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FIXED POINTS OF CONTRACTIVE DOMINATED MAPPINGS IN AN ORDERED QUASI-PARTIAL METRIC SPACES

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In this paper, we obtain some fixed point theorems for dominated mappings satisfying locally contractive conditions on a closed set in a left K-sequentially 0-complete ordered quasi-partial metric space and in a right K-sequentially 0-complete ordered quasi-partial metric space, respectively. Our results improve several well-known conventional results.

1. Introduction and Preliminaries

Fixed points results of mappings satisfying certain contractive conditions on the entire domain has been at the centre of vigorous research activity and it has a vide range of applications in different areas such as nonlinear and adoptive control systems, parameterize estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks, (see [16, 18, 27]).

Recently, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering in literature. Ran and Reurings [23] proved an analogue of Banach's fixed point theorem in metric space endowed with a partial order and gave applications to matrix equations. In this

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way, they weakened the usual contraction condition. Subsequently, Nieto et al. [21] extended the result in [23] for nondecreasing mappings and applied it to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions. Thereafter, many work related to fixed point problems have also been considered in partially ordered metric spaces (see [2, 9, 19, 21, 23]).

On the other hand, the notion of a partial metric space was introduced by G. S. Matthews in [17]. In partial metric spaces, the distance of a point from itself may not be zero. Partial metric spaces have applications in theoretical computer science (see [10]). Altun et al. [3], Aydi [4] and Paesano et al. [22] used the idea of partial metric space and partial order and gave some fixed point theorems for contractive condition on ordered partial metric spaces. Recently, Karapınar et al. [12] introduced the concept of quasi-partial metric space. Romaguera [25] has given the idea of 0-complete partial metric space. Nashine et al. [20] used this concept and proved some classical results.

From the application point of view the situation is not yet completely satisfactory because there are many situations in which the mappings are not contractive on the whole space but instead they are contractive on its subsets. However, by imposing a subtle restriction, one can establish the existence of a fixed point of such mappings. Shoaib et al. [26] proved significant results concerning the existence of fixed points of the dominated self-mappings satisfying some contractive conditions on closed ball in a 0-complete quasi-partial metric space. Other results on closed ball can be seen in [5–8, 14]. In this paper, we have obtained fixed point theorems to generalize, extend improve some classical fixed point results in [26]. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point. Our results do not exist yet even in metric spaces. An example shows how this result can be used when the corresponding results cannot.

Consistent with [1, 12, 13, 26], the following definitions and results will be needed in the sequel.

Definition 1.1 ([12, 13]). A quasi-partial metric is a function $q : X \times X \to R^+$ satisfying

(i) if $0 \le q(x,x) = q(x,y) = q(y,y)$, then $x = y$ (equality),

(ii) $q(x,x) \le q(y,x)$ (small self-distances),

(iii) $q(x,x) \le q(x,y)$ (small self-distances),

(iv) $q(x,z) + q(y,y) \le q(x,y) + q(y,z)$ (triangle inequality), for all $x,y,z \in X$.

The pair (X,q) is called a quasi-partial metric space.

Note that, if $q(x,y) = q(y,x)$ for all $x,y \in X$, then (X,q) becomes a partial metric space (X,p). Moreover if $q(x,x) = 0$ for all $x \in X$, then (X,q) and (X,p) become a quasi metric space and a metric space respectively. Also $p_q(x,y) =$

$= \frac{1}{2}[q(x,y)+q(y,x)]$, $x,y \in X$ is a partial metric on X. The function $d_{p_q} : X \times X \to R^+$ defined by $d_{p_q}(x,y) = q(x,y)+q(y,x)-q(x,x)-q(y,y)$ is a (usual) metric on X. The ball $B(x,\varepsilon)$, where $B(x,\varepsilon) = \{y \in X : q(x,y) < \varepsilon + q(x,x)\}$, is an open ball in quasi-partial metric space, for some $x \in X$ and $\varepsilon > 0$. Clearly $\overline{B(x,\varepsilon)} = \overline{\{y \in X : q(x,y) < \varepsilon + q(x,x)\}}$ is a closed subset of X.

Definition 1.2 ([26]). Let (X,q) be a quasi-partial metric space.
(a) A sequence $\{x_n\}$ in (X,q) is called 0-Cauchy if $\lim_{n,m\to\infty} q(x_n,x_m) = 0$ or $\lim_{n,m\to\infty} q(x_m,x_n) = 0$.
(b) A sequence $\{x_n\}$ in (X,q) converges to a point x if $\lim_{n\to\infty} q(x_n,x) = \lim_{n\to\infty} q(x,x_n)$ $= q(x,x) = 0$.
(c) The space (X,q) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $q(x,x) = 0$.

It is easy to see that every 0-Cauchy sequence in (X,q) is Cauchy in (X,d_{p_q}) and if (X,q) is complete, then it is 0-complete but the converse assertions do not hold. For example the space $X = [0,+\infty) \cap Q$ with $q(x,y) = |x-y|+|x|$ is a 0-complete quasi-partial metric space but it is not complete (since $d_{p_q}(x,y) = 2|x-y|$ and (X,d_{p_q}) is not complete).

Lemma 1.3 ([20]). *Every closed subset of a* 0*-complete partial metric space is* 0*-complete.*

Definition 1.4. Let X be a nonempty set. Then $(X,\preceq,q)$ is called an ordered quasi-partial metric space if:
(i) q is a quasi-partial metric on X and (ii) $\preceq$ is a partial order on X.

Definition 1.5. Let $(X,\preceq)$ be a partially ordered set. Then $x,y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 1.6 ([1]). Let $(X,\preceq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \preceq x$ for each x in X.

Example 1.7 ([1]). Let $X = [0,1]$ be endowed with the usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, therefore f is a dominated map.

2. Fixed Points of Dominated Contractive Mapping

Reilly et al. ([24]) introduced the notion of left (right) K-Cauchy sequence and left (right) K-sequentially complete spaces (see also [8, 11]). We use this concept to introduce the following definition.

Definition 2.1. Let (X,q) be a quasi-partial metric space.
(a) A sequence $\{x_n\}$ in (X,q) is called left (right) K-0-Cauchy if $\forall\ n > m$, $\lim\limits_{n,m\to\infty} q(x_m,x_n) = 0$ (respectively $\lim\limits_{n,m\to\infty} q(x_n,x_m) = 0$).
(b) The space (X,q) is called left (right) K-sequentially 0-complete if every left (right) K-0-Cauchy sequence in X converges to a point $x \in X$ such that $q(x,x) = 0$.

One can easily observe that every 0-complete quasi-partial metric space is also left K-sequentially 0-complete quasi-partial metric space but the converse does not hold always. Also, every closed subset of a left K-sequentially 0-complete quasi-partial metric space is a left K-sequentially 0-complete.

Theorem 2.2. *Let $(X,\preceq,q)$ be a left K-sequentially 0-complete ordered quasi-partial metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose that there exists $a,b \in [0,1)$ such that $a+2b < 1$ and*

$$q(Sx,Sy) \le aq(x,y) + b[q(x,Sx) + q(y,Sy)] \tag{1}$$

for all comparable elements x,y in $\overline{B(x_0,r)}$.

$$\text{and } q(x_0,Sx_0) \le (1-k)[r + q(x_0,x_0)], \tag{2}$$

where $k = \dfrac{a+b}{1-b}$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0,r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^*$ and $q(x^*,x^*) = 0$. Also, x^* is unique, if for any two points x,y in $\overline{B(x_0,r)}$ there exists a point $z \in \overline{B(x_0,r)}$ such that $z \preceq x$ and $z \preceq y$ and

$$q(x_0,Sx_0) + q(z,Sz) \le q(x_0,z) + q(Sx_0,Sz) \text{ for all } z \preceq Sx_0. \tag{3}$$

Proof. Consider a Picard sequence $x_{n+1} = Sx_n$ with initial guess x_0. As $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$. We will prove that $x_n \in \overline{B(x_0,r)}$ for all $n \in N$ by mathematical induction. By using inequality (2), we have,

$$\begin{aligned} q(x_0,x_1) &\le (1-k)[r + q(x_0,x_0)] \\ &\le r + q(x_0,x_0). \end{aligned}$$

Therefore $x_1 \in \overline{B(x_0,r)}$. Now let $x_2,\cdots,x_j \in \overline{B(x_0,r)}$ for some $j \in N$. As $x_{n+1} \preceq x_n$, so using inequality (1), we obtain

$$\begin{aligned} q(x_j,x_{j+1}) &= q(Sx_{j-1},Sx_j) \\ &\le a[q(x_{j-1},x_j))] + b[q(x_{j-1},x_j) + q(x_j,x_{j+1})] \\ q(x_j,x_{j+1}) &\le kq(x_{j-1},x_j), \end{aligned}$$

which implies that,

$$q(x_j, x_{j+1}) \leq k^2 q(x_{j-2}, x_{j-1}) \leq \cdots \leq k^j q(x_0, x_1). \tag{4}$$

Now

$$\begin{aligned} q(x_0, x_{j+1}) &\leq q(x_0, x_1) + \cdots + q(x_j, x_{j+1}) - [q(x_1, x_1) + \cdots + q(x_j, x_j)] \\ &\leq q(x_0, x_1)[1 + \cdots + k^{j-1} + k^j], \quad \text{(by (4))} \\ q(x_0, x_{j+1}) &\leq (1-k)[r + q(x_0, x_0)] \frac{(1-k^{j+1})}{1-k}, \quad \text{(by 2)}. \end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$ Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Also $x_{n+1} \preceq x_n$ for all $n \in N$. It implies that,

$$q(x_n, x_{n+1}) \leq k^n q(x_0, x_1) \text{ for all } n \in N.$$

It follows that,

$$\begin{aligned} &q(x_n, x_{n+i}) \\ &\leq q(x_n, x_{n+1}) + \cdots + q(x_{n+i-1}, x_{n+i}) - q(x_{n+1}, x_{n+1}) - \cdots - q(x_{n+i-1}, x_{n+i-1}), \\ &q(x_n, x_{n+i}) \leq k^n q(x_0, x_1)[1 + \cdots + k^{i-2} + k^{i-1}] \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Notice that the sequence $\{x_n\}$ is a left K-0-Cauchy sequence in $(\overline{B(x_0, r)}, q)$. As $\overline{B(x_0, r)}$ is closed so it is left K-sequentially 0-complete. Therefore there exists a point $x^* \in \overline{B(x_0, r)}$ with

$$q(x^*, x^*) = \lim_{n\to\infty} q(x_n, x^*) = \lim_{n\to\infty} q(x^*, x_n) = 0. \tag{5}$$

Now,

$$q(x^*, Sx^*) \leq q(x^*, x_n) + q(Sx_{n-1}, Sx^*) - q(x_n, x_n).$$

On taking limit as $n \to \infty$ and using the fact that $x^* \preceq x_n \preceq x_{n-1}$, when $x_n \to x^*$, we have,

$$\begin{aligned} q(x^*, Sx^*) &\leq \lim_{n\to\infty} [q(x^*, x_n) + aq(x_{n-1}, x^*) + b\{q(x_{n-1}, Sx_{n-1}) + q(x^*, Sx^*)\}] \\ &\leq \lim_{n\to\infty} [q(x^*, x_n) + aq(x_{n-1}, x^*) + b\{k^{n-1} q(x_0, x_1) + q(x^*, Sx^*)\}. \end{aligned}$$

Then by inequality (5), we have,

$$(1-b) q(x^*, Sx^*) \leq 0.$$

Similarly,

$$q(Sx^*, x^*) \leq 0$$

and hence $x^* = Sx^*$. □

Uniqueness: Let y be another point in $\overline{B(x_0,r)}$ such that $y = Sy$. Then ,

$$\begin{aligned} q(y,y) &= q(Sy,Sy) \leq aq(y,y) + b\{q(y,Sy) + q(y,Sy)\} \\ (1-a-2b)q(y,y) &\leq 0. \end{aligned}$$

and hence

$$q(y,y) = 0. \tag{6}$$

Now if $x^* \preceq y$, then,

$$\begin{aligned} q(x^*,y) &= q(Sx^*,Sy) \\ &\leq aq(x^*,y) + b[q(x^*,Sx^*) + q(y,Sy)] \\ (1-a)q(x^*,y) &\leq 0 \ \text{ (by (5) and (6))}. \end{aligned}$$

Similarly $q(y,x^*) \leq 0$. This shows that $x^* = y$. Now if x^* and y are not comparable then there exists a point $z \in X$ which is lower bound of both x^* and y that is $z \preceq x^*$ and $z \preceq y$. Now, we will prove that $S^n z \in \overline{B(x_0,r)}$, by using mathematical induction to apply inequality (1). By assumptions $z \preceq x^* \preceq x_n \cdots \preceq x_0$ and hence

$$\begin{aligned} q(Sx_0,Sz) &\leq aq(x_0,z) + b[q(x_0,x_1) + q(z,Sz)] \\ &\leq aq(x_0,z) + b[q(x_0,z) + q(x_1,Sz)], \quad \text{by (2.3)} \\ q(x_1,Sz) &\leq kq(x_0,z) \end{aligned} \tag{7}$$

Now,

$$\begin{aligned} q(x_0,Sz) &\leq q(x_0,x_1) + q(x_1,Sz) - q(x_1,x_1) \\ &\leq q(x_0,x_1) + kq(x_0,z), \quad \text{by (7)} \\ q(x_0,Sz) &\leq (1-k)[r + q(x_0,x_0)] + k[r + q(x_0,x_0)] = r. \end{aligned}$$

It follows that $Sz \in \overline{B(x_0,r)}$. Let $S^2 z, \ldots, S^j z \in \overline{B(x_0,r)}$ for some $j \in N$. As $S^j z \preceq S^{j-1} z \preceq \cdots \preceq z \preceq x^* \preceq x_n \cdots \preceq x_0$, then,

$$\begin{aligned} q(x_1,S^{j+1}z) &= aq(x_0,S^j z) + b[q(x_0,x_1) + q(S^j z,S^{j+1}z)] \\ &\leq aq(x_0,S^j z) + b[q(x_0,S^j z) + q(x_1,S^{j+1}z)], \text{ (by 3)} \end{aligned}$$

which implies that,

$$q(x_1,S^{j+1}z) \leq kq(x_0,S^j z) \leq k[r + q(x_0,x_0)], \quad (\text{as } S^j z \in \overline{B(x_0,r)}) \tag{8}$$

Now,

$$\begin{aligned} q(x_0,S^{j+1}z) &\leq q(x_0,x_1) + q(x_1,S^{j+1}z) \\ &\leq (1-k)[r + q(x_0,x_0)] + k[r + q(x_0,x_0)] = r \end{aligned}$$

It follows that $S^{j+1}z \in \overline{B(x_0,r)}$ and hence $S^n z \in \overline{B(x_0,r)}$. As $S^n z \preceq S^{n-1}z \preceq \cdots \preceq z$ and so

$$q(S^n z, S^{n+1}z) \leq aq(S^{n-1}z, S^n z) + b[q(S^{n-1}z, S^n z) + q(S^n z, S^{n+1}z)]$$

which implies that,

$$\begin{aligned} q(S^n z, S^{n+1}z) &\leq kq(S^{n-1}z, S^n z) \\ &\leq k^2 q(S^{n-2}z, S^{n-1}z) \leq \cdots \leq k^n q(z, Sz) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{9}$$

Now,

$$\begin{aligned} q(x^*, y) &= q(Sx^*, Sy) \\ &\leq q(Sx^*, S^{n+1}z) + q(S^{n+1}z, Sy) - q(S^{n+1}z, S^{n+1}z) \end{aligned}$$

As $S^{n-1}z \preceq x^*$ and $S^{n-1}z \preceq y$ for all $n \in N$, which further implies $S^{n-1}z \preceq S^n x^*$ and $S^{n-1}z \preceq S^n y$ for all $n \in N$ as $S^n x^* = x^*$ and $S^n y = y$ for all $n \in N$. Then,

$$\begin{aligned} q(x^*, y) \leq\ & aq(x^*, S^n z) + b\{q(x^*, Sx^*) + q(S^n z, S^{n+1}z)\} \\ & + aq(S^n z, y) + b\{q(S^n z, S^{n+1}z) + q(y, Sy)\} \end{aligned}$$

On taking limit as $n \to \infty$ and by using inequalities (6) and (9), we have,

$$\begin{aligned} q(x^*, y) &\leq \lim_{n\to\infty} [aq(x^*, S^n z) + aq(S^n z, y)] \\ &\leq \lim_{n\to\infty} [a^2 q(x^*, S^{n-1}z) + a^2 q(S^{n-1}z, y)] \\ &\ \ \vdots \\ &\leq \lim_{n\to\infty} [a^n q(x^*, Sz) + a^n q(Sz, y)] \longrightarrow 0 \end{aligned}$$

Similarly $q(y, x^*) \leq 0$. Hence $x^* = y$.

Example 2.3. Let $X = [0, +\infty) \cap Q$ be endowed with order, $x \preceq y$ if $q(x,x) \leq q(y,y)$ and let $q : X \times X \to R^+$ be the left K-sequentially 0-complete ordered quasi-partial metric on X defined by $q(x,y) = \max\{y - x, 0\} + x$. Define

$$Sx = \begin{cases} \dfrac{1}{10}x \text{ if } x \in [0,1] \cap Q \\ x - \dfrac{4}{9} \text{ if } x \in (1, \infty) \cap Q \end{cases}$$

Clearly, S is dominated mapping. Take, $a = \dfrac{7}{45}, b = \dfrac{1}{5}$, $x_0 = \dfrac{1}{2}$, $r = \dfrac{1}{2}$, then $\overline{B(x_0,r)} = [0,1] \cap Q$, we have $q(x_0, x_0) = \dfrac{1}{2}$, $k = \dfrac{a+b}{1-b} = \dfrac{4}{9}$ with

$$(1-k)[r + q(x_0, x_0)] = (1 - \frac{4}{9})[\frac{1}{2} + \frac{1}{2}] = \frac{5}{9}$$

and

$$q(x_0, Sx_0) = q(\frac{1}{2}, S(\frac{1}{2})) = q(\frac{1}{2}, \frac{1}{20}) = \frac{1}{2} < \frac{5}{9}.$$

Also if $x, y \in (1, \infty) \cap Q$, then,

$$q(Sx, Sy) = \max\{y - \frac{4}{9} - x + \frac{4}{9}, 0\} + x - \frac{4}{9} = \max\{y - x, 0\} + x - \frac{4}{9}$$

Now if $x = y$, then,

$$x - \frac{4}{9} \geq \frac{5}{9}x$$

Now if $x > y$, then,

$$x - \frac{4}{9} \geq \frac{16}{45}x + \frac{1}{5}y$$

Now if $x < y$, then,

$$y - \frac{4}{9} \geq \frac{16}{45}y + \frac{1}{5}x$$

So the contractive condition does not hold on X in each case.

Now if $x, y \in \overline{B(x_0, r)} \cap Q$, then,

$$q(Sx, Sy) = \max\{\frac{1}{10}y - \frac{1}{10}x, 0\} + \frac{1}{10}x = \frac{1}{10}q(x, y) < \frac{7}{45}q(x, y)$$
$$< aq(x, y) + b[q(x, Sx) + q(y, Sy)]$$

Also,

$$q(x_0, Sx_0) + q(z, Sz) \leq q(x_0, z) + q(Sx_0, Sz) \text{ for all } z \preceq Sx_0.$$

Therefore, all the conditions of Theorem 2.2 are satisfied. Moreover, 0 is the fixed point of S and $q(0, 0) = 0$.

In Theorem 2.2, the condition "for a nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$", the existence of lower bound and the condition (3) are imposed to restrict the condition (1) only for comparable elements. However, the following result relax these restrictions but impose the condition (1) for all elements in $\overline{B(x_0, r)}$.

Theorem 2.4. *Let* (X, q) *be a left K-sequentially* 0*-complete quasi-partial metric space,* $S : X \to X$ *be a self map and* x_0 *be an arbitrary point in* X*. Suppose there exists a and b such that,* $a + 2b < 1$ *with*

$$q(Sx, Sy) \leq aq(x, y) + b[q(x, Sx) + q(y, Sy)],$$

for all elements x, y *in* $\overline{B(x_0, r)}$ *and* $q(x_0, Sx_0) \leq (1 - k)[r + q(x_0, x_0)]$*, then there exists a unique fixed point* x^* *in* $\overline{B(x_0, r)}$ *such that* $x^* = Sx^*$ *and* $q(x^*, x^*) = 0$.

In Theorem 2.2, the condition (2) and (3) are imposed to restrict the condition (1) only for x,y in $\overline{B(x_0,r)}$ and Example 2.3 explains the utility of these restrictions. However, the following result relax the condition (2) and (3) but impose the condition (1) for all comparable elements in the whole space X.

Theorem 2.5. *Let $(X,\preceq,q)$ be a left K-sequentially 0-complete ordered quasi-partial metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose there exists a and b such that, $a+2b<1$ with*

$$q(Sx,Sy) \leq aq(x,y)+b[q(x,Sx)+q(y,Sy)],$$

for all comparable elements x,y in X.
If, for a nonincreasing sequence $\{x_n\}$in X, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in X such that $x^* = Sx^*$ and $q(x^*,x^*) = 0$. Moreover, x^* is unique, if for every pair of elements x,y in X there exists a point $z \in X$ such that $z \preceq x$ and $z \preceq y$.

In Theorem 2.2, the conditions (3) is imposed to obtain unique fixed point of a contractive mapping satisfying conditions (1). However, the following result relax restriction (3) but impose the condition (1) for $b=0$. Also we can replace left K-sequentially 0-complete ordered quasi-partial metric space by 0-complete ordered quasi-partial metric space to obtain Theorem 10 of [26] as a corollary of Theorem 2.2.

Corollary 2.6 ([26])**.** *Let $(X,\preceq,q)$ be a 0-complete ordered quasi-partial metric space, $S:X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose that there exists $a,b \in [0,1)$ such that $a+2b<1$ and*

$$q(Sx,Sy) \leq aq(x,y)$$

for all comparable elements x,y in $\overline{B(x_0,r)}$ and

$$q(x_0,Sx_0) \leq (1-k)[r+q(x_0,x_0)],$$

where $k \in [0,1)$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0,r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^*$ and $q(x^*,x^*) = 0$. Also, x^* is unique, if for any two points x,y in $\overline{B(x_0,r)}$ there exists a point $z \in \overline{B(x_0,r)}$ such that $z \preceq x$ and $z \preceq y$.

Remark 2.7. By taking $a=0$ and 0-complete ordered quasi-partial metric space instead of left K-sequentially 0-complete ordered quasi-partial metric space in Theorem 2.2 and in Theorem 2.5, we can obtain Theorem 15 and Theorem 17 of [26].

A metric version of Theorem 2.2 is given below:

Theorem 2.8. *Let* $(X, \preceq, d)$ *be a complete ordered metric space,* $S : X \to X$ *be a dominated map and* x_0 *be an arbitrary point in* X. *Suppose that there exists* $a, b \in [0,1)$ *such that* $a + 2b < 1$ *and*

$$d(Sx, Sy) \leq ad(x,y) + b[d(x,Sx) + d(y,Sy)]$$

for all comparable elements x, y *in* $\overline{B(x_0, r)}$ *and*

$$d(x_0, Sx_0) \leq (1-k)r,$$

where $k = \dfrac{a+b}{1-b}$. *If, for a nonincreasing sequence* $\{x_n\}$ *in* $\overline{B(x_0, r)}$, $\{x_n\} \to u$ *implies that* $u \preceq x_n$, *then there exists a point* x^* *in* $\overline{B(x_0, r)}$ *such that* $x^* = Sx^*$. *Moreover,* x^* *is unique, if for any two points* x, y *in* $\overline{B(x_0, r)}$ *there exists a point* $z \in \overline{B(x_0, r)}$ *such that* $z \preceq x$ *and* $z \preceq y$ *and*

$$d(x_0, Sx_0) + d(z, Sz) \leq d(x_0, z) + d(Sx_0, Sz) \text{ for all } z \preceq Sx_0.$$

Remark 2.9. The above results can easily be proved in right K-sequentially dislocated quasi metric space.

Remark 2.10. We can obtain the partial metric, quasi-metric and metric version of all theorems which are still not present in the literature.

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