{2},\\ \frac{D(y,\,F(q,\,r,\,s,\,p))+D(q,\,F(y,\,z,\,w,\,x))}{2},\\ \frac{D(z,\,F(r,\,s,\,p,\,q))+D(r,\,F(z,\,w,\,x,\,y))}{2},\\ \frac{D(w,\,F(s,\,p,\,q,\,r))+D(s,\,F(w,\,x,\,y,\,z))}{2} \end{array}\right\}\right],$$

for all $x,\,y,\,z,\,w,\,p,\,q,\,r,\,s \in X$. *Then* F *has a quadruple fixed point.*

If we put $\varphi(t) = kt$ where $0 < k < 1$ in Theorem 2.1, then we have the following result:

Corollary 2.5. *Let* $(X,\,d)$ *be a metric space. Assume* $F : X^4 \to CB(X)$ *and* $g : X \to X$ *be two mappings satisfying*

$$H(F(x,\,y,\,z,\,w),\,F(p,\,q,\,r,\,s)) \leq k\max\left\{\begin{array}{c} d(gx,\,gp),\,D(gx,\,F(x,\,y,\,z,\,w)),\,D(gp,\,F(p,\,q,\,r,\,s)),\\ d(gy,\,gq),\,D(gy,\,F(y,\,z,\,w,\,x)),\,D(gq,\,F(q,\,r,\,s,\,p)),\\ d(gz,\,gr),\,D(gz,\,F(z,\,w,\,x,\,y)),\,D(gr,\,F(r,\,s,\,p,\,q)),\\ d(gw,\,gs),\,D(gw,\,F(w,\,x,\,y,\,z)),\,D(gs,\,F(s,\,p,\,q,\,r)),\\ \frac{D(gx,\,F(p,\,q,\,r,\,s))+D(gp,\,F(x,\,y,\,z,\,w))}{2},\\ \frac{D(gy,\,F(q,\,r,\,s,\,p))+D(gq,\,F(y,\,z,\,w,\,x))}{2},\\ \frac{D(gz,\,F(r,\,s,\,p,\,q))+D(gr,\,F(z,\,w,\,x,\,y))}{2},\\ \frac{D(gw,\,F(s,\,p,\,q,\,r))+D(gs,\,F(w,\,x,\,y,\,z))}{2} \end{array}\right\},$$

for all $x,\,y,\,z,\,w,\,p,\,q,\,r,\,s \in X$, *where* $0 < k < 1$. *Furthermore assume that* $F(X^4) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have a quadruple coincidence point. Moreover,* F *and* g *have a common quadruple fixed point, if one of the following conditions holds:*

(a) F *and* g *are* w*-compatible.*

$$\lim_{n\to\infty} g^n x = p,\ \lim_{n\to\infty} g^n y = q,\ \lim_{n\to\infty} g^n z = r,\ \lim_{n\to\infty} g^n w = s$$

for some $(x,\,y,\,z,\,w) \in C\{F,\,g\}$ *and for some* $p,\,q,\,r,\,s \in X$ *and* g *is continuous at* p, q, r *and* s.

(b) g *is* F*-weakly commuting for some* $(x,\,y,\,z,\,w) \in C\{F,\,g\}$ *and* gx, gy, gz *and* gw *are fixed points of* g, *that is,* $g^2x = gx$, $g^2y = gy$, $g^2z = gz$ *and* $g^2w = gw$.

(c) *g is continuous at* x, y, z *and* w.

$$\lim_{n\to\infty} g^n p = x,\ \lim_{n\to\infty} g^n q = y,\ \lim_{n\to\infty} g^n r = z,\ \lim_{n\to\infty} g^n s = w$$

for some $(x, y, z, w) \in C\{F, g\}$ *and for some* $p, q, r, s \in X$.

(d) $g(C\{F, g\})$ *is a singleton subset of* $C\{F, g\}$.

If we put $g = I$ (I is the identity mapping) in Corollary 2.5, then we have the following result:

Corollary 2.6. *Let* (X, d) *be a complete metric space,* $F : X^4 \to CB(X)$ *be a mapping satisfying*

$$\begin{aligned} &H(F(x, y, z, w), F(p, q, r, s)) \\ &\leq k \max \left\{ \begin{array}{c} d(x, p), D(x, F(x, y, z, w)), D(p, F(p, q, r, s)), \\ d(y, q), D(y, F(y, z, w, x)), D(q, F(q, r, s, p)), \\ d(z, r), D(z, F(z, w, x, y)), D(r, F(r, s, p, q)), \\ d(w, s), D(w, F(w, x, y, z)), D(s, F(s, p, q, r)), \\ \frac{D(x, F(p, q, r, s)) + D(p, F(x, y, z, w))}{2}, \\ \frac{D(y, F(q, r, s, p)) + D(q, F(y, z, w, x))}{2}, \\ \frac{D(z, F(r, s, p, q)) + D(r, F(z, w, x, y))}{2}, \\ \frac{D(w, F(s, p, q, r)) + D(s, F(w, x, y, z))}{2} \end{array} \right\}, \end{aligned}$$

for all $x, y, z, w, p, q, r, s \in X$, *where* $0 < k < 1$. *Then* F *has a quadruple fixed point.*

REFERENCES

[1] M. Abbas - B. Ali - A. Amini-Harandi, *Common fixed point theorem for hybrid pair of mappings in Hausdorff fuzzy metric spaces*, Fixed Point Theory Appl. 2012, 2012:225.

[2] M. Abbas - L. Ciric - B. Damjanovic - M. A. Khan, *Coupled coincidence point and common fixed point theorems for hybrid pair of mappings*, Fixed Point Theory Appl. doi:10.1186/1687-1812-2012-4 (2012).

[3] S. M. Alsulami - A. Alotaibi, *Tripled coincidence points for monotone operators in partially ordered metric spaces*, Int. Math. Forum 7 (37) (2012), 1811–1824.

[4] A. Amini-Harandi - D. O'Regan, *Fixed point theorems for set-valued contraction type mappings in metric spaces*, Fixed Point Theory Appl. 2010, 7 (2010). Article ID 390183.

[5] H. Aydi - E. Karapinar - M. Postolache, *Tripled coincidence point theorems for weak φ-contractions in partially ordered metric spaces*, Fixed Point Theory Appl. 44 (2012).

[6] S. Banach, *Sur les Operations dans les Ensembles Abstraits et leur Applications aux Equations Integrales*, Fund. Math. 3 (1922), 133–181.

[7] V. Berinde, *Coupled fixed point theorems for φ-contractive mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal. 75 (2012), 3218–3228.

[8] V. Berinde - M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal. 74 (15) (2011), 4889–4897.

[9] V. Berinde - M. Borcut, *Tripled coincidence theorems of contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput. 218 (10), 5929–5936.

[10] T. G. Bhaskar V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (7) (2006), 1379–1393.

[11] P. Charoensawan, *Tripled fixed points theorems of φ-contractive mixed monotone operators on partially ordered metric spaces*, Appl. Math. Sci. 6 (105) (2012), 5229–5239.

[12] Deepmala - H. K. Pathak, *A study on some problems on existence of solutions for nonlinear functional-integral equations*, Acta Math. Sci. 33 B (5) (2013), 1305–1313.

[13] Deepmala, *A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications*, Ph.D. Thesis (2014), Pt. Ravishankar Shukla University, Raipur (Chhatisgarh) India – 492 010.

[14] B. Deshpande, *Common fixed point for set and single valued functions without continuity and compatibility*, Mathematica Moravica 11 (2007), 27–38.

[15] B. Deshpande - S. Chouhan, *Fixed points for two hybrid pairs of mappings satisfying some weaker conditions on noncomplete metric spaces*, Southeast Asian Bull. Math. 35 (2011), 851–858.

[16] B. Deshpande - A. Handa, *Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations*, Afr. Mat. DOI 10.1007/s13370-013-0204-0.

[17] B. Deshpande - A. Handa, *Application of coupled fixed point technique in solving integral equations on modified intuitionistic fuzzy metric spaces*, Adv. Fuzzy Syst. Volume 2014, Article ID 348069, 11 pages

[18] B. Deshpande - S. Sharma - A. Handa, *φ-contractive mixed monotone mappings in coupled fixed point consideration on intuitionistic fuzzy metric spaces with application to integral equations*, J. Fuzzy Math. 22 (1) (2014), 401–418.

[19] B. Deshpande - S. Sharma - A. Handa, *Common coupled fixed point theorems for nonlinear contractive condition on intuitionistic fuzzy metric spaces with application to integral equations*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 20 (3) (2013), 159–180.

[20] B. Deshpande - S. Sharma - A. Handa, *Tripled fixed point theorem for hybrid pair of mappings under generalized nonlinear contraction*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 21 (1) (2014), 23–38.

[21] B. Deshpande - A. Handa, *Quadruple fixed point theorems for multivalued mappings under $\varphi - \psi$ contraction on partially ordered metric spaces*, Southeast Asian Bull. Math. 38 (2014), 771–790.

[22] B. Deshpande - A. Handa, *Quadruple coincidence and common quadruple fixed point for hybrid pair of mappings under new contractive condition*, Mathematica Moravica 18 (2) (2014), 73–90.

[23] H. S. Ding - L. Li - S. Radenovic, *Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces*, Fixed Point Theory Appl. 2012, 2012:96.

[24] M. Jain - K. Tas - S. Kumar - N. Gupta, *Coupled common fixed point results involving a $\varphi - \psi$ contractive condition for mixed g-monotone operators in partially ordered metric spaces*, J. Inequal. Appl. 2012, 2012:285.

[25] E. Karapinar - N. V. Luong, *Quadruple fixed point theorems for nonlinear contractions*, Comput. Math. Appl. 64 (6) (2012), 1839–1848.

[26] I. Kubiaczyk - B. Deshpande, *Noncompatibility, discontinuity in consideration of common fixed point of set and single valued mappings*, Southeast Asian Bull. Math. 32 (2008), 467–474.

[27] V. Lakshmikantham - L. Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. 70 (12) (2009), 4341–4349.

[28] X. Liu, *Quadruple fixed point theorems in partially ordered metric spaces with mixed g-monotone property*, Fixed Point Theory Appl. 2013, 2013:147.

[29] W. Long - S. Shukla - S. Radenovic, *Some coupled coincidence and common fixed point results for hybrid pair of mappings in 0-complete partial metric spaces*, Fixed Point Theory Appl. 2013, 2013:145.

[30] N. V. Luong - N. X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. 74 (2011), 983–992.

[31] J. T. Markin, *Continuous dependence of fixed point sets*, Proceedings of the American Mathematical Society, 38 (1947), 545–547.

[32] L. N. Mishra - S. K. Tiwari - V. N. Mishra, *Unique fixed point theorems for generalized contractive mappings in partial metric spaces*, accepted on September 25, 2014 in Journal of Function Spaces.

[33] V. N. Mishra, *Some Problems on Approximations of Functions in Banach Spaces*, Ph.D. Thesis (2007), Indian Inst. Tech. Roorkee - 247 667, Uttarakhand, India.

[34] V. N. Mishra - M. L. Mittal - U. Singh; *On best approximation in locally convex space*, Varāhmihir J. of Math. Sci. India 6 (1) (2006), 43–48.

[35] N. Mizoguchi - W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. 141 (1989), 177–188.

[36] Z. Mustafa, *Mixed g-monotone property and quadruple fixed point theorems in partially ordered G-metric spaces using* $(\phi - \psi)$ *contractions*, Fixed Point Theory Appl. 2012, 2012:199.

[37] Z. Mustafa - H. Aydi - E. Karapinar, *Mixed g-monotone property and quadruple fixed point theorems in partially ordered metric spaces*, Fixed Point Theory Appl. 2012, 2012:71.

[38] Z. Mustafa - W. A. Shatanawi - E. Karapinar, *Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces*, J. Appl. Math. 2012, Article Id: 951912.

[39] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math. 30 (1969), 475–488.

[40] S. H. Rasouli - M. Bahrampour, *On Quadruple fixed point theorems in partially ordered metric spaces*, Int. J. Math. Stat. 13 (1) (2013).

[41] B. Samet - E. Karapinar H. Aydi - V. C. Rajic, *Discussion on some coupled fixed point theorems*, Fixed Point Theory Appl. 2013, 2013:50.

[42] B. Samet - C. Vetro, *Coupled fixed point, F-invariant set and fixed point of N-order*, Ann. Funct. Anal. 1 (2010), 46–56.

[43] S. Sharma - B. Deshpande, *Fixed point theorems for set and single valued mappings without continuity and compatibility*, Demonstr. Math. XL (3) (2007), 649–658.

[44] N. Singh - R. Jain, *Coupled coincidence and common fixed point theorems for set-valued and single-valued mappings in fuzzy metric space*, Journal of Fuzzy Set Valued Analysis, Volume 2012, Year 2012 Article ID jfsva-00129, 10 pages.

[45] W. Sintunavarat - P. Kumam - Y. J. Cho, *Coupled fixed point theorems for nonlinear contractions without mixed monotone property*, Fixed Point Theory Appl. 2012, 2012:170.

BHAVANA DESHPANDE
Bhavana Deshpande
Department of Mathematics,
Govt. P. G. Arts & Science College,
Ratlam-457001, India.
e-mail: `bhavnadeshpande@yahoo.com`

AMRISH HANDA
Amrish Handa
Department of Mathematics,
Govt. P. G. Arts & Science College,
Ratlam-457001, India.
e-mail: `amrishhanda83@gmail.com`