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ON THE STRATIFICATION OF PROJECTIVE n-SPACE BY X-RANKS, FOR A LINEARLY NORMAL ELLIPTIC CURVE $X \subset \mathbb{P}^n$

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Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. For any $P \in \mathbb{P}^n$ the X-rank of P is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. In this paper we give an almost complete description of the stratification of $\mathbb{P}^n$ by the X-rank.

1. Introduction

Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a subset $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\ \rangle$ denote the linear span. The X-rank is an extensively studied topic ([14], [8], [6], [13] and references therein). In the applications one mainly needs the cases in which X is either a Veronese embedding of a projective space or a Segre embedding of a multiprojective space. We feel that the general case gives a treasure of new projective geometry. Up to now only for rational normal curves there is a complete description of the stratification of $\mathbb{P}^n$ by X-rank ([11], [14], Theorem 5.1, [6]). Here we look at the case of elliptic linearly normal curves. For any integer $t \geq 1$ let $\sigma_t(X)$ denote the closure in $\mathbb{P}^n$ of all $(t-1)$-dimensional linear spaces spanned by t points of X. Set $\sigma_0(X) = \emptyset$. For any $P \in \mathbb{P}^n$ the

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border X-rank $b_X(P)$ is the minimal integer $t \ge 1$ such that $P \in \sigma_t(X)$, i.e. the only positive integer t such that $P \in \sigma_t(X) \setminus \sigma_{t-1}(X)$. If (as always in this paper) X is a curve, then $\dim(\sigma_t(X)) = \min\{n, 2t-1\}$ for all $t \ge 1$ ([1], Remark 1.6). Notice that $r_X(P) \ge b_X(P)$ and that equality holds at least on a non-empty open subset of $\sigma_t(X) \setminus \sigma_{t-1}(X)$, $t := b_X(P)$. Obviously $b_X(P) = 1 \iff P \in X \iff r_X(P) = 1$. Hence to compute all X-ranks it is sufficient to compute the X-ranks of all points of $\mathbb{P}^n \setminus X$. In this paper we look at the case of the linearly normal elliptic curves.

We prove the following result.

Theorem 1.1. *Fix integers $w \ge 2$ and $n \ge 2w$. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n$ with $b_X(P) = w$. Then either $r_X(P) = w$ or $r_X(P) = n+1-w$.*

Proposition 3.2 gives the existence of non-general $P \in \mathbb{P}^{2w-1} \setminus \sigma_{w-1}(X)$ such that $r_X(P) = b_X(P) + 1 = w+1$

Remark 1.2. Fix $P \in \mathbb{P}^n$ such that $b_X(P) \le n/2$, i.e. such that the border rank of P is not the maximal one. There is a unique zero-dimensional scheme $W \subset X$ such that $\deg(W) \le b_X(P)$ and $P \in \langle W \rangle$ (Proposition 2.2). We have $\deg(W) = b_X(P)$ and $P \notin \langle W' \rangle$ for any $W' \subsetneq W$. If W is reduced, then $r_X(P) = w$. If W is not reduced, then Theorem 1.1 says that $r_X(P) = n+1-w$.

Following works by A. Białynicki-Birula and A. Schinzel ([4], [5]), recently J. Jelisiejew introduced the definition of open rank for symmetric tensors, i.e. for the Veronese embeddings of projective spaces ([12]). In the general case of X-rank we may translate the definition of open rank in the following way.

Definition 1.3. Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$. For each $P \in \mathbb{P}^n$ the *open X-rank $or_X(P)$* of P is the minimal integer t such that for every proper closed subset $T \subsetneq X$ there is $S \subset X \setminus T$ with $\sharp(S) \le t$ and $P \in \langle S \rangle$.

Obviously $or_X(P) \ge r_X(P)$, but often the strict inequality holds (e.g., we have $or_X(P) > 1$ for all P).

For linearly normal elliptic curves we prove the following result.

Proposition 1.4. *Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n \setminus X$ and set $w := b_X(P)$. We have $or_X(P) \ge n+1-w$. Assume $n \ge 2w \ge 4$. There is $O \in X$ with the following property. Fix any finite set $T \subset X$ such that $O \notin T$. Then there is $S \subset X \setminus T$ such that $P \in \langle S \rangle$.*

Proposition 1.4 doesn't say if $or_X(P) = n+1-w$. The answer is YES if $w = 1$, i.e. if $P \in X$ (Lemma 2.9). The answer is YES if $w = 2$ and $n > 4$, while it is NO if $w = 2$ and $n = 4$ (Proposition 3.5). If n is odd, say $n = 2w-1$, and

P is general in $\mathbb{P}^{2w-1}$, then $or_X(P) \geq w+1$, i. e. $or_X(P) \geq n+2-w$ (Remark 3.3).

The case $n = 3$ for the rank is contained in [16]. For the open rank, see Proposition 3.6.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. This assumption is essential in our proofs when $(n,w) \neq (4,2)$, mainly to quote [9], Proposition 5.8, which is a very strong non-linear version of Bertini's theorem. In the case $n = 3$ we also check the positive characteristic case.

2. Preliminary lemmas

In this paper an elliptic curve is a smooth and connected projective curve with genus 1.

Fix any non-degenerate variety $X \subset \mathbb{P}^n$. For any $P \in \mathbb{P}^n$ let $\mathcal{S}(X,P)$ denote the set of all $S \subset X$ evincing $r_X(P)$, i. e. the set of all $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Notice that every $S \in \mathcal{S}(X,P)$ is linearly independent and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. Now assume that X is a linearly normal elliptic curve. Let $\mathcal{Z}(X,P)$ denote the set of all zero-dimensional subschemes $Z \subset X$ such that $\deg(Z) = b_X(P)$ and $P \in \langle Z \rangle$. Lemma 2.5 below gives $\mathcal{Z}(X,P) \neq \emptyset$. Fix any $Z \in \mathcal{Z}(X,P)$. Notice that Z is linearly independent (i. e. $\dim(\langle Z \rangle) = \deg(Z) - 1$) and $P \notin \langle Z' \rangle$ for any subscheme $Z' \subsetneq Z$.

Notation. Let $C \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate curve. Let $\beta(C)$ be the maximal integer such that every zero-dimensional subscheme of C with degree at most $\beta(C)$ is linearly independent.

The following lemma is just a reformulation of [2], Lemma 1.

Lemma 2.1. *Let $Y \subset \mathbb{P}^r$ be an integral variety. Fix any $P \in \mathbb{P}^r$ and two zero-dimensional subschemes A, B of Y such that $A \neq B$, $P \in \langle A \rangle$, $P \in \langle B \rangle$, $P \notin \langle A' \rangle$ for any $A' \subsetneq A$ and $P \notin \langle B' \rangle$ for any $B' \subsetneq B$. Then $h^1(\mathbb{P}^r, \mathcal{I}_{A \cup B}(1)) > 0$.*

Proposition 2.2. *Fix an integer $k \leq \lfloor \beta(C)/2 \rfloor$ and any $P \in \sigma_k(C) \setminus \sigma_{k-1}(C)$. Then there exists a unique zero-dimensional scheme $Z \subset C$ such that $\deg(Z) \leq k$ and $P \in \langle Z \rangle$. Moreover $\deg(Z) = k$ and $P \notin \langle Z' \rangle$ for all $Z' \subsetneq Z$.*

Proof. The existence part is stated in [3], Lemma 1, which in turn is just an adaptation of some parts of the beautiful paper [8] ([8], Lemma 2.1.6) or of [6], Proposition 11. For more about these schemes, see [7]. The uniqueness part is true by Lemma 2.1 and the definition of the integer $\beta(C)$. □

Lemma 2.3. *Let $X \subset \mathbb{P}^n$, $n \geq 2$, be a linearly normal elliptic curve.*

(i) We have $\beta(X) = n$. A scheme $Z \subset X$ with $\deg(Z) = n+1$ *is linearly independent if and only if $Z \notin |\mathcal{O}_X(1)|$.*

(ii) Fix zero-dimensional schemes $A, B \subset X$ such that $\deg(A) + \deg(B) \le n+1$. *If* $\deg(A) + \deg(B) = n+1$ *and $A + B \in |\mathcal{O}_X(1)|$, assume $A \cap B \ne \emptyset$, 1. e. assume $A \cup B \ne A + B$. Then $\langle A \rangle \cap \langle B \rangle = \langle A \cap B \rangle$.*

(iii) Fix zero-dimensional schemes $A, B \subset X$ such that $A \ne \emptyset$, $B \ne \emptyset$, $\mathcal{O}_X(A+B) \cong \mathcal{O}_X(1)$ and $A \cap B = \emptyset$. Then $\langle A \rangle \cap \langle B \rangle$ is a single point.

Proof. Let $F \subset X$ be a zero-dimensional subscheme. Since X is linearly normal, we have $h^1(\mathcal{I}_F(1)) = 0$ if and only if either $\deg(F) < \deg(\mathcal{O}_X(1)) = n+1$ or $\deg(F) = n+1$ and $F \notin |\mathcal{O}_X(1)|$ (use the cohomology of line bundles on an elliptic curve). Hence we get part (i). By the Grassmann's formula we also get part (ii) when either $\deg(A) + \deg(B) \le n$ or $\deg(A) + \deg(B) = n+1$ and $A + B \notin |\mathcal{O}_X(1)|$. Take A, B with $A + B \in |\mathcal{O}_X(1)|$, $A \ne \emptyset$ and $B \ne \emptyset$. Calling $A \cup B$ the minimal subscheme of X containing both A and B, we have $\deg(A \cup B) = \deg(A) + \deg(B) - \deg(A \cap B)$, while $\deg(A + B) = \deg(A) + \deg(B)$. Assume $A \cap B \ne \emptyset$, 1. e. assume $A + B \ne A \cup B$. Since $\langle A \cup B \rangle \supset (\langle A \rangle \cup \langle B \rangle)$, then $\langle A \cup B \rangle$ is the linear span of the linear spaces $\langle A \rangle$ and $\langle B \rangle$. Since $\deg(A \cup B) \le n$ and $\beta(X) = n$, we have $\dim(\langle U \rangle) = \deg(U) - 1$ for all $U \in \{A \cup B, A, B, A \cap B\}$. Grassmann's formula gives part (ii). It also gives part (iii), because $\dim(\langle A + B \rangle) = \deg(A + B) - 2$ and $\beta(X) \ge \max\{\deg(A), \deg(B)\}$. □

Lemma 2.4. *Let $X \subset \mathbb{P}^n$, $n \ge 2$, be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n$. Then either $b_X(P) = r_X(P)$ or $r_X(P) + b_X(P) \ge n+1$.*

Proof. Assume $b_X(P) < r_X(P)$. Fix W evincing $b_X(P)$ and S evincing $r_X(P)$. Assume $\sharp(S) + \deg(W) \le n$. Hence $S \cup W$ is linearly independent (Lemma 2.3). Therefore $h^1(\mathcal{I}_{W \cup S}(1)) = 0$, contradicting Lemma 2.1. □

Lemma 2.5. *Fix integers $w > 0$ and $n \ge$* $\max\{2w-1, 2\}$. *Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n$ and assume the existence of a zero-dimensional scheme $Z \subset X$ such that* $\deg(Z) = w$, *$P \in \langle Z \rangle$, while $P \notin \langle Z' \rangle$ for all $Z' \subsetneq Z$. Then $b_X(P) = w$. If $n \ge 2w$, then $\mathcal{Z}(X, P) = \{Z\}$.*

Proof. Assume $b_X(P) < w$ and take a scheme $B \in \mathcal{Z}(X, P)$ (Proposition 2.2). Hence $P \in \langle B \rangle$ and $\deg(B) \le w-1$. Since $\deg(Z) + \deg(B) \le n$, part (ii) of Lemma 2.3. Hence $\langle Z \rangle \cap \langle B \rangle = \langle Z \cap B \rangle$. We have $P \in \langle Z \rangle \cap \langle B \rangle$. Since $\deg(B) < w$, we have $Z \cap B \subsetneq Z$. Hence $P \notin \langle Z \cap B \rangle$, a contradiction. Now assume $2w \le n$ and take any $W \subset X$ such that $\deg(W) = w$ and $P \in \langle Z \rangle$. Part (ii) of Lemma 2.3 gives $\langle Z \rangle \cap \langle W \rangle = \langle Z \cap W \rangle$. Since $P \in \langle Z \rangle \cap \langle W \rangle$, $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$ and $\deg(Z) = \deg(W)$, we get $Z = W$. □

Lemma 2.6. *Fix integers $w > 0$, $n \geq 2$. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n$ such that $b_X(P) = w$. Then $or_X(P) \geq n+1-w$.*

Proof. By Lemma 2.5 there is a unique degree w scheme such that $P \in \langle W \rangle$. Fix any zero-dimensional scheme $Z \subset X$ such that $P \in \langle S \rangle$, $\deg(Z) = n-w$ and $Z \nsubseteq W$. Since $\deg(Z)+\deg(W) \leq n$, part (ii) of Lemma 2.3 gives $\langle Z \rangle \cap \langle W \rangle = \langle Z \cap W \rangle$. Since $Z \cap W \subsetneq W$, Lemma 2.4 applied to $Z \cap W$ gives $P \notin \langle Z \rangle$. $\square$

In characteristic zero the proof of [14], Proposition 5.1, gives the following result.

Lemma 2.7. *Let $Y \subset \mathbb{P}^n$ be an integral and non-degenerate curve. Fix $P \in \mathbb{P}^n \setminus Y$. In positive characteristic assume that P is not a strange point of Y. Then $or_X(P) \leq n$.*

In a few cases (e.g. when Y is a rational normal curve) the statement of Lemma 2.7 fails for some $P \in Y$.

Remark 2.8. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. A theorem of Lluis says that a plane conic in characteristic two is the only smooth strange curve ([15]). Hence, by Lemma 2.7 $or_X(P) \leq n$ for all $P \in \mathbb{P}^n \setminus X$.

Lemma 2.9. *Let $X \subset \mathbb{P}^n$, $n \geq 2$, be a linearly normal elliptic curve. For any $P \in X$ we have $or_X(P) = n$.*

Proof. Remark 2.8 gives $or_X(P) \leq n$. Fix any $S \subset X \setminus \{P\}$ such that $\sharp(S) \leq n-1$. Since $P \notin S$, part (ii) of Lemma 2.3 gives $P \notin \langle S \rangle$. Hence $or_X(P) \geq n$. $\square$

3. Proof of Theorem 1.1 and related results

Proposition 3.1. *Fix an integer $k \geq 1$, a linearly normal elliptic curve $C \subset \mathbb{P}^{2k+1}$ and $P \in \mathbb{P}^{2k+1} \setminus \sigma_k(C)$.*

(a) Either $\sharp(\mathcal{Z}(C,P)) \leq 2$ or $\mathcal{Z}(C,P)$ is infinite. We have $Z_1 \cap Z_2 = \emptyset$, $\langle Z_1 \rangle \cap \langle Z_2 \rangle = \{P\}$ and $\mathcal{O}_C(Z_1+Z_2) \cong \mathcal{O}_C(1)$ for any $Z_1, Z_2 \in \mathcal{Z}(C,P)$ such that $Z_1 \neq Z_2$.

(b) If $\sharp(\mathcal{Z}(C,P)) \neq 2$, then either $\sharp(\mathcal{Z}(C,P)) = 1$ or $\mathcal{Z}(C,P)$ is infinite. In both cases $\mathcal{O}_C(2Z) \cong \mathcal{O}_C(1)$ and $\mathcal{O}_C(Z) \cong \mathcal{O}_C(Z_1)$ for all $Z, Z_1 \in \mathcal{Z}(C,P)$.

(c) If $\mathcal{Z}(C,P)$ is infinite, then its positive-dimensional part Γ is irreducible and one-dimensional. Fix a general $Z \in \Gamma$. Either Z is reduced or there is an integer $m \geq 2$ such that $Z = mS_1$ for a reduced $S_1 \subset C$ such that $\sharp(S_1) = (k+1)/m$.

(d) If P is general, then $\sharp(\mathcal{Z}(C,P)) = 2$.

Proof. Since no non-degenerate curve is defective ([1], Remark 1.6), we have $\sigma_{k+1}(C) = \mathbb{P}^{2k+1}$ and $\dim(\sigma_k(C)) = 2k-1$. Hence $b_C(P) = k+1$. Proposition 2.2 and part (i) of Lemma 2.3 give $\mathcal{Z}(C,P) \neq \emptyset$. Fix $Z_1, Z_2 \in \mathcal{Z}(C,P)$ such that $Z_1 \neq Z_2$. Since $P \in \langle Z_1 \rangle \cap \langle Z_2 \rangle$ and $P \notin \langle E \rangle$ if $\deg(E) \leq k$, Lemma 2.3 gives $\mathcal{O}_C(Z_1 + Z_2) \cong \mathcal{O}_C(1)$ and $Z_1 \cap Z_2 = \emptyset$, proving part (a).

(i) Let $J(C,\dots,C) \subset C^{k+1} \times \mathbb{P}^{2k+1}$ be the abstract join of $k+1$ copies of C, 1. e. the closure in $C^{k+1} \times \mathbb{P}^{2k+1}$ of the set of all $(P_1,\dots,P_{k+1},P)$ such that $P_i \neq P_j$ for all $i \neq j$, the set $\{P_1,\dots,P_{k+1}\}$ is linearly independent and $P \in \langle \{P_1,\dots,P_{k+1}\} \rangle$. Since $\sigma_{k+1}(C) = \mathbb{P}^{2k+1}$, for a general P the set $\mathcal{Z}(C,P)$ is finite and its cardinality is the degree of the generically finite surjection $\pi : J(C,\dots,C) \to \mathbb{P}^{2k+1}$ induced by the projection $C^{k+1} \times \mathbb{P}^{2k+1} \to \mathbb{P}^{2k+1}$. Assume the existence of schemes $Z_1, Z_2, Z_3 \in \mathcal{Z}(C,P)$ such that $Z_i \neq Z_j$ for all $i \neq j$. Part (a) gives $Z_i \cap Z_j = \emptyset$ and $\mathcal{O}_C(Z_i + Z_j) \cong \mathcal{O}_C(1)$ for all $i \neq j$. Taking $i = 1$ and $j \in \{2,3\}$ we get $\mathcal{O}_C(Z_2) \cong \mathcal{O}_C(Z_3)$. By symmetry we get $\mathcal{O}_C(Z) \cong \mathcal{O}_C(Z_1)$ for all $Z \in \mathcal{Z}(C,P)$. Since $\mathcal{O}_C(Z_1 + Z_2) \cong \mathcal{O}_C(1)$, we also get $\mathcal{O}_C(2Z) \cong \mathcal{O}_C(1)$ for all $Z \in \mathcal{Z}(C,P)$.

(ii) Since C is not the rational normal curve of $\mathbb{P}^3$, C is not a variety with one apparent double point in the sense of [10], 1. e. $\deg(\pi) > 1$, 1. e. $\sharp(\mathcal{Z}(C,P)) \neq 1$ for a general $P \in \mathbb{P}^{2k+1}$. There are only finitely many lines bundles R on X such that $R^{\otimes 2} \cong \mathcal{O}_X(1)$. For each of these lines bundles R we have $h^0(R) = k+1$ and $\dim(\langle Z \rangle) = k$ for each $Z \in |R|$ (Lemma 2.3). Hence a dimensional count and part (i) gives $\sharp(\mathcal{Z}(X,P)) = 2$ for a general $P \in \mathbb{P}^{2k+1}$, proving part (d). Now assume $\sharp(\mathcal{Z}(C,P)) = 1$, say $\mathcal{Z}(X,P) = \{Z\}$. For a general $P \in \mathbb{P}^{2k+1}$ the two elements, say $Z_1(P)$ and $Z_2(P)$, of $\mathcal{Z}(X,P)$ satisfy $\mathcal{O}_C(Z_1(P) + Z_2(P)) \cong \mathcal{O}_C(1)$. When P goes to Q we get $\mathcal{O}_C(2Z) \cong \mathcal{O}_C(1)$ (here we are implicitly using that $\beta(X) = 2k+1 \geq k+1$ and hence that the limit of a family of degree $k+1$ subschemes of X is linearly independent). Since $\pi : J(C,\dots,C) \to \mathbb{P}^{2k+1}$ is a proper surjective morphism and $\mathbb{P}^{2k+1}$ is a normal variety, each fiber of π is either infinite or with cardinality ≤ 2. Therefore either $\sharp(\mathcal{Z}(C,P)) \leq 2$ or $\mathcal{Z}(C,P)$ is infinite.

(iii) Now assume that $\mathcal{Z}(C,P)$ is infinite. Since any two different elements of $\mathcal{Z}(C,P)$ are disjoint (see step (i)), for a general $A \in C$ there is at most one element of Γ containing A. Hence $\dim(\Gamma) = 1$, Γ is irreducible and a general point of C is contained in a unique element of Γ, 1. e. the algebraic family Γ of effective divisors of C is a so-called *involution* ([9], §5). Since any two elements of Γ are disjoint, this involution has no base points. Let Z be a general element of Γ. Either Z is reduced or there is an integer $m \geq 2$ such that $Z = mS$ with S reduced ([9], Proposition 5.8), concluding the proof of part (c). □

We may apply Lemma 2.6 at a general $P \in \mathbb{P}^{2k+1}$, because $\mathcal{Z}(C,P)$ is finite for a general P.

Proposition 3.2. *Fix an integer $k \geq 1$ and a linearly normal elliptic curve $X \subset \mathbb{P}^{2k+1}$. Then there are $Q, P \in \mathbb{P}^{2k+1}$ such that $b_X(Q) = b_X(P) = r_X(Q) = k+1$ and $r_X(P) \geq k+2$. The set of all such points Q contains a non-empty open subset of $\mathbb{P}^{2k+1}$, while the set of all such points P contains a non-empty algebraic subset of codimension 2 of $\mathbb{P}^{2k+1}$.*

Proof. Since $\sigma_{k+1}(X) = \mathbb{P}^{2k+1}$, while $\dim(\sigma_k(X)) = 2k-1$ ([1], Remark 1.6), we may take as Q a general point of $\mathbb{P}^{2k+1}$. Now we prove the existence of points $P \in \mathbb{P}^n$ such that $r_X(P) > b_X(P) = k+1$ and that the set of all P such that $b_X(P) = k+1 < r_X(P)$ contains a codimension 2 subset of $\mathbb{P}^{2k+1}$. Let $\mathcal{U}$ be the set of all degree $k+1$ schemes $Z_1 \subset X$ such that Z_1 is not reduced and $2Z_1 \notin |\mathcal{O}_X(1)|$. The set $\mathcal{U}$ is a non-empty quasi-projective integral variety of dimension $k+1$. Fix any $Z_1 \in \mathcal{U}$. Let $\mathcal{V}(Z_1)$ denote the set of all $Z_2 \in |\mathcal{O}_X(1)(-Z_1)|$ such that Z_2 is not reduced and $Z_2 \cap Z_1 = \emptyset$. The set $\mathcal{V}(Z_1)$ is a non-empty quasi-projective and integral variety of dimension k. For any $Z_2 \in \mathcal{V}(Z_1)$ part (iii) of Lemma 2.3 gives that $\langle Z_1 \rangle \cap \langle Z_2 \rangle$ is a single point, Q. If $b_X(Q) = k+1$, then $\mathcal{Z}(X,Q) = \{Z_1, Z_2\}$, because $\mathcal{O}_X(2Z_1) \ncong \mathcal{O}_X(1)$ (Part (b) of Proposition 3.1). Since neither Z_1 nor Z_2 is reduced, we get $r_X(Q) > k+1$. Varying Z_2 for a fixed Z_1 the set of all points Q obtained in this way covers a non-empty open subset of an irreducible hypersurface of $\langle Z_1 \rangle$. Assume $b_X(Q) \leq k$ and fix $W \in \mathcal{Z}(X,Q)$. Notice that $P \notin \langle W' \rangle$ for any $W' \subsetneq W$. Since $\deg(W) + \deg(Z_1) \leq n$, Lemma 2.1, Proposition 2.2 and Lemma 2.3 give the existence of $Z' \subsetneq Z$ such that $Q \in \langle Z' \rangle$. Iterating the trick taking Z' and W instead of Z_1 and W we get $W \subseteq Z'$ and hence $W \subset Z_1$. Making this construction using Z_2 and W we get $W \subset Z_2$. Since $Z_1 \cap Z_2 = \emptyset$, we obtained a contradiction. □

Remark 3.3. Fix an integer $w \geq 2$ and fix $P \in \mathbb{P}^{2w-1} \setminus \sigma_{w-1}(X)$. Fix any $W \in \mathcal{Z}(X,P)$. Assume $\mathcal{O}_X(2W) \ncong \mathcal{O}_X(1)$. This condition is satisfied for a general $P \in \mathbb{P}^{2w-1}$. By part (b) of Proposition 3.1 we have $\sharp(\mathcal{Z}(X,P)) \leq 2$ and in particular $\mathcal{Z}(X,P)$ is finite. Fix a finite set $T \subset X$ containing the support of all $Z \in \mathcal{Z}(X,P)$. By the definition of the set $\mathcal{Z}(X,P)$ there is no zero-dimensional scheme $Z \subset X \backslash T$ such that $P \in \langle Z \rangle$ and $\deg(Z) = w$. Therefore $or_X(P) \geq w+1 = n+2-w$.

Proofs of Theorem 1.1 and Proposition 1.4. Since $w \leq \lfloor (n+2)/2 \rfloor$, there are points P such that $r_X(P) = b_X(P) = w$. Fix any $W \in \sigma_w(X) \setminus \sigma_{w-1}(X)$. Since $n \geq 2w$, there is a unique scheme $W \subset X$ such that $\deg(W) = w$ and $P \in \langle W \rangle$ (Proposition 2.2 and Lemma 2.5). If W is reduced, then $r_X(P) = w$. If W is not reduced, then $r_X(P) \geq n+1-w$. Hence to prove Theorem 1.1 and Proposition 1.4 for the point P it is sufficient to prove the existence of $O \in X$ with the following property. Fix a finite set $T \subset X$ with $O \notin T$. Then there is a set $S \subset X \setminus T$ such that $\sharp(S) \le n+1-w$ and $P \in \langle S\rangle$. The point O (if any) will appear at the very end of the proof.

Set $\mathcal{S} := \{Z \in |\mathcal{O}_X(1)(-W)| : P \in \langle Z\rangle\}$. Since $\deg(\mathcal{O}_X(1)(-W)) = n+1-w \le n$, every element of $|\mathcal{O}_X(1)(-W)|$ is linearly independent. However, in the definition of the set $\mathcal{S}$ we did not prescribe that $P \notin \langle Z'\rangle$ for all $Z' \subsetneq Z$. Part (i) of Lemma 2.3 and the inequality $r_X(P) \ge n+1-w$ give that $r_X(P) = n+1-w$ if and only if there is a reduced $S \in \mathcal{S}$.

(a) Fix $E \subset X \setminus T$ such that $\sharp(E) = n-2w+1$ and $E \cap W_{red} = \emptyset$. In this step we prove the existence of an effective divisor A_E on X such that $\deg(A_E) = w$ and $E + A_E \in \mathcal{S}$. In step (b) we will also check that A_E is unique if $\mathcal{O}_X(1)(-E) \ncong \mathcal{O}_X(2W)$. Since $n \ge 2w$, we have $E \ne \emptyset$. Since $\sharp(E) < n$, E is linearly independent, 1. e. $\dim(\langle E\rangle) = n-2w$. Let $\ell_{\langle E\rangle} : \mathbb{P}^n \setminus \langle E\rangle \to \mathbb{P}^{2w-1}$ denote the linear projection from $\langle E\rangle$. Since $\sharp(E) < n+1-w$, Lemma 2.4 gives $P \notin \langle E\rangle$. Hence $\ell_{\langle E\rangle}(P)$ is a well-defined point of $\mathbb{P}^{2w-1}$. Call $X_E \subset \mathbb{P}^{2w-1}$ the closure of $\ell_{\langle E\rangle}|(X \setminus \langle E\rangle \cap X)$ in $\mathbb{P}^{2w-1}$. Since X is a smooth curve, the rational map $\ell_{\langle E\rangle}|(X \setminus \langle E\rangle \cap X)$ extends to a surjective morphism $\psi : X \to X_E$. For every $Q \in X$ the divisor $E+Q$ is linearly independent, because $\deg(E+Q) < n$. Hence E is the scheme-theoretic intersection of X with $\langle E\rangle$. Hence $\deg(X_E)\cdot\deg(\psi) = deg(X) - \deg(E) = n+1-n+2w+1 = 2w$. Since X is non-degenerate, X_E spans $\mathbb{P}^{2w-1}$ and in particular $\deg(X_E) \ge 2w-1$. Since $\deg(X_E) \ge 2w-1$ and $w \ge 2$, we get $\deg(X_E) = 2w$ and $\deg(\psi) = 1$. Since $\deg(\psi) = 1$, X_E and X are birational. Since $\deg(X_E) \le 2w$ and X_E is non-degenerate, we have $p_a(X_E) \le 1$. Since X_E is birational to X, we get that X_E is smooth and that it is a linearly normal elliptic curve. Since X and X_E are smooth curves, ψ is an isomorphism.

(b) Call $X[n-2w+1]$ the set of all $E \subset X$ such that $\sharp(E) = n-2w+1$, $E \cap W_{red} = \emptyset$ and $\mathcal{O}_X(1)(-E) \ncong \mathcal{O}_X(2W)$. For any $E \in X[n-2w+1]$ we have $\mathcal{O}_X(E + A_E + W) \cong \mathcal{O}_C(1)$ and $\mathcal{O}_X(1)(-E) \ncong \mathcal{O}_C(2W)$. Therefore $A_E \ne W$. Since $\deg(A_E) = \deg(W)$, $E \cap W = \emptyset$, $P \in \langle W\rangle \cap \langle E + A_E\rangle$ and $P \notin \langle W_1\rangle$ for any $W_1 \subsetneq W$, parts (ii) and (iii) of Lemma 2.3 give $(E + A_E) \cap W = \emptyset$ for every $E \in X[n-2w+1]$. Let $\Gamma \subseteq \mathcal{S}$ be any irreducible component of $\mathcal{S}$ containing the irreducible algebraic family $\{E + A_E\}_{E \in X[n-2w+1]}$. Let F be a general element of Γ. Remember that to prove $r_X(P) = n+1-w$ it is sufficient to find a reduced $S \in \Gamma$, while for the open rank we need S with $S \cap T = \emptyset$. Γ is an irreducible algebraic family of divisors of X. We have $\dim(\Gamma) = n-2w+1$. Fix any $E \in X[n-2w+1]$ and call $\psi : X \to X_E$ the isomorphism constructed in step (a). Set $W' := \psi(W)$. Since $\deg(E+W) = n-w+1 \le n$ and $\beta(X) = n$ (Lemma 2.3) the divisors E, W and $E+W$ are linearly independent. Since $E \cap W_{red} = \emptyset$, Grassmann's formula gives $\langle W\rangle \cap \langle E\rangle = \emptyset$. Hence $\dim(\langle W'\rangle) = w$.

Fix any subscheme W_1 of W with $\deg(W_1) = w-1$. Since $W_1 \cap \langle E\rangle = \emptyset$, we have $\dim(\langle\psi(W_1)) = \dim(\langle W'\rangle) - 1$. Since $P \in \langle W\rangle$, but $P \notin \langle W_1\rangle$, we have $\langle W\rangle = \langle W_1 \cup \{P\}\rangle$. Hence $\ell_{\langle E\rangle}(P) \notin \langle\psi(W_1)\rangle$. Varying W_1 we get $\ell_{\langle E\rangle}(P) \notin \langle W''\rangle$ for every scheme $W'' \subsetneq W'$. Hence $\ell_{\langle E\rangle}(P)$ has border rank w (Lemma 2.5). Since $\mathcal{O}_X(2W) \ncong \mathcal{O}_X(1)(-E)$ for all $E \in X[n-2w+1]$, part (b) of Proposition 3.1 with $k = w-1$ applied to the curve X_E, the point $\ell_{\langle E\rangle}(P)$ and the scheme $Z := W'$ gives that such a divisor A_E is unique. Hence Γ is an involution in the classical terminology ([9], §5). Assume for the moment that Γ has no fixed component. In particular $T \cap \gamma = \emptyset$ for a general $\gamma \in \Gamma$. We get that either F is reduced (and hence there is $S \subset X \setminus T$ with $\sharp(S) = n+1-w$ and $P \in \langle S\rangle$) or there is an integer $m \geq 2$ such that each connected component of F appears with multiplicity m ([9], Proposition 5.8). Since $F = E + A_E$ with E reduced and $\sharp(E) > \deg(A_E)$ this is obviously false. Hence we may assume that Γ has a base locus. Call D the base locus of Γ. Since E moves in $X \setminus T$, then $E \cap D = \emptyset$. Hence D is the base locus of the algebraic family $\{A_E\}_{E\in X[n-2w+1]}$. For any $E \in X[n-2w+1]$ we have $A_E \neq W$, because $\mathcal{O}_X(1)(-E) \ncong \mathcal{O}_X(2W)$. Since $E \cap W_{red} = \emptyset$, part (a) of Proposition 3.1 gives $A_E \cap W_{red} = \emptyset$. Hence $D \cap W_{red} = \emptyset$.

The irreducible algebraic family $\Gamma(-D)$ of effective divisors of X has the same dimension and it is base point free. We have $F = D + F'$ with F' general in $\Gamma(-D)$. Since $\Gamma(-D)$ is an involution without base points and whose general member has at least one reduced connected component (a connected component of E), its general member F' is reduced ([9], Proposition 5.8) and contain no point of $T \cup D \cup W_{red}$. In particular $F' \cap D = \emptyset$, $F' \subset X \setminus T$ for a general F' and $F' \cap D = \emptyset$. Therefore to get a reduced divisor $F' + D \in \mathcal{S}$ (and hence to prove that $r_X(P) \leq n+1-w$) it is sufficient to prove that D is reduced. We will even prove that $\deg(D) = 1$ (if $D \neq \emptyset$).

Claim 1. We have $D \cap W = \emptyset$.

Proof of Claim 1. Since $E \cap W = \emptyset$, it is sufficient to prove that $A_E \cap W = \emptyset$ for all W. This is true by part (a) of Proposition 3.1, because $\ell_{\langle E\rangle}(A_E)$ and $\ell_{\langle E\rangle}(W)$ are different elements of $\mathcal{Z}(X_E, \ell_{\langle E\rangle}(P))$.

Claim 2. If Γ has a base locus, then it has a unique base point and this base point appears in D with multiplicity one.

Proof of Claim 2. Assume that Γ has a base point, O, 1. e. that O is contained in the support of A_E for a general $E \in X[n-2w+1]$. Let $\ell_O : \mathbb{P}^n \setminus \{O\} \to \mathbb{P}^{n-1}$ denote the linear projection from O. Claim 1 says that $O \notin W_{red}$. Since $\beta(X) > w$, we get $O \notin \langle W\rangle$. Hence $\ell_O|W$ is an embedding, $\dim(\ell_O(\langle W\rangle)) = w-1$ and $\ell_O(P) \notin \langle W'\rangle$ for any $W' \subsetneq W$. Let X_O be the closure of $\ell_O(X \setminus \{O\})$ in $\mathbb{P}^{n-1}$.

As in step (a) we see that X_O is a linearly normal elliptic curve and that $\ell_O|(X\setminus\{O\})$ extends to an isomorphism $\psi_O : X \to X_O$. We get that $\ell_O(P)$ has border rank w with respect to X_O and that the family

$$\{\psi_O(E+(A_E-\{O\}))\}_{E\in X[n-2w+1]}$$

is an $(n-2w+1)$-dimensional family $\mathcal{F}$ of effective divisors on X_O such that $\ell_O(P)\in\langle J\rangle$ for all $J\in\mathcal{F}$. Therefore for a general $B\subset X_O$ with $\sharp(B)=n-2w$ there are infinitely many $J\in\mathcal{F}$ containing B; call $\mathcal{G}_B$ any such positive-dimensional family. Take the linear projection $\ell_{\langle B\rangle}:\mathbb{P}^{n-1}\setminus\langle B\rangle\to\mathbb{P}^{2w-1}$ from $\langle B\rangle$. Let $X_{O,B}$ be the closure in $\mathbb{P}^{2w-1}$ of $\ell_{\langle B\rangle}(X_O\setminus B)$ in $\mathbb{P}^{2w-1}$. As in step (a) we see that $X_{O,B}$ is a linearly normal elliptic curve. By construction for every $J\in\mathcal{G}_B$ the divisor $J-B$ is effective and $\ell_{\langle B\rangle}(J-B)\in\mathcal{Z}(X_{O,B},\ell_{\langle B\rangle}(\ell_O(P))$. By Proposition 3.1 any two divisors $J-B$ are disjoint. Taking $O+\psi_O^{-1}(B)$ we see that the family in the boundary of the family $\{E+A_E\}_{E\in X[n-2w+1]}$ (we only took sets E containing O and we are allowed to do it, because we checked that $O\notin W_{red}$). Hence $D-O$ is in the base locus of $\mathcal{G}_B$. Therefore $D=\{O\}$.

If $D\neq\emptyset$, then Claim 2 gives $D=O$ for some $O\in X$. In this case O is the point appearing in the statement of Proposition 1.4 and every $S\in\mathcal{S}(X,P)$ contains O. If $D=\emptyset$, then $or_X(P)=n+1-w$ and in Proposition 1.4 we may take as O any point of X. □

Remark 3.4. Fix P,X,n,w as in Theorem 1.1 and assume $or_X(P)\geq n+2-w$. By Proposition 1.4 there is $O\in X$ with the property that $O\in S$ for each $S\subset X$ such that $\sharp(S)=n+1-w$ and $P\in\langle S\rangle$. In the proof just given we also got that $O\notin W_{red}$ (this also follows from part (ii) of Lemma 2.3).

Proposition 3.5. *Let $X\subset\mathbb{P}^n$, $n\geq 4$, be a linearly normal elliptic curve. Let $\tau(X)\subset\mathbb{P}^n$ be the tangential surface of X. Fix $P\in\mathbb{P}^n$ with $b_X(P)=2$.*

(i) If $n\geq 5$, then $or_X(P)=n-1$.

(ii) Assume $n=4$. We have $3\leq or_X(P)\leq 4$.

1. *We have $or_X(P)=3$ for a general $P\in\sigma_2(X)\setminus\tau(X)$, but there are $P\in\sigma_2(X)\setminus\tau(X)$ with $or_X(P)=4$.*
2. *We have $or_X(P)=3$ for a general $P\in\tau(X)$, but there are $P\in\tau(X)\setminus X$ with $or_X(P)=4$.*
3. *The set of all $P\in\tau(X)$ with $or_X(P)=4$ has dimension 1. The set of all $P\in\sigma_2(X)\setminus\tau(X)$ with $or_X(P)=4$ has dimension 2.*
4. *The set of all $P\in\sigma_2(X)\setminus X$ with $or_X(P)=4$ is contained in the union of 4 one-dimensional families of lines of $\mathbb{P}^4$, the vertices of the rank 3 quadric hypersurfaces containing X.*

Proof. For any $Q \in \mathbb{P}^n$ let $\ell_Q : \mathbb{P}^n \setminus \{Q\} \to \mathbb{P}^{n-1}$ denote the linear projection from Q. Remark 2.8 gives $or_X(P) \le n$. Lemma 2.6 gives $or_X(P) \ge n-1$. Let $W \subset X$ be the degree two scheme such that $P \in \langle W \rangle$. Fix a finite set $T \subset X$.

(a) Assume $n = 4$. Set $Y := \ell_P(X)$. The curve Y is non-degenerate and in particular it is not a line. Since $P \notin X$, we have $5 = \deg(X) = \deg(\ell_P|X) \cdot \ell_P(X)$. Hence $\deg(\ell_P|X) = 1$, $\deg(Y) = 5$ and $\ell_P|X : X \to Y$ is the normalization map of Y. The curve Y is singular (it has at least a cusp if $P \in \tau(X)$ and at least one non-unibranch point if $P \in \sigma_2(X) \setminus \tau(X)$). An easy upper bound for the arithmetic genus of any degree 5 space curve gives $p_a(Y) \le 2$. Hence $p_a(Y) = 2$. Since $2 \cdot \deg(Y) > 2p_a(Y) - 2$, Riemann-Roch gives $h^0(\mathbb{P}^3, \mathcal{I}_Y(2)) > 0$. Since $\deg(Y) > 4$ and Y is non-degenerate, Y is not contained in two different quadric surfaces. Let M be the only quadric surface containing Y. Since Y is irreducible and non-degenerate, then M is irreducible.

(a1) First assume that the quadric surface M is smooth. Label the rulings of M so that $Y \in |\mathcal{O}_M(2,3)|$. For a general line $L \in |\mathcal{O}_M(1,0)|$ the set $L \cap Y$ is formed by 3 smooth points, none of them belonging to $\ell_P(T)$. Taking the inverse images in X of these 3 points we get $or_X(P) \le 3$.

(a2) Now assume that M is a quadric cone with vertex o. The linear projection from o maps Y onto a smooth conic. Since Y is birational to X, this map cannot be birational. Hence if M is a cone with vertex o, then o must be a smooth point of Y. Let $O \in X$ be the only point of X such that $\ell_P(O) = o$. We have $O \notin W_{red}$, because o is a smooth point of Y. Fix any $S \subset X \setminus \{O\}$ such that $\sharp(S) = 3$, $S \cap W_{red} = \emptyset$ and $P \in \langle S \rangle$. Since $\beta(X) = 4$, $P \in \langle W \rangle$ and $S \cap W_{red} = \emptyset$, part (ii) of Lemma 2.3 gives $\sharp(\ell_P(S)) = 3$. The set $\ell_P(S)$ is contained in the line $\ell_P(\langle S \rangle \setminus \{P\})$. Since every 3-secant line of Y is contained in M and every line of M contains o, we get $O \in \langle \{S, P\} \rangle$. We have $\langle \{S, P\} \rangle = \langle S \rangle$. Since $\beta(X) = 4$ and $O \in \langle S \rangle$, we have $O \in S$, contradicting our assumption $S \subset X \setminus \{O\}$. Taking as T any finite subset of X containing O we get $or_X(P) \ge 4$.

(a3) Now we analyze for which (X, P) we have $or_X(P) = 4$, 1. e. when the quadric M appearing in step (a) is singular. Let X be any elliptic curve. Fix any $\mathcal{L} \in \mathrm{Pic}^5(X)$ and use it to embed X into $\mathbb{P}^4$ as a linearly normal elliptic curve, writing $X \subset \mathbb{P}^4$ and $\mathcal{L} = \mathcal{O}_X(1)$. If $X \subset N$ with N a rank 3 quadric cone with vertex a line L (we will then project from some $P \in L$ to get a quadric cone $M \subset \mathbb{P}^3$ as in step (a2)), then the linear projection from L maps generically two to one X onto a conic and in particular L must cut quasi-transversally X and at exactly one point, U. We reverse this observation to produce any such rank 3 hyperquadric containing X. Fix any $U \in X$ and write $\mathcal{O}_X(1)(-U) = R^{\otimes 2}$ for some $R \in \mathrm{Pic}^2(X)$ (in characteristic $\neq 2$ there are exactly 4 line bundles R; in characteristic 2 there are 2 R's if X is not supersingular and one R if X is supersingular).

Use U to get an embedding j_U of $H^0(R^{\otimes 2})$ into $H^0(\mathcal{O}_X(1))$ as a hyperplane. Riemann-Roch gives $h^0(R) = 2$ and $h^0(R^{\otimes 2}) = 2$. Since $S^2(H^0(R))$ has dimension 3, the multiplication map $H^0(R) \otimes H^0(R) \to H^0(R^{\otimes 2})$ has a one-dimensional kernel. Taking a basis of $h^0(R^{\otimes 2})$ we see that this kernel corresponds to the vertex of a rank 3 quadric cone of $\mathbb{P}^3$ containing the embedding of X by the linear system $|R^{\otimes 2}|$ and, using j_U, a quadric cone of $\mathbb{P}^4$ containing X and with as its vertex a line through U. All rank 3 quadric cones of $\mathbb{P}^4$ containing X appears in this way (the line bundle R is the line bundle inducing the generically two to one morphism $\phi : X \to D$ with D a conic). Therefore they are parametrized by 4 (or 2 or 1 in characteristic 2) one-dimensional families; fix any $P' \in \sigma_2(X) \setminus X$; taking the linear projection from P' we see that two different rank 3 quadric cones cannot have vertices containing P'. Fix a rank 3 quadric cone with associated vertex L and associated degree two map ϕ. We only need to see which $P \in \sigma_2(X) \setminus X$ is contained in $(L \setminus \{U\})$ for some U and some R. We fix U and R and hence we fix L. Since $L \cap X = \{U\}$ scheme-theoretically (R^2 has no base points) L is neither a secant line nor a tangent line. Fix $O \in X \setminus \{U\}$. We have $L \cap \langle 2O \rangle \neq \emptyset$ if and only if ϕ ramifies at O (in characteristic $\neq 2$ there are 4 such points; in characteristic 2 there are 2 such points if X is not supersingular and a unique such point if X is supersingular). Fix $O_1, O_2 \in X \setminus \{U\}$ such that $O_1 \neq O_2$. We have $L \cap \langle \{O_1, O_2\} \rangle \neq \emptyset$ if and only if $\phi(O_1) = \phi(O_2)$ and hence there is a 1-dimensional family of such secant lines. Since $L \cap X = \{U\}$ scheme-theoretically, each non-empty intersection $L \cap \langle 2O \rangle$ and $L \cap \langle \{O_1, O_2\} \rangle$ is a single point.

(b) Assume $n = 5$. Assume $or_X(P) \geq 5 = n$. We take the set-up of the proof of Theorem 1.4 and Proposition 1.4. Let $\mathcal{S}(X,P)'$ be the set of all $S \subset X$ such that $\sharp(S) = 4$, $P \in \langle S \rangle$ and $S \not\supseteq W$. We claim that the proof of Theorem 1.1 gives the existence of a two-dimensional family Λ of elements of $\mathcal{Z}(X,P)$ such that $M \not\supseteq W$ for every $M \in \Lambda$ (even if W is reduced); indeed, since $n \geq 2w$, we have $E \neq \emptyset$ and hence for a general $E \in X$ we have $\mathcal{O}_X(1)(-E) \neq \mathcal{O}_X(W)$, 1. e. $A_E \neq W$. We also proved that $\dim(\mathcal{S}(X,P)') \geq 1$ (the family $\mathcal{G}_B$). Recall that D is reduced (Claim 2 of the quoted proof). Since D is reduced a general $E + A_E$ is reduced. Therefore $\dim(\Lambda \cap \mathcal{S}(X,P)') \geq 2$. By Remark 3.4 there is $O \in X \setminus W_{red}$ such that $O \in S$ for all $S \in \Lambda \cap \mathcal{S}(X,P)'$. Part (ii) of Lemma 2.3 gives $S \cap W_{red} = \emptyset$ for all $S \in \mathcal{S}(X,P)'$. Since $\beta(X) = 5 > 4$ and $P \in \langle Z \rangle$ for some scheme $Z \subset X$ with $\deg(Z) = 2$, the line $\langle \{O,P\} \rangle$ meets X quasi-transversally and only at O, 1. e. $\langle \{O,P\} \rangle \cap X = \{O\}$ (scheme-theoretic intersection). Let $\ell : \mathbb{P}^5 \setminus \langle \{O,P\} \rangle \to \mathbb{P}^3$ denote the linear projection from the line $\langle \{O,P\} \rangle$. Let Y be the closure of $\ell(X \setminus \{O\})$ in $\mathbb{P}^3$.

Since X is non-degenerate, Y is non-degenerate. Since X is smooth and $\langle\{O,P\}\rangle \cap X = \{O\}$ as schemes, ℓ induces a morphism $\psi : X \to Y$ with $5 = \deg(X) - 1 = \deg(\psi)\cdot\deg(Y)$. Hence $\deg(Y) = 5$ and ψ is the normalization map. The curve Y is singular (it has at least a cusp if P is in the tangential variety $\tau(X)$ of X, while it has at least one singular non-unibranch point if $P \in \sigma_2(X) \setminus \tau(X)$). Since $\deg(Y) = 5 > 4$, Y is contained in at most one quadric surface. Since X is the normalization of Y and Y is singular, we have $p_a(Y) \ge 2$. The bound for the arithmetic genus of any degree 5 space curve gives $p_a(Y) = 2$ and hence $\deg(Y) > 2p_a(Y) - 2$. Riemann-Roch gives $h^0(\mathbb{P}^3, \mathcal{I}_Y(2)) > 1$. Therefore Y is contained in a unique quadric surface M. Since $p_a(Y) = 2$, we also see that ψ is an isomorphism outside W_{red} and hence it is injective, with the only exception of identifying the two points of W if W is reduced. Fix any $S \in \Lambda \cap \mathcal{S}(X,P)'$ and write $S = S' \sqcup \{O\}$ with $\sharp(S') = 3$. Since $r_X(P) > 3$, we have $P \notin \langle S'\rangle$ and hence $\langle\{O,P\}\rangle \cap \langle S'\rangle$ is a single point, P_S. Therefore $L_S := \ell(\langle S'\rangle \setminus \{P_S\})$ is a line containing $\ell(S')$. By Bezout's theorem every line $L \subset \mathbb{P}^3$ with $\deg(L \cap Y) \ge 3$ is contained in the quadric surface M. Since $\deg(Y) = 5$, we have $\deg(L \cap Y) \le 5$ for each line $L \subset \mathbb{P}^3$. Since $\psi_{|X\setminus W_{red}}$ is injective, we get $L_S \ne L_{S_1}$ for general $S, S_1 \in \Lambda \cap \mathcal{S}(X,P)'$ and hence that M contains a two-dimensional family of lines, contradicting the irreducibility of M.

(c) Assume $n > 5$ and that the Proposition 3.5 is true for lower dimensional projective spaces. Fix $Q \in X \setminus T$. Let $C \subset \mathbb{P}^{n-1}$ be the closure of $\ell_Q(X \setminus \{Q\})$ in $\mathbb{P}^{n-1}$. Since X is smooth at Q, the morphism $\ell_{|X\setminus\{Q\}}$ extends to a morphism $\psi : X \to C$. Since X is smooth at Q, we have $\deg(\psi)\cdot\deg(C) = \deg(X) - 1 = n$. Since C is non-degenerate, we have $\deg(C) \ge n-1$. Therefore $\deg(\psi) = 1$ and $\deg(C) = n$. Since ψ is birational, we have $p_a(C) \ge 1$. An easy upper bound for the arithmetic genus of any degree n non-degenerate curve in $\mathbb{P}^{n-1}$ gives $p_a(C) \le 1$. Hence C is a linearly normal elliptic curve and ψ is an isomorphism. Since $b_X(P) = 2$, we have $P \ne Q$ and hence $\ell_Q(P)$ is defined. We have $\psi(Q) = \ell_Q(\langle 2Q\rangle \setminus \{Q\})$. Set $T_1 := \psi(T)$. Since $Q \notin T$, we have $T_1 = \ell_Q(T)$. Set $T_2 := T_1 \cup \{\psi(Q)\}$. By the inductive assumption there is $S \subset C \setminus T_2$ such that $\ell_Q(P) \in \langle S\rangle$ and $\sharp(S) \le n-2$. Set $B := \psi^{-1}(S) \cup \{Q\}$. Since $\psi(Q) \notin B$, B is a reduced divisor of X. We have $P \in \langle B\rangle$, $\sharp(B) \le n-1$ and $B \subset X \setminus T$. □

Proposition 3.6. *Let $X \subset \mathbb{P}^3$ be a linearly normal elliptic curve. Then $2 \le or_X(P) \le 3$ for all $P \in \mathbb{P}^3$. In characteristic $\ne 2$ there are exactly 4 points of $\mathbb{P}^3$ with $or_X(P) = 2$. In characteristic two there are exactly 2 (case X not supersingular) or exactly 1 (case X supersingular) points of $\mathbb{P}^3$ with $or_X(P) = 2$.*

Proof. If $P \in X$, then use Lemma 2.9. Assume $P \notin X$. Lemma 2.7 gives $or_X(P) \le 3$. Let Γ be the set of all $P \in \mathbb{P}^3 \setminus X$ such that there are infinitely many lines L with $\deg(L \cap X) = 2$ and $P \in L$. If $P \notin \Gamma$, then $or_X(P) > 2$, because the set of all $S \subset X$ with $\sharp(S) \le 2$ and $P \in \langle S\rangle$ is finite. If $P \in \Gamma$, then

$or_X(P) = 2$, because P is not a strange point of X (Remark 2.8) and for any finite set $T \subset X$ there is a line L though P containing two different points of $X \setminus T$. The curve X is a complete intersection of two quadrics. Therefore there is no line $L \subset \mathbb{P}^3$ with $\deg(L \cap X) \geq 3$. Therefore the linear projection from P shows that Γ is the set of all vertices of all quadric cones containing X. Every quadric cone containing X has a unique vertex, because X is irreducible. There is a bijection between these quadric cones (hence the elements of Γ) and the set of all line bundles R on X with $R^{\otimes 2} \cong \mathcal{O}_X(1)$. In characteristic $\neq 2$ there are exactly 4 such R's. In characteristic two there are 2 such R if X is not supersingular and 1 such R if R is supersingular. $\square$

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