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HERMITE-HADAMARD TYPE INEQUALITIES FOR r-CONVEX FUNCTIONS IN q-CALCULUS

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The aim of this work is to establish the q-analogue of the Hermite-Hadamard's inequalities for convex functions and r-convex functions.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a,\ b \in I$ with $a < b$. The following inequality:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

is known as the Hermite-Hadamard's inequality (see [6] for more details). It was generalized in [11] to an r-convex positive function which is defined on an interval $[a,b]$. A positive function is called r-convex on $[a,b]$, if for all $x,\ y \in [a,b]$ and $t \in [0,1]$,

$$f(tx+(1-t)y) \leq \left(t(f(x))^r + (1-t)(f(y))^r\right)^{\frac{1}{r}},\ r > 0.$$

It is obvious 1-convex functions are classical convex functions. It should be noted that if f is r-convex function, then f^r is convex function.

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For $0 < q < 1$, the q-Jackson integral from 0 to b is defined by [8]

$$\int_0^b f(x)d_qx = (1-q)b\sum_{n=0}^{\infty} f(bq^n)q^n \tag{1}$$

provided the sum converge absolutely.
The q-Jackson integral in a generic interval $[a,b]$ is given by [8]

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2}$$

In [12], the authors presented a Riemann-type q-integral by

$$\begin{aligned}\int_a^b f(x)d_q^R x &= (b-a)(1-q)\sum_{k=0}^{\infty} f(a+(b-a)q^k)q^k \\ &= (b-a)\int_0^1 f((1-t)a+tb)d_qt. \quad (3)\end{aligned}$$

For $x \in \mathbb{C}$, we note $[x]_q = \frac{1-q^x}{1-q}$ and $(x,q)_n = \prod_{k=0}^{n-1}(1-xq^k), \qquad n = 1,2,\ldots,\infty.$
The q-Binomial theorem is given by

$$\sum_{n=0}^{\infty} \frac{(a,q)_n}{(q,q)_n} z^n = \frac{(az,q)_\infty}{(z,q)_\infty}, \qquad |z| < 1. \tag{4}$$

In [10], Pachpatte established new Hadamard-type inequalities for products of convex functions:
Let $f,g:\ [a,b] \to [0,\infty)$ be convex functions on $[a,b] \subset \mathbb{R}$, $a < b$. Then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}\Big(f(a)g(a)+f(b)g(b)\Big)+\frac{1}{6}\Big(f(a)g(b)+f(b)g(a)\Big). \tag{5}$$

In [13], the authors showed the following integral inequalities of Hadamard type for r-convex functions:
Let $f : [a,b] \to [0,\infty)$ be r-convex function on $[a,b]$. Then for $0 < r \le 1$

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{r}{r+1}\left([f(a)]^r + [f(b)]^r\right)^{\frac{1}{r}}.$$

Moreover, if $f,g : [a,b] \to [0,\infty)$ be r-convex and s-convex functions respectively on $[a,b]$. Then for $0 \le r,s \le 2$:

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &\le \frac{1}{2}\frac{r}{r+2}\left([f(a)]^r + [f(b)]^r\right)^{\frac{2}{r}} \\ &+ \frac{1}{2}\frac{s}{s+2}\left([g(a)]^s + [g(b)]^s\right)^{\frac{2}{s}},\end{aligned}$$

another, if f is r-convex and g is s-convex functions. Then the following inequality holds if $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \left(\frac{[f(a)]^r + [f(b)]^r}{2}\right)^{\frac{1}{r}} \left(\frac{[g(a)]^s + [g(b)]^s}{2}\right)^{\frac{1}{s}}.$$

For other Hermite-Hadamard type inequalities, see ([1],[2],[3],[4],[7],[13]) and the references cited therein.

In this paper, we use the Riemann-type q-integral to develop some new results of Hermite-Hadamard inequality for convex functions and r-convex functions.

2. Main Results

Theorem 2.1. *Let $f,g : [a,b] \to [0,\infty)$ be two convex functions on $[a,b] \subset \mathbb{R}$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)d_q^R x \le \left(1 + \frac{1}{[3]_q} - \frac{2}{[2]_q}\right) f(a)g(a) + \frac{1}{[3]_q} f(b)g(b) + \left(\frac{1}{[2]_q} - \frac{1}{[3]_q}\right)\Big(f(a)g(b) + f(b)g(a)\Big).$$

Proof. Since f and g are convex functions on $[a,b]$ and for all $t \in [0,1]$, we have

$$f((1-t)a + tb) \le (1-t)f(a) + tf(b)$$

and

$$g((1-t)a + tb) \le (1-t)g(a) + tg(b).$$

Then

$$f((1-t)a+tb)g((1-t)a+tb) \le (1-t)^2 f(a)g(a) + t^2 f(b)g(b) + t(1-t)\Big(f(a)g(b) + f(b)g(a)\Big).$$

By integrating the inequality on $[0,1]$, we obtain

$$\int_0^1 f((1-t)a+tb)g((1-t)a+tb)d_q t \le f(a)g(a)\int_0^1 (1-t)^2 d_q t + f(b)g(b)\int_0^1 t^2 d_q t + (f(a)g(b) + f(b)g(a)\Big)\int_0^1 t(1-t)d_q t$$

Therefore

$$\frac{1}{b-a}\int_a^b f(x)g(x)d_q^R t \le \left(1 + \frac{1}{[3]_q} - \frac{2}{[2]_q}\right) f(a)g(a) + \frac{1}{[3]_q} f(b)g(b) + \left(\frac{1}{[2]_q} - \frac{1}{[3]_q}\right)\Big(f(a)g(b) + f(b)g(a)\Big).$$

Theorem 2.1 is thus proved. $\square$

Remark 2.2. If $q \to 1$ in Theorem 2.1, we obtain the inequality (5).

We need the following Lemma for Theorem 2.4:

Lemma 2.3. *For $p \geq 1$, the following inequality is valid*

$$\int_0^1 (1-t)^p d_q t \leq \frac{q}{[p+1]_q}.$$

Proof. Let

$$f(t) = (1-t)_q^{(p)} := \frac{(t,q)_\infty}{(tq^p,q)_\infty}; \qquad p \in \mathbb{R}, p \geq 1. \tag{6}$$

The q-derivative of f is given by

$$D_q f(t) = D_q(1-t)_q^{(p)} = -[p]_q (1-qt)_q^{(p-1)}. \tag{7}$$

On the other hand, we have

$$(1-t)^{-p} = 1 + \sum_{k=1}^{\infty} \frac{p(p+1)\cdots(p+k-1)}{k!} t^k, \qquad t \in [0,1], \tag{8}$$

and from the q-Binomial theorem (4)

$$\frac{1}{(1-t)_q^{(p)}} = \frac{(tq^p,q)_\infty}{(t,q)_\infty} = 1 + \sum_{k=1}^{\infty} \frac{(q^p,q)_k}{(q,q)_k} t^k. \tag{9}$$

We consider now the function

$$g_k(u) = \frac{1-u^{p+k-1}}{1-u^k}; \qquad 0 < u < 1.$$

The function g_k is increasing on $]0,1[$ and $\lim_{u \to 1^-} g_k(u) = \dfrac{p+k-1}{k}$, so

$$\frac{1-q^{p+k-1}}{1-q^k} \leq \frac{p+k-1}{k}, \ 0 < q < 1. \tag{10}$$

From the relations (8), (9) and (10), we deduce that

$$(1-t)^p \leq \frac{(t,q)_\infty}{(tq^p,q)_\infty} = (1-t)_q^{(p)}. \tag{11}$$

We integrate both sides of (11) on $[0,1]$, we obtain

$$\int_0^1 (1-t)^p d_q t \leq \int_0^1 (1-t)_q^{(p)} d_q t = \frac{q}{[p+1]_q}.$$

Lemma 2.1 is thus proved.

□

Theorem 2.4. *Let $f:[a,b]\to[0,\infty)$ be r-convex function on $[a,b]$. Then the following inequality holds for $0<r\leq 1$:*

$$\frac{1}{b-a}\int_a^b f(x)d_q^R x \leq \frac{1}{[\frac{1}{r}+1]_q}\Big([qf(a)]^r+[f(b)]^r\Big)^{\frac{1}{r}}.$$

Proof. According to the definition of r-convex, for all $t\in[0,1]$, we have

$$f((1-t)a+tb)\leq\Big((1-t)[f(a)]^r+t[f(b)]^r\Big)^{\frac{1}{r}}.$$

By integrating the inequality on $[0,1]$, we obtain

$$\int_0^1 f((1-t)a+tb)d_qt\leq\int_0^1\Big((1-t)[f(a)]^r+t[f(b)]^r\Big)^{\frac{1}{r}}d_qt.$$

Using Minkowski's inequality, we have

$$\int_0^1\Big((1-t)[f(a)]^r+t[f(b)]^r\Big)^{\frac{1}{r}}d_qt \\ \leq\left(\left(\int_0^1(1-t)^{\frac{1}{r}}d_qt\right)^r[f(a)]^r+\left(\int_0^1 t^{\frac{1}{r}}d_qt\right)^r[f(b)]^r\right)^{\frac{1}{r}},$$

and by Lemma 2.3, we have

$$\left(\int_0^1(1-t)^{\frac{1}{r}}d_qt\right)^r\leq\Big(\frac{q}{[\frac{1}{r}+1]_q}\Big)^r.$$

Thus

$$\frac{1}{b-a}\int_a^b f(x)d_q^R x\leq\left(\left(\frac{q}{[\frac{1}{r}+1]_q}\right)^r[f(a)]^r+\frac{1}{[\frac{1}{r}+1]_q^r}[f(b)]^r\right)^{\frac{1}{r}} \\ \leq\frac{1}{[\frac{1}{r}+1]_q}\left(q^r[f(a)]^r+[f(b)]^r\right)^{\frac{1}{r}}.$$

Theorem 2.4 is thus proved. □

Theorem 2.5. *Let $f,g:[a,b]\to[0,\infty)$ be r-convex and s-convex functions respectively on $[a,b]$. Then the following inequality holds for $0<r,s\leq 2$:*

$$\frac{2}{b-a}\int_a^b f(x)g(x)d_q^R x\leq\Big(\frac{1}{[\frac{2}{r}+1]_q}\Big)\Big([q^{\frac{1}{2}}f(a)]^r+[f(b)]^r\Big)^{\frac{2}{r}} \\ +\Big(\frac{1}{[\frac{2}{s}+1]_q}\Big)\Big([q^{\frac{1}{2}}g(a)]^s+[g(b)]^s\Big)^{\frac{2}{s}}.$$

Proof. From the definition of r-convex function we can write:

$$f((1-t)a+tb) \leq ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}}$$

and

$$g((1-t)a+tb) \leq ((1-t)[g(a)]^s + t[g(b)]^s)^{\frac{1}{s}}$$

for all $t \in [0,1]$ and $r,s > 0$.
Then

$$\begin{aligned} & f((1-t)a+tb)g((1-t)a+tb) \\ & \qquad \leq \Big((1-t)[f(a)]^r + t[f(b)]^r\Big)^{\frac{1}{r}} \Big((1-t)[g(a)]^s + t[g(b)]^s\Big)^{\frac{1}{s}}. \end{aligned}$$

Integrating both sides with respect to t on $[0,1]$, we obtain

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)g(x)d_q^R x \\ & \qquad \leq \int_0^1 ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}} ((1-t)[g(a)]^s + t[g(b)]^s)^{\frac{1}{s}} d_q t. \end{aligned}$$

Using Cauchy's inequality, we have

$$\begin{aligned} & \int_0^1 ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}} ((1-t)[g(a)]^s + t[g(b)]^s)^{\frac{1}{s}} d_q t \\ & \leq \frac{1}{2}\int_0^1 \Big((1-t)[f(a)]^r + t[f(b)]^r\Big)^{\frac{2}{r}} d_q t + \frac{1}{2}\int_0^1 \Big((1-t)[g(a)]^s + t[g(b)]^s\Big)^{\frac{2}{s}} d_q t. \end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned} & \int_0^1 ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{2}{r}} d_q t \\ & \leq \Big(\Big(\int_0^1 (1-t)^{\frac{2}{r}} d_q t\Big)^{\frac{r}{2}} [f(a)]^r + \Big(\int_0^1 t^{\frac{2}{r}} d_q t\Big)^{\frac{r}{2}} [f(b)]^r\Big)^{\frac{2}{r}} \\ & = \Big(\Big(\frac{q}{[\frac{2}{r}+1]_q}\Big)^{\frac{r}{2}} [f(a)]^r + \frac{1}{[\frac{2}{r}+1]_q} [f(b)]^r\Big)^{\frac{2}{r}}. \end{aligned}$$

Similarly we have

$$\begin{aligned} & \int_0^1 ((1-t)[g(a)]^s + t[g(b)]^s)^{\frac{2}{s}} d_q t \\ & \leq \Big(\Big(\int_0^1 (1-t)^{\frac{2}{s}} d_q t\Big)^{\frac{s}{2}} [g(a)]^s + \Big(\int_0^1 t^{\frac{2}{s}} d_q t\Big)^{\frac{s}{2}} [g(b)]^s\Big)^{\frac{2}{s}} \\ & = \Big(\Big(\frac{q}{[\frac{2}{s}+1]_q}\Big)^{\frac{s}{2}} [g(a)]^s + \frac{1}{[\frac{2}{s}+1]_q} [g(b)]^s\Big)^{\frac{2}{s}}. \end{aligned}$$

Thus,

$$\frac{2}{b-a}\int_a^b f(x)g(x)d_q^R x \le \left(\left(\frac{q}{[\frac{2}{r}+1]_q}\right)^{\frac{r}{2}}[f(a)]^r + \frac{1}{[\frac{2}{r}+1]_q}[f(b)]^r\right)^{\frac{2}{r}} + \left(\left(\frac{q}{[\frac{2}{s}+1]_q}\right)^{\frac{s}{2}}[g(a)]^s + \frac{1}{[\frac{2}{s}+1]_q}[g(b)]^s\right)^{\frac{2}{s}}.$$

Theorem 2.5 is thus proved. □

Theorem 2.6. *Let $f,g:[a,b]\to[0,\infty)$ be r-convex and s-convex functions respectively on $[a,b]$. Then the following inequality holds if $r>1$ and $\frac{1}{r}+\frac{1}{s}=1$:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)d_q^R x \le \left(\frac{q[f(a)]^r+[f(b)]^r}{[2]_q}\right)^{\frac{1}{r}}\left(\frac{q[g(a)]^s+[g(b)]^s}{[2]_q}\right)^{\frac{1}{s}}.$$

Proof. Since f is r-convex function and g is s-convex function, we have

$$f((1-t)a+tb)\le((1-t)[f(a)]^r+t[f(b)]^r)^{\frac{1}{r}}$$

and

$$g((1-t)a+tb)\le((1-t)[g(a)]^s+t[g(b)]^s)^{\frac{1}{s}}$$

for all $t\in[0,1]$ and $r,s>0$.
Integrate t over $[0,1]$, we obtain

$$\int_0^1 f((1-t)a+tb)g((1-t)a+tb)d_qt \le \int_0^1((1-t)[f(a)]^r+t[f(b)]^r)^{\frac{1}{r}}((1-t)[g(a)]^s+t[g(b)]^s)^{\frac{1}{s}}\,d_qt.$$

Using q-analogue of Hölder type inequality (see[9]), we have

$$\begin{aligned}&\int_0^1((1-t)[f(a)]^r+t[f(b)]^r)^{\frac{1}{r}}((1-t)[g(a)]^s+t[g(b)]^s)^{\frac{1}{s}}\,d_qt\\ &\le\left(\int_0^1(1-t)[f(a)]^r+t[f(b)]^r d_qt\right)^{\frac{1}{r}}\left(\int_0^1(1-t)[g(a)]^s+t[g(b)]^s d_qt\right)^{\frac{1}{s}}\\ &=\left(\frac{q[f(a)]^r+[f(b)]^r}{[2]_q}\right)^{\frac{1}{r}}\left(\frac{q[g(a)]^s+[g(b)]^s}{[2]_q}\right)^{\frac{1}{s}}.\end{aligned}$$

Thus

$$\frac{1}{b-a}\int_a^b f(t)g(t)d_q^R t \le \left(\frac{q[f(a)]^r+[f(b)]^r}{[2]_q}\right)^{\frac{1}{r}}\left(\frac{q[g(a)]^s+[g(b)]^s}{[2]_q}\right)^{\frac{1}{s}}.$$

Theorem 2.6 is thus proved. □

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