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DISTANCE TWO LABELING ON SPECIAL FAMILY OF GRAPHS

MUTHALI MURUGAN

An $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 2$ if x and y are adjacent and $|f(x)-f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. A k-$L(2,1)$-labeling is an $L(2,1)$-labeling $f : V(G) \rightarrow \{0,\ldots,k\}$, and we are interested to find the minimum k among all possible assignments. This invariant, the minimum k, is known as the $L(2,1)$-labeling number or λ-number and is denoted by $\lambda(G)$. In this paper, we determine the λ-number for the coronas $P_m \circ P_n, P_m \circ C_n, P_m \circ K_{1,n}$ and $P_m \circ W_n$ and find an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any two graphs such that G_2 has an injective $L(2,1)$-labeling and also we prove that the bound is attainable when G_1 and G_2 are complete. Also we present an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any two graphs.

1. Introduction

The unprecedented growth of wireless communication made the study of assigning proper radio frequencies to these communication networks more popular. The interference by unconstrained transmitters will interrupt the communication. In the channel assignment problem, we assign a channel (non-negative integer) to each television or radio transmitters located at various places so that

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we do not have any interference in the communication. The original notion of distance two labeling can be seen in the context of frequency assignment, where 'close' transmitters must receive different frequencies and 'very close' transmitters must receive frequencies that are at least two frequencies apart so that they can avoid interference. Due to its practical importance, the distance two labeling problem has been widely studied. Distance two labeling is also known as $L(2,1)$-labeling. An $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 2$ if x and y are adjacent and $|f(x)-f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. A k-$L(2,1)$-labeling is an $L(2,1)$-labeling $f : V(G) \to \{0,\ldots,k\}$, and we are interested to find the minimum k among all possible assignments. This invariant, the minimum k, is known as the $L(2,1)$-labeling number or λ-number and is denoted by $\lambda(G)$. The generalization of this concept is as below.

For positive integers k, d_1, d_2, a k-$L(d_1,d_2)$-labeling of a graph G is a function $f : V(G) \to \{0,1,2,\ldots,k\}$ such that $|f(u)-f(v)| \geq d_i$ whenever the distance between u and v in G, $d_G(u,v) = i$, for $i = 1,2$. The $L(d_1,d_2)$-number of G, $\lambda_{d_1,d_2}(G)$, is the smallest k such that there exists a k-$L(d_1,d_2)$-labeling of G.

2. Some Existing Results

Distance two labeling or $L(2,1)$-labeling has received the attention of many researchers and here we present some important existing results.

- In [1] Griggs and Yeh have discussed $L(2,1)$-labeling for path, cycle, tree and cube. They have derived results for the graphs of diameter 2. They have shown that the $\lambda(T)$ for trees with maximum degree $\Delta \geq 1$ is either $\Delta+1$ or $\Delta+2$.

- Chang and Kuo [2] provided an algorithm to obtain $\lambda(T)$.

- Vaidya and Bantava [3] have discussed $L(2,1)$-labeling of cacti.

- Vaidya et.al. [4] have discussed $L(2,1)$-labeling in the context of some graph operations.

- Yeh [5] have discussed the $L(2,1)$-labeling on various class of graphs like trees, cycles, chordal graphs, Cartesian products of graphs etc.,

- Griggs and Yeh [1] proved that if a graph G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta+2$, where Δ is the maximum degree of G.

- Griggs and Yeh [1] posed a conjecture that $\lambda(G) \leq \Delta^2$ for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G, and they proved that $\lambda(G) \leq \Delta^2 + 2\Delta$ at the same time.
- Chang and Kuo [6] proved that $\lambda(G) \leq \Delta^2 + \Delta$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Kral and Skrekovski [7] proved that $\lambda(G) \leq \Delta^2 + \Delta - 1$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Goncalves [8] proved that $\lambda(G) \leq \Delta^2 + \Delta - 2$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.

In spite of all the efforts the conjecture posed by Griggs and Yeh is still open. Also many results on trees are available and strict bounds of λ are found for trees. So in this paper, we concentrate on graphs with cycles.

3. Results

In this section, we determine the λ-number for the coronas $P_m \circ P_n, P_m \circ C_n, P_m \circ K_{1,n}$ and $P_m \circ W_n$ and find an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any two graphs such that G_2 has an injective $L(2,1)$-labeling and the bound is attainable when G_1 and G_2 are complete. Also we present an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any two graphs.

Definition 3.1. Let G_1 and G_2 be two graphs with $V(G_1) = \{u_0, u_1, \ldots, u_{m-1}\}$ and $V(G_2) = \{v_0, v_1, \ldots, v_{n-1}\}$. The corona $G_1 \circ G_2$ is the graph with

$$V(G_1 \circ G_2) = V(G_1) \cup \{v_{i,j} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$$

and

$$E(G_1 \circ G_2) = E(G_1) \cup \left\{v_{i,j_1} v_{i,j_2} : v_{j_1} v_{j_2} \in E(G_2)\right\}_{i=0}^{m-1} \cup \{u_i v_{i,j} : 0 \leq j \leq n-1\}_{i=0}^{m-1}$$

Definition 3.2. An injective $L(2,1)$-labeling is called an $L'(2,1)$-labeling.

A k-$L'(2,1)$-labeling is an $L'(2,1)$-labeling $f : V(G) \to \{0, \ldots, k\}$, and we are interested to find the minimum k among all possible assignments. This invariant, the minimum k, is known as the $L'(2,1)$-labeling number or λ'-number and is denoted by $\lambda'(G)$.

Definition 3.3. Let f be a labeling of a graph G. The number of occurrence of a label less one is called the multiplicity of the label in f and the sum of the multiplicity of labels of f is called the multiplicity of f.

Theorem 3.4. *For the corona* $P_m \circ P_n, m,n \geq 5, \lambda(P_m \circ P_n) = n+4 = \Delta+2$.

Proof. Consider the corona $P_m \circ P_n, m,n \geq 5$. Let $V(P_m) = \{u_0, u_1, \ldots, u_{m-1}\}$ and $V(P_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. Define $f : V(P_m \circ P_n) \to N \cup \{0\}$ such that

$$f(u_i) = 2(i \bmod 3), \;\; 0 \leq i \leq m-1.$$

Suppose $i \equiv 0,1(\bmod 3)$, define

$$f(v_{i,j}) = \begin{cases} 5 & \text{if} \quad j=0 \\ f(v_{i,j-2})+1 & \text{if} \quad j \geq 1 \text{ and } j \text{ is even} \\ f(v_{i,j-1})+\lceil \frac{n}{2} \rceil & \text{if} \quad j \geq 1 \text{ and } j \text{ is odd} \end{cases}$$

Suppose $i \equiv 2(\bmod 3)$, define

$$f(v_{i,j}) = \begin{cases} 1 & \text{if} \quad j=0 \\ 6 & \text{if} \quad j=1 \\ f(v_{i,j-2})+1 & \text{if} \quad j \geq 2 \text{ and } j \text{ is odd} \\ f(v_{i,j-1})+\lfloor \frac{n}{2} \rfloor & \text{if} \quad j \geq 2 \text{ and } j \text{ is even} \end{cases}$$

Now we prove that f is a distance two labeling. When $d(u_i, u_j)$ is 1 or 2, we have $|f(u_i) - f(u_j)|$ is 2 or 4. When $d(v_{i,j}, v_{i,k}) = 1$, we have $|f(v_{i,j}) - f(v_{i,k})| = \lceil \frac{n}{2} \rceil$ or $\lceil \frac{n}{2} \rceil - 1$ or $\lfloor \frac{n}{2} \rfloor - 1$ or 5. When $d(v_{i,j}, v_{i,k}) = 2$, we have $|f(v_{i,j}) - f(v_{i,k})| = 1$ or $\lfloor \frac{n}{2} \rfloor + 5$. When $d(u_i, v_{i,j}) = 1$, clearly $|f(u_i) - f(v_{i,j})| \geq 2$. When $d(u_i, v_{\alpha,j}) = 2$, $i \neq \alpha$, clearly $|f(u_i) - f(v_{\alpha,j})| \geq 1$, since no label of u_i occur as a label on $v_{\alpha,j}$. Thus, for any two vertices of a_i, b_j of $P_m \circ P_n$, $|f(a_i) - f(b_j)| \geq 2$ when $d(a_i, b_j) = 1$ and $|f(a_i) - f(b_j)| \geq 1$ when $d(a_i, b_j) = 2$. Hence f is a distance two labeling.

When n is even and $i \equiv 0,1(\bmod 3)$, the maximum label occurs on $v_{i,n-1}$ by construction of f and

$$\begin{aligned} f(v_{i,n-1}) &= f(v_{i,n-2}) + \left\lceil \frac{n}{2} \right\rceil \\ &= f(v_{i,0}) + \frac{n-2}{2} + \left\lceil \frac{n}{2} \right\rceil \\ &= 5 + \frac{n-2}{2} + \frac{n}{2} \\ &= n+4 \\ &= n+2+2 \\ &= \Delta+2. \end{aligned}$$

When n is even and $i \equiv 2 (\mathrm{mod}\ 3)$, the maximum label occurs on $v_{i,n-2}$ by construction of f and

$$\begin{aligned}
f(v_{i,n-2}) &= f(v_{i,n-3}) + \left\lceil \frac{n}{2} \right\rceil \\
&= f(v_{i,1}) + \frac{n-4}{2} + \left\lceil \frac{n}{2} \right\rceil \\
&= 6 + \frac{n-4}{2} + \frac{n}{2} \\
&= n+4 \\
&= n+2+2 \\
&= \Delta + 2.
\end{aligned}$$

When n is odd and $i \equiv 0,1 (\mathrm{mod}\ 3)$, the maximum label occurs on $v_{i,n-2}$ by construction of f and

$$\begin{aligned}
f(v_{i,n-2}) &= f(v_{i,n-3}) + \left\lceil \frac{n}{2} \right\rceil \\
&= f(v_{i,0}) + \frac{n-3}{2} + \frac{n+1}{2} \\
&= 5 + n - 1 \\
&= n+4 \\
&= n+2+2 \\
&= \Delta + 2.
\end{aligned}$$

When n is odd and $i \equiv 2 (\mathrm{mod}\ 3)$, the maximum label occurs on $v_{i,n-1}$ by construction of f and

$$\begin{aligned}
f(v_{i,n-1}) &= f(v_{i,n-2}) + \left\lfloor \frac{n}{2} \right\rfloor \\
&= f(v_{i,1}) + \frac{n-3}{2} + \frac{n-1}{2} \\
&= 6 + n - 2 \\
&= n+4 \\
&= n+2+2 \\
&= \Delta + 2.
\end{aligned}$$

Therefore $\lambda(P_m \circ P_n) \leq n+4 = \Delta+2$. Since $P_m \circ P_n$ contains three vertices of degree $\Delta = n+2$ such that one of them is adjacent to the other two, we have $\lambda(P_m \circ P_n) \geq n+4 = \Delta+2$. Hence $\lambda(P_m \circ P_n) = n+4 = \Delta+2$. □

Theorem 3.5. *For the corona $P_m \circ C_n, m \geq 5, n \geq 6, \lambda(P_m \circ C_n) = n+4 = \Delta+2$.*

Proof. Consider the corona $P_m \circ C_n, m \geq 5, n \geq 6$. Let

$$V(P_m) = \{u_0, u_1, \ldots, u_{m-1}\}.$$

Define $f : V(P_m \circ C_n) \to N \cup \{0\}$ such that

$$f(u_i) = n + 4 - 2(i \bmod 3), \;\; 0 \leq i \leq m-1.$$

Case 1 n is even.

Name the first $\frac{n}{2}$ vertices of C_n as $v_1, v_2, \ldots, v_{\frac{n}{2}}$ and the remaining vertices as $w_1, w_2, \ldots, w_{\frac{n}{2}}$. Define f to the vertices of C_n which are adjacent to u_i, for $i \equiv 0, 1 (\text{mod } 3)$ such that

$$f(v_i) = 2i - 2, \text{ for } i = 1, 2, 3, \ldots, \frac{n}{2} \quad \text{and}$$
$$f(w_i) = 2i - 1, \text{ for } i = 1, 2, 3, \ldots, \frac{n}{2}$$

Define f to the vertices of C_n which are adjacent to u_i, for $i \equiv 2 (\text{mod } 3)$ such that

$$f(v_i) = 2i - 2, \text{ for } i = 1, 2, 3, \ldots, \frac{n}{2} \quad \text{and}$$
$$f(w_i) = 2i - 1, \text{ for } i = 1, 2, 3, \ldots, \frac{n}{2} - 1$$
$$f(w_i) = n + 3, \text{ for } i = \frac{n}{2}$$

Now we prove that this f is a distance two labeling.

Since u_is are labeled with $n+4, n+2, n$ consecutively and repeatedly we have,

$$|f(u_i) - f(u_j)| = 2 \text{ or } 4,$$

when $d(u_i, u_j) = 1$ and $d(u_i, u_j) = 2$.

For the vertices a_i, b_j of C_n such that $d(a_i, b_j) = 1$, we have

$$|f(a_i) - f(b_j)| = 2 \text{ or } n-3 \text{ or } n-1, \text{ or } 6, \text{ or } n+3.$$

Since no vertex label repeats on C_n, we have

$$|f(a_i) - f(b_j)| \geq 1,$$

when $d(a_i, b_j) = 2$.

Among the vertices of C_n which are adjacent to u_i, $i \equiv 0, 1 (\text{mod } 3)$, the highest label is $n-1$; among the vertices of C_n which are adjacent to u_i,

$i \equiv 2(\text{mod } 3)$, the highest label is $n+3$ and since no label of u_i occur as a label in the other C_n, we have, $|f(u_i) - f(c_j)| \geq 2$, when $d(u_i, c_j) = 1$ and $|f(u_i) - f(c_j)| \geq 1$, when $d(u_i, c_j) = 2$ where c_j is a vertex of C_n.

Hence for any two vertices s_i, t_j of $P_m \circ C_n$, $|f(s_i) - f(t_j)| \geq 2$, when $d(s_i, t_j) = 1$ and $|f(s_i) - f(t_j)| \geq 1$, when $d(s_i, t_j) = 2$. Hence f is a distance two labeling.

Since $n+4$ is the maximum label we have used, $\lambda(P_m \circ C_n) \leq n+4$. Since $P_m \circ C_n$ contains three vertices of degree $\Delta = n+2$ such that one of them is adjacent to the other two, we have $\lambda(P_m \circ C_n) \geq \Delta + 2 = n+4$. Hence $\lambda(P_m \circ C_n) = n+4 = \Delta + 2$.

Case 2 n is odd.

Name the first $\frac{n+1}{2}$ vertices of C_n as $v_1, v_2, \ldots, v_{\frac{n+1}{2}}$ and the remaining vertices as $w_1, w_2, \ldots, w_{\frac{n-1}{2}}$. Define f to the vertices of C_n which are adjacent to u_i, for $i \equiv 0, 1(\text{mod } 3)$ such that

$$f(v_i) = 2i - 2, \text{ for } i = 1, 2, 3, \ldots, \frac{n+1}{2} \quad \text{and}$$
$$f(w_i) = 2i - 1, \text{ for } i = 1, 2, 3, \ldots, \frac{n-1}{2}$$

Define f to the vertices of C_n which are adjacent to u_i, for $i \equiv 2(\text{mod } 3)$ such that

$$f(v_i) = 2i - 2, \text{ for } i = 1, 2, 3, \ldots, \frac{n+1}{2} - 1$$
$$f(v_{\frac{n+1}{2}}) = n+3$$
$$f(w_i) = 2i - 1, \text{ for } i = 1, 2, 3, \ldots, \frac{n-1}{2}$$

Now we prove that this f is a distance two labeling.

Since u_is are labeled with $n+4, n+2, n$ consecutively and repeatedly we have,

$$|f(u_i) - f(u_j)| = 2 \text{ or } 4,$$

when $d(u_i, u_j) = 1$ and $d(u_i, u_j) = 2$.

For the vertices a_i, b_j of C_n such that $d(a_i, b_j) = 1$, we have

$$|f(a_i) - f(b_j)| = 2 \text{ or } n-2, \text{ or } 6, \text{ or } n+2.$$

Since no vertex label repeat on C_n, we have

$$|f(a_i) - f(b_j)| \geq 1,$$

when $d(a_i, b_j) = 2$.

Among the vertices of C_n which are adjacent to u_i, $i \equiv 0, 1 (\text{mod } 3)$, the highest label is $n-1$; among the vertices of C_n which are adjacent to u_i, $i \equiv 2 (\text{mod } 3)$, the highest label is $n+3$ and since no label of u_i occur as a label in the other C_n, we have, $|f(u_i) - f(c_j)| \geq 2$, when $d(u_i, c_j) = 1$ and $|f(u_i) - f(c_j)| \geq 1$, when $d(u_i, c_j) = 2$ where c_j is a vertex of C_n.

Hence for any two vertices s_i, t_j of $P_m \circ C_n$, $|f(s_i) - f(t_j)| \geq 2$, when $d(s_i, t_j) = 1$ and $|f(s_i) - f(t_j)| \geq 1$, when $d(s_i, t_j) = 2$. Hence f is a distance two labeling.

Since $n+4$ is the maximum label we have used, $\lambda(P_m \circ C_n) \leq n+4$. Since $P_m \circ C_n$ contains three vertices of degree $\Delta = n+2$ such that one of them is adjacent to the other two, we have $\lambda(P_m \circ C_n) \geq \Delta + 2 = n+4$. Hence $\lambda(P_m \circ C_n) = n+4 = \Delta + 2$.

Hence the theorem. □

Theorem 3.6. *For the corona $P_m \circ K_{1,n}, m \geq 5, n \geq 3, \lambda(P_m \circ K_{1,n}) = n+5 = \Delta + 2$.*

Proof. Consider the corona $P_m \circ K_{1,n}, m \geq 5, n \geq 3$. Let

$$V(P_m) = \{u_0, u_1, \ldots, u_{m-1}\} \text{ and } V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$$

where v_0 is the central vertex of $K_{1,n}$. Define $f : V(P_m \circ K_{1,n}) \to N \cup \{0\}$ such that

$$f(u_i) = \begin{cases} n+5 & \text{if} \quad i \text{ mod } 4 = 0 \\ n+3 & \text{if} \quad i \text{ mod } 4 = 1 \\ 1 & \text{if} \quad i \text{ mod } 4 = 2 \\ 4 & \text{if} \quad i \text{ mod } 4 = 3 \end{cases}$$

Define f to the vertices of $K_{1,n}$ which are adjacent to u_i, for $i \equiv 0 \ (\text{ mod } 4)$ such that

$$\begin{aligned} f(w) &= 0 \\ f(v_1) &= 2 \\ f(v_2) &= 3 \\ f(v_i) &= i+2, \ \ i = 3, 4, \ldots, n. \end{aligned}$$

Define f to the vertices of $K_{1,n}$ which are adjacent to u_i, for $i \equiv 1 \ (\text{ mod } 4)$ such that

$$\begin{aligned} f(w) &= 0 \\ f(v_i) &= i+1, \ \ i = 1, 2, \ldots, n. \end{aligned}$$

Define f to the vertices of $K_{1,n}$ which are adjacent to u_i, for $i \equiv 2 \pmod 4$ such that

$$\begin{aligned} f(w) &= 3 \\ f(v_i) &= 4+i \;\; i = 1,2,\ldots,n-2 \\ f(v_{n-1}) &= n+4 \\ f(v_n) &= n+5. \end{aligned}$$

Define f to the vertices of $K_{1,n}$ which are adjacent to u_i, for $i \equiv 3 \pmod 4$ such that

$$\begin{aligned} f(w) &= 0 \\ f(v_1) &= 2 \\ f(v_i) &= 4+i, \;\; i = 2,3,\ldots,n. \end{aligned}$$

Now we prove that f is a distance two labeling.

Since u_is are labeled with $n+5, n+3, 1, 4$ consecutively and repeatedly we have,

$$|f(u_i) - f(u_j)| = 2 \text{ or } 3, \text{ or } n+1, \text{ or } n+2,$$

when $d(u_i, u_j) = 1$;

$$|f(u_i) - f(u_j)| = n+4 \text{ or } n-1,$$

when $d(u_i, u_j) = 2$.

By construction of f,

$$\begin{aligned} |f(v_i) - f(v_j)| &\geq 2 \quad \text{when } d(v_i, v_j) = 1 \text{ and} \\ |f(v_i) - f(v_j)| &\geq 1 \quad \text{when } d(v_i, v_j) = 2 \end{aligned}$$

Since no label of u_i occur as a label in the corresponding $K_{1,n}$, each u_i keeps a minimum label difference 2 with respect to the corresponding $K_{1,n}$ and each u_i occur in other $K_{1,n}$ at a distance 3, we have for any two vertices a_i, b_j of $P_m \circ K_{1,n}$, $|f(a_i) - f(b_j)| \geq 2$, when $d(a_i, b_j) = 1$ and $|f(a_i) - f(b_j)| \geq 1$, when $d(a_i, b_j) = 2$. Hence f is a distance two labeling.

Since $n+5$ is the maximum label we have used, $\lambda(P_m \circ K_{1,n}) \leq n+5$. Since $P_m \circ K_{1,n}$ contains three vertices of degree $\Delta = n+3$ such that one of them is adjacent to the other two, we have $\lambda(P_m \circ K_{1,n}) \geq \Delta + 2 = n+5$. Hence $\lambda(P_m \circ K_{1,n}) = n+5 = \Delta+2$. □

Theorem 3.7. *For the corona $P_m \circ W_n, m \geq 5, n \geq 6, \lambda(P_m \circ W_n) = n+4 = \Delta+2$.*

Proof. Consider the corona $P_m \circ W_n, m \geq 5, n \geq 6$. Let

$$V(P_m) = \{u_0, u_1, \ldots, u_{m-1}\}.$$

Define $f : V(P_m \circ W_n) \to N \cup \{0\}$ such that

$$f(u_i) = \begin{cases} 0 & \text{if} \quad i \bmod 3 = 0 \\ 2 & \text{if} \quad i \bmod 3 = 1 \\ n+3 & \text{if} \quad i \bmod 3 = 2 \end{cases}$$

Case 1 n is even.

Define f to the vertices of W_n which are adjacent to u_i, for $i \equiv 0, 1 (\bmod 3)$ as follows:
Let the central vertex be v_0 and the vertices of C_n be $v_1, v_2, \ldots, v_{n-1}$,

$$\begin{aligned} f(v_0) &= n+4 \\ f(v_1) &= 4 \\ f(v_2) &= \frac{n+6}{2} + 1 \\ f(v_i) &= f(v_1) + \frac{i-1}{2}, \text{ if } i \text{ is odd and } i = 3, 5, \ldots, n-1 \\ f(v_i) &= f(v_2) + \frac{i-2}{2}, \text{ if } i \text{ is even and } i = 4, 6, \ldots, n-2 \end{aligned}$$

Define f to the vertices of W_n which are adjacent to u_i, for $i \equiv 2 (\bmod 3)$ as follows:
Let the central vertex be v_0 and the vertices of C_n be $v_1, v_2, \ldots, v_{\frac{n}{2}}$, and $w_1, w_2, \ldots, w_{\frac{n}{2}-1}$.

$$\begin{aligned} f(v_0) &= 1 \\ f(v_i) &= 2i+1 \text{ if } i = 1, 2, \ldots, \frac{n}{2} \\ f(w_i) &= 2i+2 \text{ if } i = 1, 2, \ldots, \frac{n}{2} - 1 \end{aligned}$$

Now we prove that this f is a distance two labeling.

Since u_is are labeled with $0, 2, n+3$ consecutively and repeatedly we have,

$$|f(u_i) - f(u_j)| = 2 \text{ or } n+1 \text{ or } n+3,$$

when $d(u_i, u_j) = 1$ and $d(u_i, u_j) = 2$. By construction of f, for $v_i, v_j \in W_n$, $|f(v_i) - f(v_j)| \geq 2$ when $d(v_i, v_j) = 1$ and $|f(v_i) - f(v_j)| \geq 1$ when $d(v_i, v_j) = 2$.

Since no label of u_is occur as a label in the corresponding W_n, each u_i maintains a minimum label difference 2 with respect to the corresponding W_n and no label of u_is occur in the different W_n, we have, for any two vertices s_i, t_j of $P_m \circ W_n$, $|f(s_i) - f(t_j)| \geq 2$ when $d(s_i, t_j) = 1$ and $|f(s_i) - f(t_j)| \geq 1$ when $d(s_i, t_j) = 2$. Hence f is a distance two labeling.

Since $n+4$ is the maximum label we have used, $\lambda(P_m \circ W_n) \leq n+4$. Since $P_m \circ W_n$ contains three vertices of degree $\Delta = n+2$ such that one of them is adjacent to the other two, we have $\lambda(P_m \circ W_n) \geq \Delta + 2 = n+4$. Hence $\lambda(P_m \circ W_n) = n+4 = \Delta + 2$. Hence the theorem follows in this case.

Case 2 n is odd.

Define f to the vertices of W_n which are adjacent to u_i, for $i \equiv 0, 1 \pmod 3$ as follows:

Let the central vertex be v_0 and the vertices of C_n be $v_1, v_2, \ldots, v_{n-1}$,

$$\begin{aligned}
f(v_0) &= n+4 \\
f(v_1) &= 4 \\
f(v_2) &= \frac{n+5}{2} + 1 \\
f(v_i) &= f(v_1) + \frac{i-1}{2}, \text{ if } i \text{ is odd and } i = 3, 5, \ldots, n-2 \\
f(v_i) &= f(v_2) + \frac{i-2}{2}, \text{ if } i \text{ is even and } i = 4, 6, \ldots, n-1
\end{aligned}$$

Define f to the vertices of W_n which are adjacent to u_i, for $i \equiv 2 \pmod 3$ as follows:

Let the central vertex be v_0 and the vertices of C_n be $v_1, v_2, \ldots, v_{\frac{n-1}{2}}$, and $w_1, w_2, \ldots, w_{\frac{n-1}{2}}$.

$$\begin{aligned}
f(v_0) &= 1 \\
f(v_i) &= 2i+1 \text{ if } i = 1, 2, \ldots, \frac{n-1}{2} \\
f(w_i) &= 2i+2 \text{ if } i = 1, 2, \ldots, \frac{n-1}{2}
\end{aligned}$$

Now we prove that this f is a distance two labeling.

Since u_is are labeled with $0, 2, n+3$ consecutively and repeatedly we have,

$$|f(u_i) - f(u_j)| = 2 \text{ or } n+1 \text{ or } n+3,$$

when $d(u_i, u_j) = 1$ and $d(u_i, u_j) = 2$. By construction of f, for $v_i, v_j \in W_n$, $|f(v_i) - f(v_j)| \geq 2$ when $d(v_i, v_j) = 1$ and $|f(v_i) - f(v_j)| \geq 1$ when $d(v_i, v_j) = 2$.

Since no label of u_is occur in the corresponding W_n, each u_i maintains a minimum label difference 2 with respect to the corresponding W_n and no label of u_is occur in the different W_n, we have, for any two vertices s_i, t_j of $P_m \circ W_n$, $|f(s_i) - f(t_j)| \geq 2$ when $d(s_i, t_j) = 1$ and $|f(s_i) - f(t_j)| \geq 1$ when $d(s_i, t_j) = 2$. Hence f is a distance two labeling.

Since $n+4$ is the maximum label we have used, $\lambda(P_m \circ W_n) \leq n+4$. Since $P_m \circ W_n$ contains three vertices of degree $\Delta = n+2$ such that one of them is adjacent to the other two, we have $\lambda(P_m \circ W_n) \geq \Delta + 2 = n+4$. Hence $\lambda(P_m \circ W_n) = n+4 = \Delta + 2$. Hence the theorem follows in this case also.

Hence the theorem. □

Theorem 3.8. *For any two graphs G_1 and G_2, $\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda'(G_2) + 2$ and the bound is attainable when G_1 and G_2 are complete.*

Proof. Let f_1 be the $L(2,1)$-labeling of G_1 corresponding to $\lambda(G_1)$ and f_2 be the injective $L(2,1)$-labeling of G_2 corresponding to $\lambda'(G_2)$. Set $V(G_1) = \{u_1, u_2, \ldots, u_m\}$, $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ and define a labeling f on $V(G_1 \circ G_2)$:

$$f(u_i) = f_1(u_i)$$
$$f(v_i) = f_2(v_i) + \lambda(G_1) + 2,$$

for all v_i in all copies. Clearly f is a $L(2,1)$-labeling for $G_1 \circ G_2$. Hence

$$\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda'(G_2) + 2.$$

Now let us assume that G_1 and G_2 are complete. Since G_1 is complete on m vertices, any $L(2,1)$-labeling of $G_1 \circ G_2$ needs $2m$ distinct labels for the vertices of G_1 and a different set of $2n$ labels for the vertices of G_2. Since we can use the label zero also,

$$\lambda(G_1 \circ G_2) \geq 2m + 2n - 2 = 2m - 2 + 2n - 2 + 2 = \lambda(G_1) + \lambda'(G_2) + 2.$$

That is, $\lambda(G_1 \circ G_2) = \lambda(G_1) + \lambda'(G_2) + 2$. □

Theorem 3.9. *For any two graphs G_1 and G_2,*

$$\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda(G_2) + m + 2,$$

where m is the multiplicity of the $L(2,1)$-labeling corresponding to $\lambda(G_2)$.

Proof. Let f_1 be the $L(2,1)$-labeling of G_1 corresponding to $\lambda(G_1)$, f_2 be the $L(2,1)$-labeling of G_2 corresponding to $\lambda(G_2)$ and m be the multiplicity of f_2. Let $V(G_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$. If f_2 is injective, then by the above theorem, $\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda(G_2) + 2$ and since $m = 0$ in this case, the theorem is true. Otherwise, we rename the vertices of G_2 as below.

Let $k = \lambda(G_2)$ and let n_i denotes the multiplicity of the label i of f_2. For $i = 0, 1, 2, \ldots, k$ and $j = 0, 1, 2, \ldots, n_i$ let $\{v_{i,j}\}$ denote the set of all vertices of G_2 which receive the colour i in f_2 and these sets form a partition of $V(G_2)$. We note that for some i, this set may be empty. Hence the multiplicity of f_2 is $n_0 + n_1 + \cdots + n_k$.

Define f_2' on $V(G_2)$ as below. For $i = 0, 1, 2, \ldots, k$ and $j = 0, 1, 2, \ldots, n_i$, let

$$f_2'(v_{i,j}) = i + (n_{i-k} + n_{i-(k-1)} + \cdots + n_{i-1}) + j$$

where n_α is zero, when $\alpha < 0$. Since f_2 is an $L(2,1)$-labeling of G_2, f_2' is also an $L(2,1)$-labeling and strictly increasing and $k + (n_0 + n_1 + \cdots + n_{k-1}) + n_k = \lambda(G_2) + n_0 + n_1 + \cdots + n_k$ is its maximum label. Now, we define a new labeling f on $V(G_1 \circ G_2)$ by

$$\begin{aligned} f(u_i) &= f_1(u_i), \quad i = 1, 2, \ldots, m \quad \text{and} \\ f(v_{i,j}) &= f_2'(v_{i,j}) + \lambda(G_1) + 2, \end{aligned}$$

for all $v_{i,j}$ in all copies for $i = 0, 1, 2, \ldots, k$ and $j = 0, 1, 2, \ldots, n_i$. Clearly f is an $L(2,1)$-labeling for $G_1 \circ G_2$ and

$$\lambda(G_1 \circ G_2) \leq \lambda(G_2) + n_0 + n_1 + \cdots + n_k + \lambda(G_1) + 2.$$

Hence

$$\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda(G_2) + m + 2$$

where m is the multiplicity of the $L(2,1)$-labeling corresponding to $\lambda(G_2)$. □

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MUTHALI MURUGAN
School of Science
Tamil Nadu Open University
577, Anna Salai, Chennai - 600 015, India
e-mail: muruganganesan@yahoo.in