LE MATEMATICHE
Vol. LXX (2015) – Fasc. II, pp. 63–77
doi: 10.4418/2015.70.2.5

ON q^2-ANALOGUE SOBOLEV TYPE SPACES

AHMED SAOUDI - AHMED FITOUHI

This paper is devoted to define the q^2-Sobolev type spaces on $\mathbb{R}_q$ by using the q^2-analogue Fourier transform and its inverse. In particular, we provide the readers by some embedding results with these spaces. Moreover we study the related q^2-potential analysis and some of its properties.

1. Introduction

The Sobolev spaces were introduced firstly by Sergei Lvovich Sobolev in 30s of the previous century to search for weak solutions, before being taken by Laurent Schwartz. Their importance comes from the fact that solutions of partial differential equations are naturally found in Sobolev spaces, rather than in spaces of continuous functions and with the derivatives understood in the classical way defined in [6]

$$\forall m \in \mathbb{N}^*, \forall 1 \leq p \leq \infty, \quad \mathcal{W}^{m,p}(\mathbb{R}) = \left\{ u \in L^p(\mathbb{R}), \partial_x^k u \in L^p(\mathbb{R}); 0 \leq k \leq m \right\}.$$

Their use and the study of their properties were facilitated by the theory of distributions and Fourier analysis. The Sobolev space $\mathcal{W}^{s,p}(\mathbb{R})$ is defined in [1] by the use of the classical Fourier transform as the set of all tempered distribution u such that its classical Fourier transform $\mathcal{F}(u)$ satisfying

Entrato in redazione: 8 giugno 2014
AMS 2010 Subject Classification: 33D15, 46E35, 43A32, 33D60.
Keywords: Rubin's operator, q^2-analogue Fourier analysis, q^2-analogue Soblev spaces, q^2-analogue potential spaces.

$$(1+|\xi|^2)^{s/2}\mathcal{F}(u) \in L^p(\mathbb{R}).$$

Generalization of the Sobolev space have been studied by replacing the classical Fourier transform by a generalized one. In the context of differential-differences operators, the generalized Sobolev has already been studied in various settings in [3, 4, 12] and in the context of q-differential-differences operators, has already been studied in [5, 11, 13]. This paper is an attempt to fill this gap by generalizing the Sobolev spaces associated with the q-Rubin's operator introduced in [5].

This paper is organized as follows. In section 2, we present some preliminary results and notations that will be useful in the sequel. In section 3, we recall the main results about the harmonic analysis associated with the q-theory that will be frequently used in this work. In section 4, we introduce and investigate Sobolev spaces associated with q^2-analogue Fourier transform. Finally, we deal with the q^2-potential spaces in section 5.

2. Preliminaries

In this section we give a summary of the mathematical notations and definitions used in the q-theory. We refer the reader to the book of Gasper and Rahmen [7] and the book of Kac and Cheung [10] for the definitions, notations and properties on q-hypergeometric functions. Other recent references in q-theory, we refer the reader to [2].
For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_n = \begin{cases} 1 & \text{if} \quad n=0 \\ \prod_{k=0}^{n-1}(1-aq^k) & \text{if} \quad n \in \mathbb{N} \end{cases} \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \qquad (1)$$

$$(a_1,a_2,\ldots,a_p;q)_n = (a_1;q)_n(a_2;q)_n\ldots(a_P;q)_n, \quad n=0,1,\ldots\infty. \qquad (2)$$

The *q-number* or basic *nubmer* is denoted by

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}. \qquad (3)$$

The *q-number* factorial and the corresponding *q-number* factorial shifted are defined for a non-negative integer n respectively by

$$[n]_q = \prod_{k=1}^{n}[k]_q, \qquad [a]_{q;n} = \prod_{k=0}^{n-1}[a+k]_q. \qquad (4)$$

We denote the q-Gamma function by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{5}$$

For $0 < q < 1$ we denote

$$\mathbb{R}_q = \{\pm q^n; n \in \mathbb{Z}\} \quad \text{and} \quad \tilde{\mathbb{R}}_q = \{\pm q^n; n \in \mathbb{Z}\} \cup \{0\}.$$

The q-Jackson integral is defined in [7] by

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx, \tag{6}$$

where

$$\int_0^a f(x)d_qx = a(1-q)\sum_{n=0}^{+\infty} f(aq^n)q^n, \tag{7}$$

and

$$\int_{-\infty}^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} \{f(q^n)+f(-q^n)\}q^n, \tag{8}$$

provided the sums converge absolutely.
When f is continuous on $[0,a]$, one can show that

$$\lim_{q\to 1}\int_0^a f(x)d_qx = \int_0^a f(x)dx.$$

The q-trigonometric functions are defined as

$$\cos(x;q^2) := \sum_{n=0}^{+\infty}(-1)^n b_{2n}(x;q^2) \tag{9}$$

and

$$\sin(x;q^2) := \sum_{n=0}^{+\infty}(-1)^n b_{2n+1}(x;q^2), \tag{10}$$

where

$$b_n(x;q^2) = \frac{q^{[\frac{n}{2}]([\frac{n}{2}]+1)}}{n!_q}x^n. \tag{11}$$

These two functions introduced the notion of ∂_q-adapted q^2-analogue exponential function given by

$$e(z;q^2) := \cos(-iz;q^2) + i\sin(-iz;q^2) = \sum_{n=0}^{+\infty} b_n(x;q^2), \tag{12}$$

satisfying the following inequality

$$|\exp(ix;q^2)| \leq \frac{2}{(q;q)_\infty}, \quad \forall x \in \mathbb{R}_q. \tag{13}$$

$\exp(z;q^2)$ is absolutely convergent for all z in the plane since both of its component functions are, and we have $\lim\limits_{q\to 1^-} e(z;q^2) = e^z$ point-wise and uniformly on compacts.
The q^2-analogue differential operator is defined by

$$\partial_q f(z) = \begin{cases} \dfrac{f(q^{-1}z)+f(-q^{-1}z)-f(qz)+f(-qz)-2f(-z)}{2(1-q)z} & \text{if} \quad z \neq 0 \\ \\ \lim\limits_{z\to 0} \partial_q f(z) \quad \text{in} \quad \mathbb{R}_q & \text{if} \quad z = 0. \end{cases}$$

3. Harmonic analysis associated with Rubin operator

In the sequel, we denote by

- $\mathcal{C}_q^p(\mathbb{R}_q)$ the space of functions p times q-differentiable on $\tilde{\mathbb{R}}_q$, such that for all $0 \leq k \leq p$, $\partial_q^k f$ is continuous on $\tilde{\mathbb{R}}_q$.
- $\mathcal{D}_q(\mathbb{R}_q)$ the space of functions infinitely q-differentiable on $\mathbb{R}_q$ with compact supports.
- $\mathcal{S}_q(\mathbb{R}_q)$ the q-analogue of Schwartz space of functions defined on $\mathbb{R}_q$ and satisfying

  $$\forall n,m \in \mathbb{N} \quad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q;0\leq k\leq n} \left|(1+x)^m \partial_q^n f(x)\right| < \infty$$

  and

  $$\lim_{x\to 0} \partial_q f(x) \quad (\text{in } \mathbb{R}_q) \quad \text{exists.}$$

  $\mathcal{S}_q(\mathbb{R}_q)$ is equipped with the induced topology defined by the semi-norms $P_{n,m,q}$.
- $\mathcal{S'}_q(\mathbb{R}_q)$ the space of temperate distributions on $\mathbb{R}_q$. It is the topological dual of $\mathcal{S}_q(\mathbb{R}_q)$.
- $L_q^p(\mathbb{R}_q), 1 \leq p < \infty$, the set of all functions defined on $\mathbb{R}_q$ such that

  $$\|f\|_{L_q^p(\mathbb{R}_q)} = \left\{\int_{-\infty}^{\infty} |f(x)|^p d_q x\right\}^{\frac{1}{p}} < \infty. \tag{14}$$

- $L_q^\infty(\mathbb{R}_q)$, the set of all functions defined on $\mathbb{R}_q$ such that

$$\|f\|_{L_q^\infty(\mathbb{R}_q)} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty. \tag{15}$$

- The q^2-analogue Fourier transform introduced by Richard L. Rubin in [15] as

$$\mathcal{F}_q(f)(x) := K\int_{-\infty}^{\infty} f(t)e(-itx;q^2)d_qt, \quad x \in \tilde{\mathbb{R}}_q \tag{16}$$

where

$$K = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})}.$$

Letting $q \uparrow 1$ subject to the condition

$$\frac{\log(1-q)}{\log(q)} \in 2\mathbb{Z}, \tag{17}$$

gives at least formally, the classical Fourier Transform. In the remainder of this paper, we assume that (17) holds.

It was shown in [9, 15] that the q^2-analogue Fourier transform $\mathcal{F}_q$ verifies the following properties:

Lemma 3.1. *1. If $f(u), uf(u) \in L_q^1(\mathbb{R}_q)$, then*

$$\partial_q(\mathcal{F}_q)(f)(x) = \mathcal{F}_q(-iuf(u))(x).$$

2. If $f, \partial_q f \in L_q^1(\mathbb{R})$, then $\mathcal{F}_q(\partial_q(f))(x) = ix\mathcal{F}_q(f)(x)$.

3. $\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ (resp $L^2(\mathbb{R}_q)$) onto itself and we have

$$\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{2,q}, \quad \forall f \in L^2(\mathbb{R}_q)$$

and

$$\forall t \in \mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{\infty} \mathcal{F}_q(f)e(itx;q^2)d_qt.$$

Definition 3.2 (see [15])**.** The q-translation operator $\mathcal{T}_{q,x}$, $x \in \mathbb{R}_q$ is defined on $L_q^1(\mathbb{R}_q)$ by

$$\mathcal{T}_{q,y}(f)(x) = \int_{-\infty}^{\infty} \mathcal{F}_q(f)(t)e(ity;q^2)e(itx;q^2)d_qt, \quad y \in \mathbb{R}_q, \tag{18}$$

$$\mathcal{T}_{q,0}(f)(x) = f(x). \tag{19}$$

It verifies the following properties (see [9, 14, 15]).

Proposition 3.3. *1. For $f,g \in L_q^1(\mathbb{R}_q)$, we have*

$$\mathcal{T}_{q,y}f(x) = \mathcal{T}_{q,x}f(y), \quad \forall x,y \in \mathbb{R}_q.$$

2. $T_{q,y}f$ is an isomorphism for $f \in L_q^2(\mathbb{R}_q)$ onto itself and we have

$$\|\mathcal{T}_{q,y}f\|_{L_q^2(\mathbb{R}_q)} \leq \frac{2}{(q,q)_\infty}\|f\|_{L_q^2(\mathbb{R}_q)}, \quad \forall x \in \tilde{\mathbb{R}}_q.$$

3. Let $f \in L_q^2(\mathbb{R}_q)$, then

$$\mathcal{F}_q(\mathcal{T}_{q,y}f)(\lambda) = e(i\lambda y; q^2)F_q(f)(\lambda).$$

4. If $f, \partial_q f \in (L^1 \cap L^2)d_q$, then

$$\partial_q[\mathcal{T}_{q,y}f](x) = [\mathcal{T}_{q,y}(\partial_q f)](x).$$

Definition 3.4. The q-convolution product is defined by using the q-translation operator for $f \in L_q^2(\mathbb{R}_q)$ and $g \in L_q^1(\mathbb{R}_q)$, as follows

$$f *_q g(y) = K\int_{-\infty}^{\infty} \mathcal{T}_{q,y}f(-x)g(x)d_qx.$$

We have

$$f *_q g = g *_q f,$$
$$\forall f,g \in (L^1 \cap L^2)d_q, \quad \mathcal{F}_q(f *_q g) = \mathcal{F}_q(f)\mathcal{F}_q(g),$$

and

$$\forall f,g \in \mathcal{S}_q(\mathbb{R}_q), \quad f *_q g \in \mathcal{S}_q(\mathbb{R}_q).$$

Definition 3.5 (see[5])**.** The q^2-analogue Fourier transform of a distribution u in $\mathcal{S}_q'(\mathbb{R}_q)$ is defined by

$$\langle \mathcal{F}_q(u), \varphi \rangle = \langle u, \mathcal{F}_q(\varphi) \rangle, \quad \forall \varphi \in \mathcal{S}_q(\mathbb{R}_q). \tag{20}$$

Proposition 3.6 (see[5])**.** *The q^2-analogue Fourier transform is a topological isomorphism from $\mathcal{S}_q'(\mathbb{R}_q)$ onto itself.*

Let τ be in $\mathcal{S}_q'(\mathbb{R}_q)$. We define the distribution $\partial_q\tau$, by

$$\langle \partial_q\tau, \psi \rangle = -\langle \tau, \partial_q\psi \rangle, \quad \forall \psi \in \mathcal{S}_q(\mathbb{R}_q). \tag{21}$$

Hence, if we denote the q^2-analogue Laplace operator $\Delta_q := \partial_q^2$ we deduce

$$\langle \Delta_q\tau, \psi \rangle = \langle \tau, \Delta_q\psi \rangle, \quad \forall \psi \in \mathcal{S}_q(\mathbb{R}_q). \tag{22}$$

These distributions satisfy the following properties

$$\mathcal{F}_q(\partial_q\tau) = iy\mathcal{F}_q(\tau), \tag{23}$$
$$\mathcal{F}_q(\Delta_q^n\tau) = (-1)^n y^{2n}\mathcal{F}_q(\tau), \quad \forall n \in \mathbb{N}. \tag{24}$$

4. q^2-analogue Sobolev type spaces

In this section, we generalize the Sobolev spaces associated with q-Rubin operator introduced in [5] and we establish their main properties.

Definition 4.1. Let $s \in \mathbb{R}$ and $1 \leq p < \infty$, we define the space $\mathcal{W}_q^{s,p}(\mathbb{R}_q)$ as

$$\left\{ u \in \mathcal{S}'_q(\mathbb{R}_q) : (1+|\xi|^2)^{\frac{s}{2}} \mathcal{F}_q(u) \in L_q^p(\mathbb{R}_q) \right\}.$$

We provide this space with the norm

$$\|u\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)} := \left(\int_{\mathbb{R}_q} (1+\xi^2)^{\frac{sp}{2}} |\mathcal{F}_q(u)(\xi)|^2 d q \right)^{\frac{1}{p}}.$$

Remark 4.2. For $p=2$, we use the notation $\mathcal{H}_q^s(\mathbb{R}_q)$ instead of $\mathcal{W}_q^{s,2}(\mathbb{R}_q)$.

Proposition 4.3.
1. *Let $1 \leq p < \infty$. The space $\mathcal{W}_q^{s,p}(\mathbb{R}_q)$ provided with the norm $\|u\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)}$ is a Banach space.*
2. *Let $1 \leq p < \infty$ and s_1, s_2 in $\mathbb{R}$ such that $s_1 > s_2$ then*
$$\mathcal{W}_q^{s_1,p}(\mathbb{R}_q) \hookrightarrow \mathcal{W}_q^{s_2,p}(\mathbb{R}_q).$$
3. *Let $s \in \mathbb{R}$ and $1 \leq p < \infty$, then $\mathcal{S}_q(\mathbb{R}_q) \subset \mathcal{W}_q^{s,p}(\mathbb{R}_q)$.*
4. *For $(s,p) \in \mathbb{R} \times [1,\infty)$ the operator ∂_q is continuous from $\mathcal{W}_q^{s,p}(\mathbb{R}_q)$ into $\mathcal{W}_q^{s-1,p}(\mathbb{R}_q)$ and*
$$\|\partial_q u\|_{\mathcal{W}_q^{s-1,p}(\mathbb{R}_q)} \leq \|u\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)}.$$

Proof. It is clear that $L^p(\mathbb{R},(1+|\xi|^2)^{\frac{sp}{2}} d_q\xi)$ is complete and since $\mathcal{F}_q$ is an isomorphism from $\mathcal{S}'_q(\mathbb{R}_q)$ onto $\mathcal{S}'_q(\mathbb{R}_q), \mathcal{W}_q^{s,p}(\mathbb{R}_q)$ is then a Banach space.

The results 2. and 3. are immediately taken from definition of the generalized Sobolev space. By Lemma 3.1, we deduce 4. □

Proposition 4.4. *Let $1 \leq p < \infty$ and s_1, s, s_2 be three real numbers: $s_1 < s < s_2$. Then, for all $\varepsilon > 0$, there exists a non-negative constant K_ε such that for all u in $\mathcal{W}_q^{s,p}(\mathbb{R}_q)$*

$$\|u\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)} \leq K_\varepsilon \|u\|_{\mathcal{W}_q^{s_1,p}(\mathbb{R}_q)} + \varepsilon \|u\|_{\mathcal{W}_q^{s_2,p}(\mathbb{R}_q)}.$$

Proof. For all $s_1 < s < s_2$, there exists $t \in (0,1)$ such that $s = (1-t)s_1 + t s_2$. Moreover, using Holder inequality it is easy to see

$$\|u\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)} \leq \|u\|_{\mathcal{W}_q^{s_1,p}(\mathbb{R}_q)}^{1-t} \|u\|_{\mathcal{W}_q^{s_2,p}(\mathbb{R}_q)}^{t}.$$

Therefore

$$\begin{aligned}\|u\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)} &\leq \left(\varepsilon^{-\frac{t}{1-t}}\|u\|_{\mathcal{W}_q^{s_1,p}(\mathbb{R}_q)}\right)^{1-t}\left(\varepsilon\|u\|_{\mathcal{W}_q^{s_2,p}(\mathbb{R}_q)}\right)^{t}\\ &\leq \varepsilon^{-\frac{t}{1-t}}\|u\|_{\mathcal{W}_q^{s_1,p}(\mathbb{R}_q)}+\varepsilon\|u\|_{\mathcal{W}_q^{s_2,p}(\mathbb{R}_q)}.\end{aligned}$$

Hence, the proof is completed for $K_\varepsilon = \varepsilon^{-\frac{t}{1-t}}$. □

After that, we will characterize the spaces $\mathcal{W}_q^{s,p}(\mathbb{R}_q)$, for $s = m$, a positive integer.

Proposition 4.5. *Let $m \in \mathbb{N}$, then for $1 \leq p < \infty$*

$$\mathcal{W}_q^{m,p}(\mathbb{R}_q) = \left\{u \in \mathcal{S}_q'(\mathbb{R}_q) : \mathcal{F}_q(\partial_q^j u) \in L^p(\mathbb{R}_q), 0 \leq j \leq m\right\}.$$

Proof. Let $u \in \mathcal{W}_q^{m,p}(\mathbb{R}_q)$. Then using the Lemma 1, we have

$$\mathcal{F}_q(\partial_q u)(\xi) = -i\xi\mathcal{F}_q(u)(\xi),\ u \in \mathcal{S}_q'(\mathbb{R}_q).$$

and

$$\begin{aligned}\forall 0 \leq j \leq m, \int_{\mathbb{R}_q} |\mathcal{F}_q(\partial_q^j u)(\xi)|^p d_q\xi &= \int_{\mathbb{R}_q} |(-i\xi)^j\mathcal{F}_q(u)(\xi)|^p d_q\xi\\ &\leq \int_{\mathbb{R}_q}(1+|\xi|^2)^{\frac{mp}{2}}|\mathcal{F}_q(u)(\xi)|^p d_q\xi < \infty.\end{aligned}$$

Conversely, we assume now that $\mathcal{F}_q(\partial_q^j u) \in L_q^p(\mathbb{R}_q), \forall 0 \leq j \leq m$. It is easy to see that there exists $K > 0$ such that $(1+\xi^2)^{\frac{mp}{2}} \leq \sum_{j=0}^{m}|\xi|^{pj}$.

Then

$$\begin{aligned}\int_{\mathbb{R}_q}(1+|\xi|^2)^{\frac{mp}{2}}|\mathcal{F}_q(u)(\xi)|^p d_q\xi &\leq K\sum_{j=0}^{m}\int_{\mathbb{R}_q}|(-i\xi)^j\mathcal{F}_q(u)(\xi)|^p d_q\xi\\ &= K\sum_{j=0}^{m}\int_{\mathbb{R}_q}|\mathcal{F}_q(\partial_q^j u)(\xi)|^p d_q\xi < \infty.\end{aligned}$$

□

Remark 4.6. 1. For $m \in \mathbb{N}$ and $p = 2$, we find the particular case studied in

$$\mathcal{W}_q^{m,2}(\mathbb{R}_q) = \left\{f \in L_q^2(\mathbb{R}_q) : \partial_q^j f \in L_q^2(\mathbb{R}_q), j = 0, \ldots, m\right\}.$$

2. $\mathcal{H}_q^0(\mathbb{R}_q) = L_q^2(\mathbb{R}_q)$.

Corollary 4.7. *For all $(s,p) \in \mathbb{R} \times [1,+\infty]$, we have $\mathcal{S}_q(\mathbb{R}_q) \subset \mathcal{W}_q^{s,p}(\mathbb{R}_q)$.*

Example 4.8. Let $(s,p) \in \mathbb{R} \times [1,+\infty]$ such that $ps < -1$, then for any x in $\mathbb{R}_q$ the q^2-analogue Dirac distribution δ_x defined by

$$\delta_x(y) = \begin{cases} [(1-q)x]^{-1} & \text{if} \quad x = y, \\ 0 & \text{if} \quad x \neq y \end{cases}$$

belongs to $\mathcal{W}_q^{s,p}(\mathbb{R}_q)$.

In fact, for any $\varphi \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$\langle \mathcal{F}_q(\delta_x), \varphi \rangle = \langle \delta_x, \mathcal{F}_q(\varphi) \rangle = \mathcal{F}_q(\varphi)(x) = K \int_{-\infty}^{\infty} \varphi(t) e(-itx;q^2) d_q t.$$

Hence

$$\mathcal{F}_q(\varphi)(x) = K e(-itx;q^2),$$

thank to (13), we obtain

$$\int_{-\infty}^{+\infty} (1+\xi^2)^{\frac{sp}{2}} |\mathcal{F}_q(u)(\xi)|^p d_q\xi \leq \frac{K^p}{(q,q)_\infty^{\frac{p}{2}}} \int_{-\infty}^{+\infty} (1+\xi^2)^{\frac{sp}{2}} d_q\xi$$

$$\leq +\infty.$$

Proposition 4.9. *For $(s,p) \in \mathbb{R} \times [1,+\infty[$, the map*

$$\begin{aligned} T : (\mathcal{W}_q^{s,p}(\mathbb{R}_q), \|.\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)}) &\longrightarrow (L^p(\mathbb{R}_q), \|.\|_{L^p(\mathbb{R}_q)}) \\ u &\longmapsto \psi(u) = (1+\xi^2)^{\frac{s}{2}} \mathcal{F}_q(u) \end{aligned}$$

is an isometric isomorphism.

Proof. Let u be in $L_q^p(\mathbb{R}_q)$. It's clear that $(1+\xi^2)^{-\frac{s}{2}} u \in \mathcal{S}_q'(\mathbb{R}_q)$ and since the q^2-analogue Fourier transform is an isomorphism from $\mathcal{S}_q'(\mathbb{R}_q)$ onto itself, there exists an unique $v \in \mathcal{S}_q'(\mathbb{R}_q)$ such that

$$\mathcal{F}_q(u) = (1+\xi^2)^{-\frac{s}{2}} u.$$

Hence

$$u = (1+\xi^2)^{\frac{s}{2}} \mathcal{F}_q(u) = \psi(u).$$

□

Proposition 4.10. *Let m be in $\mathbb{N}$. The q^2-Sobolev space $H_q^{-2m}(\mathbb{R}_q)$ is spanned by the*

$$\left\{(\partial_q)^k u; u \in L_q^2(\mathbb{R}_q); 0 \leq k \leq 2m\right\}.$$

Moreover, for all $T \in H_q^{-2m}(\mathbb{R}_q)$ there exists $u \in L_q^2(\mathbb{R}_q)$ such that

$$T = (1-\Delta_q)^m u \quad \textit{with} \quad \Delta_q = \partial_q^2.$$

Proof. We have $L_q^2(\mathbb{R}_q) = H_q^0(\mathbb{R}_q)$, then

$$(-\Delta_q)^k u \in H_q^{-2m}(\mathbb{R}_q), \quad \forall u \in L_q^2(\mathbb{R}_q).$$

Let now $T \in H_q^{-2m}(\mathbb{R}_q)$, then there exists $u \in L_q^2(\mathbb{R}_q)$ such that

$$(1+\xi^2)^{-m}\mathcal{F}_q(T) = \mathcal{F}_q(u)$$

so, that implies

$$\mathcal{F}_q(T) = (1+\xi^2)^m \mathcal{F}_q(u) = \mathcal{F}_q((1-\Delta_q)^m u).$$

□

Applications

1. We consider the q-differential-difference equation

$$P(-\Delta_q)(g) = u, \tag{25}$$

where P a polynomial of degree m.
If $u \in \mathcal{H}_q^s(\mathbb{R}_q)$, the solutions of (25) is in $\mathcal{H}_q^{s+2m}(\mathbb{R}_q)$. In fact, by a simple computation, we deduce from the relation (24) that

$$\mathcal{F}_q(u)(\xi) = \mathcal{F}_q(P(-\Delta_q)(g))(\xi) = P(\xi^2)\mathcal{F}_q(g)(\xi).$$

As $u \in \mathcal{H}_q^s(\mathbb{R}_q)$, we deduce

$$\int_{-\infty}^{+\infty} (1+\xi^2)^s |P(\xi^2)|^2 |\mathcal{F}_q(g)(\xi)|^2 d_q\xi < +\infty.$$

On the other hand, we have

$$(1+\xi^2)^{s+m}|\mathcal{F}_q(g)(\xi)|^2 \sim |\mathcal{F}_q(g)(\xi)|^2 \quad (\xi \longrightarrow 0)$$

and

$$(1+\xi^2)^{s+m}|\mathcal{F}_q(g)(\xi)|^2 \sim (1+\xi^2)^s |P(\xi^2)|^2 |\mathcal{F}_q(g)(\xi)|^2 \quad (\xi \longrightarrow 0).$$

Hence

$$\int_{-\infty}^{+\infty} (1+\xi^2)^{s+m} |\mathcal{F}_q(g)(\xi)|^2 d_q\xi < +\infty.$$

2. Let consider now the following q-differential-difference equation

$$\Delta_q u - \lambda u = f, \tag{26}$$

where λ is a given positive number.
If f is in $\mathcal{H}_q^s(\mathbb{R}_q)$ then $u \in \mathcal{H}_q^{s+2}(\mathbb{R}_q)$. In fact, by applying the q^2-analogue Fourier transform $\mathcal{F}_q$, (26) becomes

$$-(\lambda+\xi^2)\mathcal{F}_q(u)(\xi) = \mathcal{F}_q(f)(\xi).$$

Let f be in $\mathcal{S}'_q(\mathbb{R}_q)$, as $\mathcal{F}_q$ is an involution in $\mathcal{S}'_q(\mathbb{R}_q)$, we have a unique solution $u \in \mathcal{S}'_q(\mathbb{R}_q)$ for this equation given by

$$\mathcal{F}_q(u)(\xi) = -(\lambda+\xi^2)^{-1}\mathcal{F}_q(f)(\xi).$$

Hence, we have

$$(1+\xi^2)^{\frac{s+2}{2}}\mathcal{F}_q(u)(\xi) = -\frac{1+\xi^2}{\lambda+\xi^2}(1+\xi^2)^{\frac{s}{2}}\mathcal{F}_q(f)(\xi).$$

Using the fact that the function $\xi \mapsto \frac{1+\xi^2}{\lambda+\xi^2}$ is bounded, we obtain that

$$\int_0^{+\infty}(1+\xi^2)^{s+2}|\mathcal{F}_q(u)(\xi)|^2 d_q\xi \leq C_{q,\lambda}\|f\|_{\mathcal{H}_q^s},\ \ C_{q,\lambda} = ess\sup_{\xi\in\mathbb{R}_q}|\frac{1+\xi^2}{\lambda+\xi^2}|$$

which gives the result .

5. The q^2-potential spaces

In this section, we generalize and investigate the q^2-potential associated to the q^2-analogue Fourier Transform studied in [5]. As an application, it is shown that for given $T \in \mathcal{B}_q^{s,p}(\mathbb{R}_q)$, the solution of $(1-\Delta_q)^k u = T$, belongs to $\mathcal{B}_q^{s+2k,p}(\mathbb{R}_q)$.

Definition 5.1 (see [5])**.** For $u \in \mathcal{S}'_q(\mathbb{R}_q)$ and $s \in \mathbb{R}$, the q^2-potential operator $\mathcal{P}_q^s$ of order s is defined as follow

$$\begin{aligned}\mathcal{P}_q^s : \mathcal{S}'_q(\mathbb{R}_q) &\longrightarrow \mathcal{S}'_q(\mathbb{R}_q)\\ T &\longmapsto (\mathcal{F}_q)^{-1}((1+\xi^2)^{-\frac{s}{2}}\mathcal{F}_q(T)).\end{aligned}$$

Proposition 5.2 (see [5])**.** *1.* $\mathcal{P}_q^s o \mathcal{P}_q^t = \mathcal{P}_q^{s+t}, \forall s,t \in \mathbb{R}.$

2. $\mathcal{P}_q^0 = id_{\mathcal{S}'_q(\mathbb{R}_q)}$.

3. *For all s in $\mathbb{R}$, the q^2-potential $\mathcal{P}_q^s$ is a topological isomorphism from $\mathcal{S}'_q(\mathbb{R}_q)$ onto itself. Its inverse is given by $\mathcal{P}_q^{-s}$.*

Definition 5.3. For all $(s,p) \in \mathbb{R} \times [1,+\infty[$, we define the q^2-potential space as

$$\mathcal{B}_q^{s,p}(\mathbb{R}_q) = \left\{T \in \mathcal{S}'_q(\mathbb{R}_q), \mathcal{P}_q^{-s}(T) \in L_q^p(\mathbb{R}_q)\right\}.$$

In other words

$$\mathcal{B}_q^{s,p}(\mathbb{R}_q) = \mathcal{P}_q^s(L_q^p(\mathbb{R}_q)).$$

Remark 5.4. For all $(s,p) \in \mathbb{R} \times [1,+\infty[$, one can easily see that $\mathcal{S}_q(\mathbb{R}_q) \subset \mathcal{B}_q^{s,p}(\mathbb{R}_q)$.

Proposition 5.5. *For all $(s,p) \in \mathbb{R} \times [1,+\infty[$, we have the following properties*

1. *The map $T \mapsto \|T\|_{\mathcal{B}_q^{s,p}(\mathbb{R}_q)} = \|\mathcal{P}_q^{-s}(T)\|_{L_q^p}$ defines a norm on $\mathcal{B}_q^{s,p}(\mathbb{R}_q)$.*

2. *The operator $\mathcal{P}_s^q$ is an isometric isomorphism from $L_q^p(\mathbb{R}_q)$ onto $(\mathcal{B}_q^{s,p}(\mathbb{R}_q), \|.\|_{\mathcal{B}_q^{s,p}(\mathbb{R}_q)})$. Its inverse is given by $\mathcal{P}_q^{-s}$.*

3. *$(\mathcal{B}_q^{s,p}(\mathbb{R}_q), \|.\|_{\mathcal{B}_q^{s,p}(\mathbb{R}_q)})$ is a Banach space.*

4. *$\mathcal{P}_q^t(\mathcal{B}_q^{s,p}(\mathbb{R}_q)) = \mathcal{B}_q^{s+t,p}(\mathbb{R}_q)$, for all $s,t \in \mathbb{R}$. Moreover, $\mathcal{P}_q^t$ is an isometric isomorphism from $\mathcal{B}_q^{s,p}(\mathbb{R}_q)$ on $\mathcal{B}_q^{s+t,p}(\mathbb{R}_q)$. Its inverse is given by $\mathcal{P}_q^{-s}$.*

5. *$\mathcal{B}_q^{s,2}(\mathbb{R}_q) = H_q^{\frac{s}{2}}(\mathbb{R}_q)$.*

Proof. One can easily see that the properties $1,2,3,4$ are obvious and the property 5 can be deduced directly from the q-Plancherel theorem. □

Theorem 5.6. *Let $s > 1$, then for all $p \in [1,+\infty[, \mathcal{B}_q^{s,p}(\mathbb{R}_q) \subset L_q^p(\mathbb{R}_q)$ and the canonical injection $\mathcal{B}_q^{s,p}(\mathbb{R}_q) \hookrightarrow L_q^p(\mathbb{R}_q)$ is continuous, that is there exists a positive constant $C_{p,s}$ such that*

$$\|T\|_{L_q^p(\mathbb{R}_q)} \le C_{p,s}\|T\|_{\mathcal{B}_q^{s,p}(\mathbb{R}_q)}, \quad \textit{For all} \quad T \in \mathcal{B}_q^{s,p}(\mathbb{R}_q).$$

Proof. Since the map $\xi \mapsto (1+\xi^2)^{-\frac{s}{2}}$ belongs to $L_q^1(\mathbb{R}_q) \cap L_q^\infty(\mathbb{R}_q)$, then using the inversion theorem for the q^2-analogue Fourier transform property, there exists $k_s \in L_q^1(\mathbb{R}_q)$ such that $\mathcal{F}_q(k_s)(\xi) = (1+\xi^2)^{-\frac{s}{2}}$, so by Lemma 3.1, we have

$$\forall f \in L_q^p(\mathbb{R}_q),\ (1+\xi^2)^{-\frac{s}{2}}\mathcal{F}_q(f)(\xi) = \mathcal{F}_q(k_s)(\xi).\mathcal{F}_q(f)(\xi) = \mathcal{F}_q(k_s *_q f)(\xi).$$

Hence

$$\forall f \in L_q^p(\mathbb{R}_q),\ \mathcal{P}_q^{-s}(f) = (k_s *_q f).$$

Using Lemma 3.1 we find

$$\forall f \in L^p_q(\mathbb{R}_q), \quad \|\mathcal{P}_q^{-s}(f)\|_{L^p_q} \le \|k_s\|_{L^1_q} \|f\|_{L^p_q},$$

or equivalently

$$\forall T \in \mathcal{B}^{s,p}_q(\mathbb{R}_q), \quad \|T\|_{L^p_q} \le C_{p,s} \|T\|_{\mathcal{B}^{s,p}_q}.$$

This gives the result, where $C_{p,s} = \|k_s\|_{L^1_q(\mathbb{R}_q)}$. □

Corollary 5.7. *For any $p \in [1, +\infty[$ and $s, t \in \mathbb{R}$ such that $s \ge t+1$, we have $\mathcal{B}^{s,p}_q(\mathbb{R}_q) \subset \mathcal{B}^{t,p}_q(\mathbb{R}_q)$ and the canonical injection $\mathcal{B}^{s,p}_q(\mathbb{R}_q) \hookrightarrow \mathcal{B}^{t,p}_q(\mathbb{R}_q)$ is continuous. Moreover,*

$$\forall f \in \mathcal{B}^{s,p}_q(\mathbb{R}_q), \quad \|f\|_{\mathcal{B}^{s,p}_q(\mathbb{R}_q)} \le C_{p,s} \|f\|_{\mathcal{B}^{s,p}_q(\mathbb{R}_q)},$$

where $C_{p,s}$ is the positive constant given in the latter theorem.

Proof. Let $f \in \mathcal{B}^{s,p}_q(\mathbb{R}_q)$ so $\mathcal{P}_q^{-t}(f) \in \mathcal{B}^{s-t,p}_q(\mathbb{R}_q)$ and by the Theorem 5.6, we deduce

$$\mathcal{P}_q^{-t}(f) \in L^p_q(\mathbb{R}_q) \quad \text{and} \quad \|\mathcal{P}_q^{-t}(f)\|_{L^p_q(\mathbb{R}_q)} \le C_{p,s} \|f\|_{\mathcal{B}^{s-t,p}_q(\mathbb{R}_q)}.$$

Using now property 4 of Proposition 5.5, we obtain

$$\|\mathcal{P}_q^{-t}(f)\|_{L^p_q(\mathbb{R}_q)} \le C_{p,s} \|f\|_{\mathcal{B}^{s,p}_q(\mathbb{R}_q)}.$$

This shows that f belongs to $\mathcal{B}^{t,p}_q(\mathbb{R}_q)$ and that

$$\|f\|_{\mathcal{B}^{t,p}_q(\mathbb{R}_q)} \le C_{p,s} \|f\|_{\mathcal{B}^{s,p}_q(\mathbb{R}_q)}$$

which achieves the proof. □

Application

We study the regularity in $S'_q(\mathbb{R}_q)$ of the solution of the following q-differential-difference equation

$$(1 - \Delta_q)^k U = T$$

with $k \in \mathbb{N}$ and $T \in S'_q(\mathbb{R}_q)$. After applying the q^2-analogue Fourier transform, this equation becomes

$$(1 + \xi^2)^k \mathcal{F}_q(U) = \mathcal{F}_q(T).$$

Or the q^2-analogue Fourier transform is an involution from $S'_q(\mathbb{R}_q)$ onto itself, so we deduce that the above differential equation has in $S'_q(\mathbb{R}_q)$ a unique solution given by

$$U = (\mathcal{F}_q)^{-1}\left((1+\xi^2)^{-k}\mathcal{F}_q(T)\right).$$

Now, by Proposition 5.5 we deduce that $U \in \mathcal{B}_q^{s+2k,p}(\mathbb{R}_q)$ whenever $T \in \mathcal{B}_q^{s,p}(\mathbb{R}_q)$, and in particular $U \in L_q^2(\mathbb{R}_q)$.
Moreover if

$$T = \sum_{p=0}^{k} \lambda_p(-\Delta_q)^p f_p,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are constants and $f_1, f_2, \ldots, f_k \in L_q^2(\mathbb{R}_q)$ because of in this case $T \in \mathcal{B}_q^{-2k,p}(\mathbb{R}_q)$ by virtue of Proposition 5.5 and consequently $U \in \mathcal{B}_q^{0,p}(\mathbb{R}_q) = L_q^2(\mathbb{R}_q)$.

Acknowledgements

The authors are very grateful to the referee for the constructive and valuable comments and recommendations, and for making us pay attention to certain references.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] K. Aomoto - M. Kita, *Theory of Hypergeometric Functions*, Springer Monographs in Mathematics, Springer-Verlag, Tokyo, 2011.

[3] H. Ben Mohamed - N. Bettaibi - S. H. Jah, *Sobolev Type Spaces Asociated with the Weinstein Operator*, Int. Journal of Math. Analysis 15 (28) (2011), 1353–1373.

[4] N. Ben Salem - A. Ould Ahmed Salem, *Sobolev type spaces associated with Jacobi-Dunkl operator*, Frac. Calc. and Special Funct 7 (1) (2004), 523–548.

[5] N. Bettaibi - M. M. Chaffar - A. Fitouhi, *Sobolev type spaces associated with the q-Rubin's operator*, Le Matematiche 69 (1) (2014), 36–56.

[6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[7] G. Gasper - M. Rahman, *Basic hypergeomertic series*, Encyclopedia of mathematics and its appplication 35, Cambridge Univ. Press, Cambridge, UK, 1990.

[8] A. Fitouhi - N. Bettaibi - K. Mezlini, *On a q-Analogue One-Dimentional Heat Equation*, Bulletin of Math. Analysis and Applications 4 (2) (2012), 145–173.

[9] A. Fitouhi - R. H. Bettaieb, *Wavelet Transforms in the q^2-Analogue Fourier Analysis*, Mathematical Sciences Research Journal 19 (9) (2008), 202–214.

[10] V. Kac - P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

[11] H. Mejjaoli - N. Sraeib, *Generalized sobolev spaces in quantum calculus and applications*, Journal of Inequalities and Special Functions 1 (4) (2012), 43–64.

[12] H. Mejjaoli - K. Trimèche, *Hypoellipticity and hypoanaliticity of the Dunkl Laplacian operator*, Integral Transforms and Special Functions 15 (6) (2004), 523–548.

[13] A. Nemri - B. Selmi, *Sobolev type spaces in quantum calculus*, Journal of Mathematical Analysis and Applications 359 (2) (2009), 588–601.

[14] R. L. Rubin, *A q^2-Analogue Operator for q^2-Analogue Fourier Analysis*, Journal of Mathematical Analysis and Applications 212 (2) (1997), 571–5825.

[15] R. L. Rubin, *Duhamel Solutions of non-Homogeneous q^2-Analogue Wave Equation*, Proceedings of the American Mathematical Society 135 (3) (2007), 777–785.

AHMED SAOUDI
Department of Mathematics
Faculty of Sciences of Tunis
Tunis-El Manar University, Tunisia
e-mail: ahmed.saoudi@ipeim.rnu.tn

AHMED FITOUHI
Department of Mathematics
Faculty of Sciences of Tunis
Tunis-El Manar University, Tunisia
e-mail: ahmed.fitouhi@fst.rnu.tn