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AN UPPER BOUND TO THE SECOND HANKEL DETERMINANT FOR PRE-STARLIKE FUNCTIONS OF ORDER α

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The objective of this paper is to obtain an upper bound to the second Hankel determinant $H_2(2)$ for functions f and its inverse f^{-1} when f belongs to the well known class of pre-starlike functions of order α $(0 \leq \alpha \leq 1)$, using Toeplitz determinants.

1. Introduction

Let A denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For any two analytic functions g and h respectively with their expansions as $g(z) = \sum_{k=0}^{\infty} a_k z^k$ and $h(z) = \sum_{k=0}^{\infty} b_k z^k$, the Hadamard product or convolution of $g(z)$ and $h(z)$ is defined as the power series

$$(g * h)(z) = \sum_{k=0}^{\infty} a_k b_k z^k \tag{2}$$

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The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [13] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{3}$$

This determinant has been considered by several authors in the literature. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali [2] found sharp bounds to the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. In this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{4}$$

Janteng, Halim and Darus [8] have considered the functional $|a_2 a_4 - a_3^2|$ and found a sharp upper bound for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and have shown that $|a_2 a_4 - a_3^2| \leq 1$ and $|a_2 a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors [1, 3, 4, 7, 10, 11, 15, 18].

Motivated by the above mentioned results obtained by different authors in this direction, in this paper, using convolution technique, we seek an upper bound to the non-linear functional $|a_2 a_4 - a_3^2|$ for the functions f and its inverse f^{-1} when f belongs to the class of pre-starlike functions of order α $(0 \leq \alpha < 1)$, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be starlike function of order α $(0 \leq \alpha \leq 1)$, denoted by $f \in ST(\alpha)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad \forall z \in E. \tag{5}$$

It is observed that for $\alpha = 0$, we get $ST(0) = ST$. It follows that $ST(\alpha) \subset ST$, for $(0 \leq \alpha < 1)$, ST(1)= {z} and $ST(\alpha) \subseteq ST(\beta)$, for $\alpha \geq \beta$.

Definition 1.2. A function $f(z) \in A$ is said to be convex function of order α $(0 \leq \alpha \leq 1)$, denoted by $f \in CV(\alpha)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad \forall z \in E. \tag{6}$$

It can be noted that for $\alpha = 0$, we get $CV(0) = CV$. It follows that $CV(\alpha) \subset CV$, for $(0 \leq \alpha < 1)$ and $CV(1) = \{z\}$.

Definition 1.3. A function $f \in A$ is said to be in the class of pre-starlike functions of order α $(0 \leq \alpha < 1)$, denoted by R_α, if and only if

$$f(z) * s_\alpha(z) \in ST(\alpha), \qquad \forall z \in E, \tag{7}$$

where $*$ denotes the convolution of two analytic functions and $s_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$ is the extremal function for the class $ST(\alpha)$.

The class R_α was introduced and studied by Ruscheweyh [14]. Let

$$c(\alpha, n) = \frac{\prod_{k=2}^{n}(k - 2\alpha)}{(n-1)!} \quad \text{for} \quad n = 2, 3, \ldots \tag{8}$$

so that $s_\alpha(z)$ can be written in the form

$$s_\alpha(z) = z + \sum_{n=2}^{\infty} c(\alpha, n) z^n, \tag{9}$$

note that $c(\alpha, n)$ is a decreasing function of α with

$$\lim_{n\to\infty} c(\alpha, n) = \begin{cases} \infty, & \text{if } \alpha < \frac{1}{2}, \\ 1, & \text{if } \alpha = \frac{1}{2}, \\ 0, & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

Ruscheweyh (see [17]) also showed that a necessary and sufficient condition for f to be in R_α is that the functional

$$G(\alpha, z) = \frac{f(z) * \frac{s_\alpha(z)}{(1-z)}}{f(z) * s_\alpha(z)},$$

satisfy $Re\, G(\alpha, z) > \frac{1}{2}$, $\forall z \in E$. Since $s_1(z) = z$, we say that f is pre-starlike function of order 1, if and only if

$$Re \frac{f(z)}{z} > \frac{1}{2}, \qquad \forall z \in E. \tag{10}$$

Note that $R_0 = CV(0)$ and $R_{\frac{1}{2}} = ST(\frac{1}{2})$.
It was shown that $R_\alpha \subset R_\beta$, for $0 \leq \alpha < \beta \leq 1$, which generalizes the well-known result that $CV(0) \subset ST(\frac{1}{2})$.
Some preliminary Lemmas required for proving our results are in the following section.

2. Preliminary Results

Let $\mathscr{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{11}$$

which are regular in the open unit disc E and satisfy $Re\, p(z) > 0$ for any $z \in E$. Here $p(z)$ is called Carathéodory function [5].

Lemma 2.1 ([12, 16]). *If $p \in \mathscr{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2 ([6]). *The power series for p given in (11) converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3 \ldots$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \left(\frac{1+z}{1-z}\right)$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [6] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we obtain

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2] \geq 0,$$

it is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \text{ for some } x, |x| \leq 1; \tag{12}$$

$$\text{and } D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1 c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{13}$$

Simplifying the relations (12) and (13), we get

$$4c_3 = \{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\}, \text{ with } |z| \leq 1. \tag{14}$$

To obtain our results, we refer to the classical method devised by Libera and Zlotkiewicz [9].

3. Main Results

Theorem 3.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_\alpha$ *then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8(1-\alpha)}, \ \textit{for} \quad \left(0 \leq \alpha < \frac{1}{2}\right).$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_\alpha$, from Definition 1.3, we have

$$f(z) * s_\alpha(z) \in ST(\alpha), \qquad \forall z \in E. \tag{15}$$

By the convolution, we have

$$g(z) = f(z) * s_\alpha(z) = \left\{ z + \sum_{n=2}^{\infty} a_n z^n \right\} * \left\{ z + \sum_{n=2}^{\infty} c(\alpha,n) z^n \right\}$$
$$= z + \sum_{n=2}^{\infty} c(\alpha,n) a_n z^n. \tag{16}$$

Since $g(z) \in ST(\alpha)$, from Definition 1.1, there exists an analytic function $p \in \mathscr{P}$ in the open unit disc E with $p(0) = 1$ and $Re\, p(z) > 0$ such that

$$\frac{zg'(z) - \alpha g(z)}{(1-\alpha)g(z)} = p(z) \Leftrightarrow zg'(z) - \alpha g(z) = (1-\alpha)g(z)p(z). \tag{17}$$

Replacing the values of $g(z)$, $g'(z)$ from (16) and $p(z)$ with their equivalent series expressions in (17), we have

$$z\left\{1 + \sum_{n=2}^{\infty} c(\alpha,n) n a_n z^{n-1}\right\} - \alpha \left\{ z + \sum_{n=2}^{\infty} c(\alpha,n) a_n z^n \right\}$$
$$= (1-\alpha) \left\{ z + \sum_{n=2}^{\infty} c(\alpha,n) a_n z^n \right\} \left\{ 1 + \sum_{n=1}^{\infty} c_n z^n \right\}.$$

Upon simplification, we obtain

$$c(2,\alpha)a_2+2c(3,\alpha)a_3z+3c(4,\alpha)a_4z^2+\ldots=(1-\alpha)\times$$
$$\left[c_1+\{c_2+c(2,\alpha)c_1a_2\}z+\{c_3+c(2,\alpha)c_2a_2+c(3,\alpha)c_1a_3\}z^3+\ldots\right]. \quad (18)$$

Equating the coefficients of like powers of z^0, z and z^2 respectively on both sides of (18), after simplifying, we get

$$a_2=\frac{c_1}{2};\ \ a_3=\frac{1}{2(3-2\alpha)}\left\{c_2+(1-\alpha)c_1^2\right\};$$
$$a_4=\frac{1}{4(2-\alpha)(3-2\alpha)}\left\{2c_3+3(1-\alpha)c_1c_2+(1-\alpha)^2c_1^3\right\}. \quad (19)$$

Considering, second Hankel functional $|a_2a_4-a_3^2|$ for the function $f\in R_\alpha$ and substituting the values of a_2,a_3 and a_4 from (19), we have

$$|a_2a_4-a_3^2|=\left|\frac{c_1}{2}\frac{1}{4(2-\alpha)(3-2\alpha)}\left\{2c_3+3(1-\alpha)c_1c_2+(1-\alpha)^2c_1^3\right\}\right.$$
$$\left.-\frac{1}{4(3-2\alpha)^2}\left\{c_2+(1-\alpha)c_1^2\right\}^2\right|.$$

Upon simplification, we obtain

$$|a_2a_4-a_3^2|=\frac{1}{8(2-\alpha)(3-2\alpha)^2}|2(3-2\alpha)c_1c_3+(1-\alpha)(1-2\alpha)c_1^2c_2$$
$$-2(2-\alpha)c_2^2-(1-\alpha)^2c_1^4|.$$

The above expression is equivalent to

$$|a_2a_4-a_3^2|=\frac{1}{8(2-\alpha)(3-2\alpha)^2}\times|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|, \quad (20)$$

where $d_1=2(3-2\alpha); d_2=(1-\alpha)(1-2\alpha); d_3=-2(2-\alpha); d_4=-(1-\alpha)^2$. (21)

Substituting the values of c_2 and c_3 from (12) and (14) respectively from Lemma 2.2 on the right-hand side of (20), we have

$$|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|$$
$$=|d_1c_1\times\frac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\}+$$
$$d_2c_1^2\times\frac{1}{2}\{c_1^2+x(4-c_1^2)\}+d_3\times\frac{1}{4}\{c_1^2+x(4-c_1^2)\}^2+d_4c_1^4|. \quad (22)$$

Using the facts that $|z|<1$ and $|xa+yb| \leq |x||a|+|y||b|$, where x, y, a and b are real numbers, on the right-hand side of the above expression, after simplifying, we get

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \leq |(d_1+2d_2+d_3+4d_4)c_1^4+2d_1c_1(4-c_1^2)+ \\ 2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x|-\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\}(4-c_1^2)|x|^2|. \quad (23)$$

Using the values of d_1, d_2, d_3 and d_4 from the relation (21), upon simplification, we obtain

$$d_1+2d_2+d_3+4d_4=0;\ \ d_1=2(3-2\alpha);\ \ d_1+d_2+d_3=2\alpha^2-5\alpha+3. \quad (24)$$

$$(d_1+d_3)c_1^2+2d_1c_1-4d_3=(2-2\alpha)c_1^2+4(3-2\alpha)c_1+8(2-\alpha). \quad (25)$$

Consider

$$(2-2\alpha)c_1^2+4(3-2\alpha)c_1+8(2-\alpha) \\ =(2-2\alpha)\left\{c_1^2+\frac{4(3-2\alpha)}{(2-2\alpha)}c_1+\frac{8(2-\alpha)}{(2-2\alpha)}\right\} \quad (26)$$

After simplifying, the expression (26) is equivalent to

$$(2-2\alpha)c_1^2+4(3-2\alpha)c_1+8(2-\alpha)=(2-2\alpha) \\ \cdot\left[c_1+\left\{\frac{2(3-2\alpha)}{(2-2\alpha)}+\frac{2}{(2-2\alpha)}\right\}\right]\left[c_1+\left\{\frac{2(3-2\alpha)}{(2-2\alpha)}-\frac{2}{(2-2\alpha)}\right\}\right]. \quad (27)$$

Since $c_1 \in [0,2]$, noting that $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a,b \geq 0$ on the right-hand side of (27), which simplifies to

$$\left\{(2-2\alpha)c_1^2+4(3-2\alpha)c_1+8(2-\alpha)\right\} \\ \geq \left\{(2-2\alpha)c_1^2-4(3-2\alpha)c_1+8(2-\alpha)\right\}. \quad (28)$$

From the relations (25) and (28), we get

$$-\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\} \leq -\left\{(2-2\alpha)c_1^2-4(3-2\alpha)c_1+8(2-\alpha)\right\}. \quad (29)$$

Substituting the calculated values from (24) and (29) on the right-hand side of (23), we have

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \leq |4(3-2\alpha)c_1(4-c_1^2)+ \\ 2(2\alpha^2-5\alpha+3)c_1^2(4-c_1^2)|x| \\ -\left\{(2-2\alpha)c_1^2-4(3-2\alpha)c_1+8(2-\alpha)\right\}(4-c_1^2)|x|^2|.$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right hand side of the above inequality, we obtain

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq [4(3-2\alpha)c(4-c^2)+ \\ 2(2\alpha^2-5\alpha+3)c^2(4-c^2)\mu \\ +\left\{(2-2\alpha)c^2-4(3-2\alpha)c+8(2-\alpha)\right\}(4-c^2)\mu^2] \\ = F(c,\mu), \ \text{for} \ 0 \leq \mu = |x| \leq 1, \quad (30) \end{aligned}$$

where $F(c,\mu) = [4(3-2\alpha)c + 2(2\alpha^2-5\alpha+3)c^2\mu$
$$+\left\{(2-2\alpha)c^2-4(3-2\alpha)c+8(2-\alpha)\right\}\mu^2] \times (4-c^2). \quad (31)$$

Further, we maximize the function $F(c,\mu)$ in the closed region $[0,1]\times[0,2]$. Differentiating $F(c,\mu)$ given in (31) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = [2(2\alpha^2-5\alpha+3)c^2 \\ +2\left\{(2-2\alpha)c^2-4(3-2\alpha)c+8(2-\alpha)\right\}\mu] \times (4-c^2). \quad (32)$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and $0 \leq \alpha < \frac{1}{2}$, from (32), we observe that $\frac{\partial F}{\partial \mu} > 0$, which implies that $F(c,\mu)$ is an increasing function of μ and hence, there exists no point of maximum in the interior of the closed region $[0,1]\times[0,2]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0\leq\mu\leq1} F(c,\mu) = F(c,1) = G(c). \quad (33)$$

Therefore, replacing μ by 1 in (31), upon simplification, we obtain

$$G(c) = -4(2-\alpha)\left\{(1-\alpha)c^4 - 2(1-2\alpha)c^2 - 8\right\}, \quad (34)$$

$$G'(c) = -16(2-\alpha)\left\{(1-\alpha)c^3 - (1-2\alpha)c\right\}, \quad (35)$$

$$G''(c) = -16(2-\alpha)\left\{3(1-\alpha)c^2 - (1-2\alpha)\right\}. \quad (36)$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From (35), we get

$$-16(2-\alpha)c\left\{(1-\alpha)c^2-(1-2\alpha)\right\} = 0. \quad (37)$$

We now discuss the following cases.
Case 1: If $c = 0$, then, from (36), we obtain

$$G''(c) = 16(2-\alpha)(1-2\alpha) > 0, \ \text{for} \ 0 \leq \alpha < \frac{1}{2}.$$

From the second derivative test, $G(c)$ has minimum value at $c = 0$.
Case 2: If $c \neq 0$, then, from (37), we get

$$c^2 = \frac{(1-2\alpha)}{(1-\alpha)} = 2 - \frac{1}{(1-\alpha)}. \tag{38}$$

Substituting the value of c^2 from (38) in (36), which simplifies to

$$G''(c) = -32(2-\alpha)(1-2\alpha) < 0, \text{ for } 0 \leq \alpha < \frac{1}{2}.$$

By the second derivative test, $G(c)$ has maximum value at c, where c^2 given by (38). Substituting the value of c^2 in (34), which simplifies to

$$\max_{0\leq c\leq 2} G(c) = \frac{4(2-\alpha)(3-2\alpha)^2}{(1-\alpha)}. \tag{39}$$

Considering, the maximum value of $G(c)$ only at c^2, from (30) and (39), upon simplification, we obtain

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \frac{(2-\alpha)(3-2\alpha)^2}{(1-\alpha)}. \tag{40}$$

Simplifying the expressions (20) and (40), we obtain

$$|a_2a_4 - a_3^2| \leq \frac{1}{8(1-\alpha)}. \tag{41}$$

This completes the proof of Theorem 3.1. □

Remark 3.2. For the choice of $\alpha = 0$, from (41), we obtain $|a_2a_4 - a_3^2| \leq \frac{1}{8}$. This inequality is sharp and this result coincides with that of Janteng , Halim and Darus [8].

Theorem 3.3. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_\alpha$ $(0 \leq \alpha < 1)$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$*, is the inverse function of* f*, then*

$$|t_2t_4 - t_3^2| \leq \frac{1}{(2-\alpha)(4-3\alpha)}.$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_\alpha$, from the Definition of inverse function of f, we have

$$w = f\left\{f^{-1}(w)\right\}. \tag{42}$$

Using the expression for $f(z)$, the relation (42) is equivalent to

$$w = f\left\{f^{-1}(w)\right\} = f^{-1}(w) + \sum_{n=2}^{\infty} a_n \left\{f^{-1}(w)\right\}^n$$
$$= \left\{f^{-1}(w)\right\} + a_2\left\{f^{-1}(w)\right\}^2 + a_3\left\{f^{-1}(w)\right\}^3 + \dots \quad (43)$$

Using the expression for $f^{-1}(w)$ in (43), we have

$$w = (w + t_2w^2 + t_3w^3 + \dots) + a_2(w + t_2w^2 + t_3w^3 + \dots)^2 +$$
$$a_3(w + t_2w^2 + t_3w^3 + \dots)^3 + a_4(w + t_2w^2 + t_3w^3 + \dots)^4 + \dots$$

Upon simplification, we obtain

$$(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 +$$
$$(t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 + \dots = 0. \quad (44)$$

Equating the coefficients of like powers of w^2, w^3 and w^4 on both sides of (44) respectively, further simplification gives

$$t_2 = -a_2;\ t_3 = -a_3 + 2a_2^2;\ t_4 = -a_4 + 5a_2a_3 - 5a_2^3. \quad (45)$$

Using the values of a_2, a_3 and a_4 from (19) along with (45), upon simplification, we obtain

$$t_2 = -\frac{c_1}{2};\ \ t_3 = -\frac{1}{2(3-2\alpha)}\left\{c_2 - (2-\alpha)c_1^2\right\};$$
$$t_4 = -\frac{1}{8(2-\alpha)(3-2\alpha)}\left\{4c_3 - 2(7-2\alpha)c_1c_2 + (2\alpha^2 - 9\alpha + 12)c_1^3\right\}. \quad (46)$$

Substituting the values of t_2, t_3 and t_4 from (46) in the second Hankel functional $|t_2t_4 - t_3^2|$ for the inverse function of $f \in R_\alpha$, after simplifying, we get

$$|t_2t_4 - t_3^2| = \frac{1}{16(2-\alpha)(3-2\alpha)^2} \times$$
$$|4(3-2\alpha)c_1c_3 + (8\alpha - 10)c_1^2c_2 - 4(2-\alpha)c_2^2 + (4-3\alpha)c_1^4|.$$

The above expression is equivalent to

$$|t_2t_4 - t_3^2| = \frac{1}{16(2-\alpha)(3-2\alpha)^2}|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|, \quad (47)$$

where $d_1 = 4(3-2\alpha)$; $d_2 = (8\alpha - 10)$; $d_3 = -4(2-\alpha)$; $d_4 = (4-3\alpha)$. (48)

Substituting the values of c_2 and c_3 from (12) and (14) respectively from Lemma 2.2 on the right-hand side of (47), applying the same procedure as described in Theorem 3.1, we obtain

$$\begin{aligned} 4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \le |(d_1+2d_2+d_3+4d_4)c_1^4+2d_1c_1(4-c_1^2) \\ +2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x| \\ -\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\}(4-c_1^2)|x|^2|. \quad (49) \end{aligned}$$

Using the values of d_1, d_2, d_3 and d_4 from the relation (48), upon simplification, we obtain

$$d_1+2d_2+d_3+4d_4=0;\ d_1=4(3-2\alpha);\ d_1+d_2+d_3=2(2\alpha-3). \quad (50)$$

$$(d_1+d_3)c_1^2+2d_1c_1-4d_3=4\left\{(1-\alpha)c_1^2+2(3-2\alpha)c_1+4(2-\alpha)\right\}. \quad (51)$$

Since $c_1 \in [0,2]$, using the same procedure as described in Theorem 3.1, we get

$$\begin{aligned} &-\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\} \\ &\qquad \le -4\left\{(1-\alpha)c_1^2-2(3-2\alpha)c_1+4(2-\alpha)\right\}. \quad (52) \end{aligned}$$

Substituting the calculated values from (50) and (52) on the right-hand side of (49), we have

$$\begin{aligned} 4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \le |8(3-2\alpha)c_1(4-c_1^2)+ \\ 4(2\alpha-3)c_1^2(4-c_1^2)|x| \\ -4\left\{(1-\alpha)c_1^2-2(3-2\alpha)c_1+4(2-\alpha)\right\}(4-c_1^2)|x|^2|. \end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, which semplifies to

$$\begin{aligned} &|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \le [2(3-2\alpha)c(4-c^2)+ \\ &(3-2\alpha)c^2(4-c^2)\mu+\left\{(1-\alpha)c^2-2(3-2\alpha)c+4(2-\alpha)\right\}(4-c^2)\mu^2] \\ &\qquad = F(c,\mu), \text{ for } 0 \le \mu = |x| \le 1, \quad (53) \end{aligned}$$

where $F(c,\mu) = [2(3-2\alpha)c+(3-2\alpha)c^2\mu$

$$+\left\{(1-\alpha)c^2-2(3-2\alpha)c+4(2-\alpha)\right\}\mu^2](4-c^2). \quad (54)$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (54) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = [(3-2\alpha)c^2+2\left\{(1-\alpha)c^2-2(3-2\alpha)c+4(2-\alpha)\right\}\mu](4-c^2). \quad (55)$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and $0 \leq \alpha < 1$, from (55), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ and and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. Further, for fixed $c \in [0,2]$, we have

$$\max_{0\leq\mu\leq 1} F(c,\mu) = F(c,1) = G(c). \tag{56}$$

Therefore, from (54) and (56), upon simplification, we obtain

$$G(c) = -(4-3\alpha)c^4 + 8(1-\alpha)c^2 + 16(2-\alpha), \tag{57}$$

$$G'(c) = -4(4-3\alpha)c^3 + 16(1-\alpha)c, \tag{58}$$

$$G''(c) = -12(4-3\alpha)c^2 + 16(1-\alpha). \tag{59}$$

For extreme values of $G(c)$, consider $G'(c) = 0$. From (58), we have

$$-4c\left\{(4-3\alpha)c^2 - 4(1-\alpha)\right\} = 0. \tag{60}$$

We now discuss the following cases.
Case 1: If $c = 0$, then, from (59), we obtain

$$G''(c) = 16(1-\alpha) > 0, \text{ for } 0 \leq \alpha < 1.$$

From the second derivative test, $G(c)$ has minimum value at $c = 0$.
Case 2: If $c \neq 0$, then, from (60), we get

$$c^2 = \frac{4(1-\alpha)}{4-3\alpha} = \frac{4}{3}\left\{1 - \frac{1}{(4-3\alpha)}\right\} > 0, \text{ for } 0 \leq \alpha < 1. \tag{61}$$

Substituting the value of c^2 in (59), which simplifies to

$$G''(c) = -32(1-\alpha) < 0, \text{ for } 0 \leq \alpha < 1.$$

By the second derivative test, $G(c)$ has maximum value at c, where c^2 is given in (61). Using the value of c^2 in (57), after simplifying, we get

$$\max_{0\leq c\leq 2} G(c) = \frac{16(2\alpha-3)^2}{(4-3\alpha)}. \tag{62}$$

Considering, the maximum value of $G(c)$ only at c^2, from (53) and (62), we obtain

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \frac{16(2\alpha-3)^2}{(4-3\alpha)}. \tag{63}$$

Simplifying the relations (47) and (63), we get

$$|t_2t_4 - t_3^2| \leq \frac{1}{(2-\alpha)(4-3\alpha)}. \tag{64}$$

This completes the proof of our Theorem 3.3. □

Remark 3.4. Choosing $\alpha = 0$, we have $R_0 = CV$, for which, from (64), we obtain $|t_2t_4 - t_3^2| \leq \frac{1}{8}$.

Special Remark. For $\alpha = 0$, we have $R_0 = CV$, from Theorems 3.1 and 3.3 we observe that $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ and $|t_2t_4 - t_3^2| \leq \frac{1}{8}$. From this, we conclude that the upper bound to the second Hankel functional for the function f and its inverse is the same, provided $f \in CV$.

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