LE MATEMATICHE
Vol. LXX (2015) – Fasc. II, pp. 147–159
doi: 10.4418/2015.70.2.10

INTEGRAL TRANSFORMS OF THE S-FUNCTIONS

RAM KISHORE SAXENA - JITENDRA DAIYA

The object of this paper is to introduce a new special function, which will be called S-function. This function is an extension of the generalized Mittag-Leffler function due to Prabhakar [8], generalized Mittag-Leffler function introduced by Srivastava and Tomovski [15] and M-series given by Sharma and Jain [14], various integral transform of this function such as Euler transform, Laplace transform, Whittaker transform, K-transform are derived. The results obtained are useful in applied problems of science, engineering and technology.

1. Introduction

The k-Pochhammer symbol was introduced in [1] in the form:

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k), \tag{1}$$

$$(x)_{(n+r)q,k} = (x)_{rq,k}(x+qrk)_{nq,k}, \tag{2}$$

where $x \in C,\ k \in R \ and\ n \in N$.

Entrato in redazione: 29 luglio 2014

AMS 2010 Subject Classification: 26A33, 33C60.
Keywords: Euler transform, Laplace transform, Whittaker transform, K-transform, fractional Fourier transform, generalized k-Mittag-Leffler function, generalized Wright function.
Corresponding author Jitendra Daiya.

Proposition 1.1. *Let $\gamma \in C$ and $k, s \in R$, then the following identity holds:*

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1} \Gamma_k\left(\frac{k\gamma}{s}\right) \tag{3}$$

and in particular

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right) \tag{4}$$

Proposition 1.2. *Let $\gamma \in C$, $k, s \in R$ and $n \in N$, then the following identity holds*

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq} \left(\frac{k\gamma}{s}\right)_{nq,k}, \tag{5}$$

and in particular

$$(\gamma)_{nq,k} = k^{nq} \left(\frac{\gamma}{k}\right)_{nq} \tag{6}$$

Note 1.3. For further details of k-Pochammer symbol, k-special functions and fractional Fourier transforms one can refer to the papers by Romero et al [9, 10].

2. The S-Function

Definition 2.1. The S-function introduced by the authors is defined as follows

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_{n\tau,k}}{(b_1)_n \ldots (b_q)_n \Gamma_k(n\alpha+\beta)} \frac{x^n}{n!} \tag{7}$$

$k \in R$; $\alpha, \beta, \gamma, \tau \in C$; $Re(\alpha) > 0$, $a_i\,(i = 1,2,\ldots,p)$, $b_j\,(j = 1,2,\ldots,q)$, $Re(\alpha) > kRe(\tau)$ and $p < q+1$. The Pochhammer symbol $(\lambda)_\mu, (\lambda, \mu \in C)$ with $(1)_n = n!$ for $n \in N$ defined in terms of gamma function as (also see [15, p. 199]).

$$(\lambda)_\mu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1(\mu = 0; x \in C\backslash\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+\mu-1)(\mu = n \in N; x \in C) \end{cases}$$

Special cases

(i) when $p = q = 0$ in equation (7) it reduces to generalized k-Mittag-Leffler function, defined by Saxena et al. [12].

$$\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}(-;-;x) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)} \frac{x^n}{n!} = E^{\gamma,\tau}_{k,\alpha,\beta}(x) \tag{8}$$

where $Re\left(\frac{\alpha}{k} - \tau\right) > p - q$.

(ii) For τ = q, equation (7) yields

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,q,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_{nq,k}}{(b_1)_n \ldots (b_q)_n \Gamma_k(n\alpha+\beta)} \frac{x^n}{n!} \tag{9}$$

where $Re(\alpha) > kp$.

(iii) Similarly for $\tau = 1$, equation (7) yields

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,1,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n(\gamma)_{n,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(n\alpha+\beta)}\frac{x^n}{n!}$$
$$= \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) \quad (10)$$

where $Re(\alpha) > kp$.

(iv) When k = 1, equation (10) yields K-function, defined by Sharma [13]

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,1,1)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n(\gamma)_n}{(b_1)_n\ldots(b_q)_n\Gamma(n\alpha+\beta)}\frac{x^n}{n!}$$
$$= \underset{(p,q)}{\overset{(\alpha,\beta,\gamma)}{K}}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) \quad (11)$$

where $Re(\alpha) > p-q$

(v) If we set $\gamma = 1$, in equation (11), it reduces to the generalized M-Series defined by Sharma and Jain [14]

$$\underset{(p,q)}{\overset{(\alpha,\beta,1,1,1)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n x^n}{(b_1)_n\ldots(b_q)_n\Gamma(n\alpha+\beta)}$$
$$= \underset{(p,q)}{\overset{(\alpha,\beta)}{M}}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) \quad (12)$$

where $Re(\alpha) > p-q-1$.

(vi) When $p = q = 0$ in equation (12), it reduces to generalized Mittag-Leffler function, defined by Mittag-Leffler [7]

$$\underset{(0,0)}{\overset{(\alpha,\beta,1,1,1)}{S}}(-;-;x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(x) \quad (13)$$

where $\alpha,\beta \in C; Re(\alpha) > 0, Re(\beta) > 0$.
If we take $\beta = 1$, equation (13) reduces to the Mittag-Leffler function defined in [6].
The following results are well-known

$$\int_0^{\infty} e^{-pz}z^{e-1}dz = \frac{\Gamma(e)}{pe}, Re(e) > 0, Re(p) > 0 \quad (14)$$

and

$$\int_0^\infty z^{\alpha-1}(1-z)^{\beta-1}dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, Re(\alpha) > 0, Re(\beta) > 0. \tag{15}$$

The following results are also needed in the analysis that follows:

Definition 2.2 (Euler Transform [2])**.** The Euler transform of a function $f(z)$ is defined as

$$B\{f(z);a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz \quad a,b \in C, Re(a) > 0, Re(b) > 0 \tag{16}$$

Definition 2.3 (Laplace Transform [2])**.** The Laplace transform of a function $f(t)$, denoted by $F(s)$, is defined by the equation

$$F(s) = (Lf)(s) = L\{f(t);s\} = \int_0^\infty e^{-st}f(t)dt \quad Re(s) > 0 \tag{17}$$

provided the integral (17) is convergent and that the function $f(t)$ is continuous for $t \geq 0$ and of exponential order as $t \to \infty$, (17) may be symbolically written as

$$F(s) = L\{f(t);s\} \text{ or } f(t) = L^{-1}\{F(s);t\} \tag{18}$$

Definition 2.4 (Whittaker Transform)**.**

$$\int_0^\infty e^{-1/2}t^{\upsilon-1}W_{\lambda,\mu}(t)dt = \frac{\Gamma(1/2+\mu+\upsilon)\Gamma(1/2-\mu+\upsilon)}{\Gamma(1-\lambda+\upsilon)} \tag{19}$$

where $Re(\upsilon \pm \mu) > -1/2$ and the Whittaker function $W_{\lambda,\mu}(z)$ is defined in [3] (also see Mathai et al. [5]).

$$W_{\mu,\upsilon}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\lambda-\mu\right)}M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\lambda\right)}M_{\lambda,-\mu}(z) \tag{20}$$

where $M_{\lambda,\mu}(z)$is defined by

$$M_{\lambda,\mu}(z) = z^{1/2+\mu}e^{-1/2z}{}_1F_1\left(\frac{1}{2}+\mu-\lambda;2\mu+1;z\right). \tag{21}$$

Definition 2.5 (K-Transform)**.** This transform is defined by the following integral equation [3]

$$\Re_\upsilon[f(x);p] = g[p;\upsilon] = \int_0^\infty (px)^{1/2}K_\upsilon(px)f(x)dx \tag{22}$$

where $Re(p) > 0; K_\upsilon(x)$ is the Bessel function of the second kind defined by [3, p. 332]

$$K_\upsilon(z) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,\upsilon}(2z)$$

where $W_{0,\upsilon}(\cdot)$ is the Whittaker function defined in (20)

The following result given in Mathai et al. [5, p. 54, Eq. 2.37] will be used in evaluating the integrals:

$$\int_0^\infty t^{\rho-1}K_\upsilon(ax)dx = 2^{\rho-2}a^{-\rho}\Gamma\left(\frac{\rho\pm\upsilon}{2}\right); Re(a)>0; Re(\rho\pm\upsilon)>0. \quad (23)$$

Definition 2.6. Let $u=u(t)$ be a function of the space$S(R)$, the Schwartzian space of the function that decay rapidly at infinity together with all derivatives. The Fourier transform is defined by the integral

$$\hat{u}(\omega) = \Im[u](\omega) = \int_R U(t)\exp(i\omega t)dt \quad (24)$$

and the inverse Fourier transform can be defined by

$$\Im^{-1}[\hat{u}](t) = \frac{1}{2\pi}\int_R \hat{u}(\omega)\exp(-i\omega t)d\omega. \quad (25)$$

Definition 2.7 (Lizorkin space)**.** Let $V(R)$ be the set of functions

$$V(R) = \left\{\upsilon \in S(R) : \upsilon^{(n)}(0) = 0,\ n = 0,1,2,\dots\right\}. \quad (26)$$

The Lizorkin space of function $\phi(R)$ is defined as

$$\phi(R) = \{\varphi \in S(R) : \Im[\varphi] \in V(R)\}. \quad (27)$$

Definition 2.8. Let u be a function belonging to $\phi(R)$. The Fractional Fourier transform of the order $\alpha, 0<\alpha\leq 1$ is defined by

$$\hat{u}_\alpha(\omega) = \Im_\alpha[u](\omega) = \int_R e^{i\omega^{1/\alpha}t}u(t)dt \quad (28)$$

If put $\alpha=1$, equation (28) reduces to the conventional Fourier transform and for $\omega>0$, it reduces to the Fractional Fourier Transform defined by Luchko et al. [4].

Lemma 2.9. *Let u be a function of the space $\phi(R)$, let α ($0<\alpha\leq 1$) be a real number, then*

$$\Im_\alpha[u](\omega) = \Im[u](x), for x = \omega^{1/\alpha} \quad (29)$$

The inverse Fractional Fourier transform of the order $\alpha, 0<\alpha\leq 1, u\in\phi(R)$ is defined as

$$\Im_\alpha^{-1}\{\hat{u}_\alpha(\omega)\}(t) = \frac{1}{2\pi\alpha}\int_R e^{-i\omega^{1/\alpha}t}\hat{u}_\alpha(\omega)\omega^{\frac{1-\alpha}{\alpha}}d\omega \quad (30)$$

The generalized Wright hypergeometric function ${}_p\Psi_q(z)$ *is defined by Wright [16–18] in the following form:*

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array};Z\right]=\sum_{n=0}^{\infty}\frac{\left[\prod_{i=1}^{p}\Gamma(a_i+A_in)\right]}{\left[\prod_{j=1}^{q}\Gamma\left(b_j+B_jn\right)\right]}\frac{z^n}{n!} \tag{31}$$

where $a_i,b_j\in C$ *and* $A_i,B_j\in R\,(i=1,\ldots,p;j=1,\ldots,q)$ *and the defining series* (31) *converges for*

$$\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i>-1.$$

This paper deals with the evaluation of the Euler transform, Laplace transform, Whittaker transform, K-transform and fractional Fourier transforms of the S-function defined by (7). The results obtained in this paper are the generalizations of the results given by Saxena [11] and Saxena et al. [12] and others.

Theorem 2.10 (Euler Transform)**.** *If* $k\in R$; $\alpha,\beta,\gamma,\eta,\delta\in C$; $Re(\alpha)>0$, $Re(\beta)>0$, $a_i\,(i=1,2,\ldots,p)$, $b_j\,(j=1,2,\ldots,q)$ *and* $\tau\in C$, *then*

$$\int_0^1 z^{\eta-1}(1-z)^{\delta-1}\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;xz^{\sigma})\,dz$$

$$=\frac{k^{1-\frac{\beta}{k}}\Gamma(\delta)}{\Gamma\left(\frac{\gamma}{k}\right)}{}_{p+2}\Psi_{q+2}\left[\begin{array}{c}a_1\ldots a_p;\left(\frac{\gamma}{k},\tau\right),(\eta,\sigma)\\ b_1\ldots b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\eta+\delta,\sigma)\end{array};k^{\tau-\frac{\alpha}{k}}x\right] \tag{32}$$

where $Re(\eta)>0, Re(\delta)>0, \sigma>0, Re(\alpha)>kRe(\tau)$.

Proof. Using equation (7) and (16), it gives

$$\int_0^1 z^{\eta-1}(1-z)^{\delta-1}\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;xz^{\sigma})\,dz$$

$$=\int_0^1 z^{\eta-1}(1-z)^{\delta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_{n\tau,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k\left[n\alpha+\beta\right]}\frac{(xz^{\sigma})^n}{n!}dz$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_{n\tau,k}x^n}{(b_1)_n\ldots(b_q)_n\Gamma_k\left[n\alpha+\beta\right]n!}\int_0^1 z^{\sigma n+\eta-1}(1-z)^{\delta-1}dz$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_{n\tau,k}x^n}{(b_1)_n\ldots(b_q)_n\Gamma_k\left[n\alpha+\beta\right]n!}\frac{\Gamma(\sigma n+\eta)\Gamma(\delta)}{\Gamma(\sigma n+\eta+\delta)}$$

Using equation (4) and (6), it becomes

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_nx^nk^{n\tau}\left(\frac{\gamma}{k}\right)_{n\tau}}{(b_1)_n\ldots(b_q)_n(k)^{\frac{n\alpha+\beta}{k}-1}\Gamma\left[\frac{n\alpha+\beta}{k}\right]n!}\frac{\Gamma(\sigma n+\eta)\Gamma(\delta)}{\Gamma(\sigma n+\eta+\delta)}$$

This completes the proof of the Theorem. □

Corollary 2.11. *For $\tau = q$, equation* (32) *reduces in the following form*

$$\int_0^1 z^{\eta-1}(1-z)^{\delta-1} \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,q,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;xz^{\sigma})\,dz$$

$$= \frac{k^{1-\frac{\beta}{k}}\Gamma(\delta)}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_{p+2}\Psi_{q+2}\left[\begin{array}{l} a_1\ldots a_p;\left(\frac{\gamma}{k},q\right),(\eta,\sigma) \\ b_1\ldots b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\eta+\delta,\sigma)\end{array};k^{q-\frac{\alpha}{k}}x\right] \tag{33}$$

Corollary 2.12. *When $p = q = 0$ equation* (32) *reduces to generalized k-Mittag-Leffler function defined by Saxena et al. [12]*

$$\int_0^1 z^{\eta-1}(1-z)^{\delta-1}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^{\sigma})\,dz$$

$$= \frac{k^{1-\frac{\beta}{k}}\Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_2\Psi_2\left[\begin{array}{l} \left(\frac{\gamma}{k},\tau\right),(\eta,\sigma) \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),(\eta+\delta,\sigma)\end{array};k^{\tau-\frac{\alpha}{k}}x\right] \tag{34}$$

Corollary 2.13. *For $\tau = k = 1$, equation* (34) *reduces in the following form given by Saxena [11, p. 79, Eq. 4.6]*

$$\int_0^1 z^{\eta-1}(1-z)^{\delta-1}E_{\alpha,\beta}^{\gamma}(xz^{\sigma})\,dz = \frac{\Gamma(\delta)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{l} (\gamma,1),(\eta,\sigma) \\ (\beta,\alpha),(\eta+\delta,\sigma)\end{array};x\right] \tag{35}$$

where $Re(\eta) > 0$, $Re(\delta) > 0$, $\sigma > 0$.

Theorem 2.14 (Laplace Transform)**.** *If $k \in R$; $\alpha,\beta,\gamma,\eta \in C$; $Re(\alpha) > 0$, $Re(\beta) > 0$, $a_i\,(i = 1,2,\ldots,p)$, $b_j\,(j = 1,2,\ldots,q)$ and $\tau \in C$, $\left|\frac{x}{s^{\sigma}}\right| < 1$, then*

$$\int_0^\infty z^{\eta-1}e^{-sz} \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;xz^{\sigma})\,dz$$

$$= \frac{(k)^{1-\frac{\beta}{k}}s^{-\eta}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_{p+2}\Psi_{q+1}\left[\begin{array}{l} a_1\ldots a_p;\left(\frac{\gamma}{k},\tau\right),(\eta,\sigma) \\ b_1\ldots b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right),\end{array};\frac{k^{\tau-\frac{\alpha}{k}}x}{s^{\sigma}}\right] \tag{36}$$

where $Re(\eta) > 0$, $Re(\delta) > 0$, $\sigma > 0$, $Re(\alpha) > kRe(\tau)$.

The proof of (36) can be developed on similar lines if we use the result Laplace integral (14) and beta formula (15).

Corollary 2.15. *For $\tau = q$ equation* (36) *reduces in the following form*

$$\int_0^\infty z^{\eta-1}e^{-sz} \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,q,k)}{S}}(a_1,\ldots,a_p;b_1,\ldots,b_q;xz^{\sigma})\,dz$$

$$= \frac{(k)^{1-\frac{\beta}{k}}s^{-\eta}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_{p+2}\Psi_{q+1}\left[\begin{array}{l} a_1\ldots a_p;\left(\frac{\gamma}{k},q\right),(\eta,\sigma) \\ b_1\ldots b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right),\end{array};\frac{k^{q-\frac{\alpha}{k}}x}{s^{\sigma}}\right] \tag{37}$$

Corollary 2.16. *When $p = q = 0$, equation (36) reduces to the generalized k-Mittag-Leffler function defined by Saxena et al. [12]*

$$\int_0^\infty z^{\eta-1} e^{-sz} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)\, dz$$

$$= \frac{(k)^{1-\frac{\beta}{k}} s^{-\eta}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_1 \left[\begin{array}{l} \left(\frac{\gamma}{k}, \tau\right), (\eta, \sigma) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), \end{array} ; \frac{k^{\tau-\frac{\alpha}{k}} x}{s^\sigma} \right] \tag{38}$$

Corollary 2.17. *For $\tau = k = 1$, equation (38) reduces in the following form given by Saxena [11, p. 79, Eq. 4.4]*

$$\int_0^\infty z^{\eta-1} e^{-sz} E_{\alpha,\beta}^{\gamma}(xz^\sigma)\, dz = \frac{s^{-\eta}}{\Gamma(\gamma)} {}_2\Psi_1 \left[\begin{array}{l} (\gamma, 1), (\eta, \sigma) \\ (0,1), (\beta, \alpha), \end{array} ; \frac{x}{s^\sigma} \right] \tag{39}$$

where $Re(\eta) > 0, Re(\delta) > 0, \sigma > 0$.

Theorem 2.18 (Whittaker Transform)**.** *If $k \in R$; $\alpha, \beta, \gamma \in C$; $Re(\rho) > 0$, $Re(\alpha) > 0$, $Re(\rho \pm \mu) > -1/2$ and $\tau \in C$ then*

$$\int_0^\infty t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} \left(a_1, \ldots, a_p; b_1, \ldots, b_q; wt^\delta\right) dt$$

$$= \frac{k^{1-\frac{\beta}{k}} p^{-p}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_{p+3}\Psi_{q+2} \left[\begin{array}{l} a_1 \ldots a_p; \left(\frac{\gamma}{k}, \tau\right), (1/2 \pm \mu + \rho, \delta) \\ b_1 \ldots b_q; \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (1 - \lambda + \rho, \delta) \end{array} ; \frac{k^{\tau-\frac{\alpha}{k}} w}{p^\delta} \right] \tag{40}$$

where $Re(e) > |Re(\mu)| - \frac{1}{2}$, $Re(p) > 0$, $\left|\frac{k^{\tau-\frac{\alpha}{k}} w}{p^\delta}\right| < 1$.

The result (40) can be established in the same way if we use the integral (19) instead of (14).

Corollary 2.19. *For $\tau = q$, equation (40) reduces in the following form*

$$\int_0^\infty t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,q,k)}{S}} \left(a_1, \ldots, a_p; b_1, \ldots, b_q; wt^\delta\right) dt$$

$$= \frac{k^{1-\frac{\beta}{k}} p^{-p}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_{p+3}\Psi_{q+2} \left[\begin{array}{l} a_1 \ldots a_p; \left(\frac{\gamma}{k}, q\right), (1/2 \pm \mu + \rho, \delta) \\ b_1 \ldots b_q; \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (1 - \lambda + \rho, \delta) \end{array} ; \frac{k^{q-\frac{\alpha}{k}} w}{p^\delta} \right] \tag{41}$$

where $Re\left(\rho + \mu + \frac{1}{2} + 1\right) > 0$, $Re(p) > 0$, $\left|\frac{k^{q-\frac{\alpha}{k}}}{p^\delta}\right| < 1$.

Corollary 2.20. *When $p = q = 0$, equation* (40) *reduces to the generalized k-Mittag-Leffler function defined by Saxena et al. [12].*

$$\int_0^\infty t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta}^{\gamma,\tau}\left(wt^\delta\right) dt$$

$$= \frac{k^{1-\frac{\beta}{k}} p^{-p}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_3\Psi_2 \left[\begin{array}{c} \left(\frac{\gamma}{k}, \tau\right), (1/2 \pm \mu + \rho, \delta) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (1 - \lambda + \rho, \delta) \end{array} ; \frac{k^{\tau - \frac{\alpha}{k}} w}{p^\delta} \right] \tag{42}$$

where $Re(e) > |Re(\mu)| - \frac{1}{2}$, $Re(p) > 0$, $\left|\frac{k^{\tau-\frac{\alpha}{k}}}{p^\delta}\right| < 1$.

Corollary 2.21. *For $\tau = k = 1$, equation* (42) *reduces in the following form given by Saxena [11, p. 79, Eq. 4.2]*

$$\int_0^\infty t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{\alpha,\beta}^{\gamma}\left(wt^\delta\right) dt$$

$$= \frac{p^{-p}}{\Gamma(\gamma)} {}_3\Psi_2 \left[\begin{array}{c} (\gamma, 1), (1/2 \pm \mu + \rho, \delta) \\ (\beta, \alpha), (1 - \lambda + \rho, \delta) \end{array} ; \frac{w}{p^\delta} \right] \tag{43}$$

where $Re(\rho) > |Re(\mu)| - \frac{1}{2}$, $Re(p) > 0$, $\left|\frac{w}{p^\delta}\right| < 1$.

Theorem 2.22 (k-Transform)**.** *If $k \in R$; $\alpha, \beta, \gamma, \delta, \rho \in C$; $Re(\alpha) > 0$, $Re(\beta) > 0$, $a_i \,(i = 1, 2, \ldots, p)$, $b_j \,(j = 1, 2, \ldots, q)$ and $\tau \in C$, then*

$$\int_0^\infty t^{\rho-1} K_\upsilon(at) \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} \left(a_1, \ldots, a_p; b_1, \ldots, b_q; xt^{2\delta}\right) dt$$

$$= \frac{2^{\rho-2} a^{-\rho} k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_{p+3}\Psi_{q+1} \left[\begin{array}{l} a_1 \ldots a_p; \left(\frac{\gamma}{k}, \tau\right), \left(\frac{\rho \pm \upsilon}{2}, \delta\right) \\ b_1 \ldots b_q; \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), \end{array} ; \frac{4x}{a^2} \right] \tag{44}$$

where $Re(\rho \pm \upsilon) > 0$, $Re(a) > 0$, $Re(\alpha) > kRe(\tau)$.

Equation (44) can be proved in a similar manner if we use the instead of (23).

Corollary 2.23. *For $\tau = q$ equation* (44) *reduces in the following form*

$$\int_0^\infty t^{\rho-1} K_\upsilon(at) \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,q,k)}{S}} \left(a_1, \ldots, a_p; b_1, \ldots, b_q; xt^{2\delta}\right) dt$$

$$= \frac{2^{\rho-2} a^{-\rho} k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_{p+3}\Psi_{q+1} \left[\begin{array}{l} a_1 \ldots a_p; \left(\frac{\gamma}{k}, q\right), \left(\frac{\rho \pm \upsilon}{2}, \delta\right) \\ b_1 \ldots b_q; \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), \end{array} ; \frac{4x}{a^2} \right] \tag{45}$$

where $Re(\rho \pm \upsilon) > 0$, $\delta > 0$, $Re(\alpha) > kRe(\tau)$.

Corollary 2.24. *When $p = q = 0$, equation* (44) *reduces to the generalized k-Mittag-Leffler function*

$$\int_0^\infty t^{\rho-1} K_\upsilon(at) E_{k,\alpha,\beta}^{\gamma,\tau}\left(xt^{2\delta}\right) dt$$

$$= \frac{2^{\rho-2} a^{-\rho} k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_3\Psi_1 \left[\begin{array}{l} \left(\frac{\gamma}{k}, \tau\right), \left(\frac{\rho \pm \upsilon}{2}, \delta\right) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), \end{array} ; \frac{4x}{a^2} \right] \tag{46}$$

where $Re(\rho \pm \upsilon) > 0$, $Re(a) > 0$, $\delta > 0$.

Corollary 2.25. *For $\tau = q = 1$, equation* (46) *reduces in the following form given by Saxena [11, p. 79, Eq. 4.7]*

$$\int_0^\infty t^{\rho-1} K_\upsilon(at) E_{\alpha,\beta}^{\gamma}\left(xt^{2\delta}\right) dt$$

$$= \frac{2^{\rho-2} a^{-\rho}}{\Gamma(\gamma)} {}_3\Psi_1 \left[\begin{array}{l} (\gamma, 1), \left(\frac{\rho \pm \upsilon}{2}, \delta\right) \\ (\beta, \alpha), \end{array} ; \frac{4x}{a^2} \right] \tag{47}$$

where $Re(\rho \pm \upsilon) > 0$, $Re(a) > 0$, $\delta > 0$, $|\arg x| < \frac{\pi}{2}$.

3. Fractional Fourier Transform (FFT) of S-Function

Theorem 3.1. *If $k \in R$; $\alpha, \beta, \gamma \in C$; $Re(\alpha) > 0$, $Re(\beta) > 0$, $a_i\, (i = 1, 2, \ldots, p)$, $b_j\, (j = 1, 2, \ldots, q)$ and $\tau \in C$, for FFT of order ς of the S-function for $t < 0$, is given by*

$$\Im_\varsigma \left[\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} (a_1, \ldots, a_p; b_1, \ldots, b_q; t) \right] (\omega)$$

$$= \frac{(k)^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (k)^{\tau - \frac{\alpha}{k}} \Gamma\left(\frac{\gamma}{k} + n\tau\right) (i)^{-n-1} \omega^{-(n+1)/\varsigma} (-1)^{-n}}{(b_1)_n \ldots (b_q)_n \Gamma_k \left[\frac{\alpha}{k} n + \frac{\beta}{k}\right]} \tag{48}$$

where $\varsigma > 0$, $w > 0$, $Re(\alpha) > kRe(\tau)$.

Proof. Using equation (7) and (28) and gamma function formula, it gives

$$\Im_\varsigma \left[\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} (a_1, \ldots, a_p; b_1, \ldots, b_q; t) \right] (\omega)$$

$$= \int_R e^{i\omega^{1/\varsigma} t} \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} (a_1, \ldots, a_p; b_1, \ldots, b_q; t)\, dt$$

$$= \int_R e^{i\omega^{1/\varsigma}t} \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n (\gamma)_{n\tau,k}}{(b_1)_n \dots (b_q)_n \Gamma_k [n\alpha+\beta]} \frac{t^n}{n!} dt$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n (\gamma)_{n\tau,k}}{(b_1)_n \dots (b_q)_n \Gamma_k [n\alpha+\beta] n!} \int_R e^{i\omega^{1/\varsigma}t} t^n dt$$

If we set $i\omega^{1/\varsigma}t = -\xi$, then

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n (\gamma)_{n\tau,k}}{(b_1)_n \dots (b_q)_n \Gamma_k [n\alpha+\beta] n!} \int_{-\infty}^{0} e^{-\xi} \left(\frac{-\xi}{i\omega^{1/\varsigma}}\right)^n \left(\frac{-d\xi}{i\omega^{1/\varsigma}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n (\gamma)_{n\tau,k}}{(b_1)_n \dots (b_q)_n \Gamma_k [n\alpha+\beta] (i)^{n+1} \omega^{(n+1)/\varsigma} (-1)^n n!} \int_0^{\infty} e^{-\xi} \xi^n d\xi$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n (\gamma)_{n\tau,k} \Gamma(n+1)}{(b_1)_n \dots (b_q)_n \Gamma_k [n\alpha+\beta] (i)^{n+1} \omega^{(n+1)/\varsigma} (-1)^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n (\gamma)_{n\tau,k} (i)^{-n-1} \omega^{-(n+1)/\varsigma} (-1)^{-n}}{(b_1)_n \dots (b_q)_n \Gamma_k [n\alpha+\beta] n!} \Gamma(n+1)$$

In view of the equations (4) and (6), the above line

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n k^{n\tau} \left(\frac{\gamma}{k}\right)_{n\tau,k} (i)^{-n-1} \omega^{-(n+1)/\varsigma} (-1)^{-n}}{(b_1)_n \dots (b_q)_n (k)^{\frac{n\alpha+\beta}{k}-1} \Gamma_k \left[\frac{\alpha}{k} n + \frac{\beta}{k}\right]}$$

This completes the proof of the Theorem. □

Corollary 3.2. *For $\tau = q$, equation* (48) *reduces in the following form*

$$\Im_\varsigma \left[\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} (a_1,\dots,a_p;b_1,\dots,b_q;t) \right] (\omega) \tag{49}$$

$$= \frac{(k)^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n (k)^{\left(q-\frac{\alpha}{k}\right)n} \Gamma\left(\frac{\gamma}{k}+nq\right) (i)^{-n-1} \omega^{-(n+1)/\varsigma} (-1)^{-n}}{(b_1)_n \dots (b_q)_n \Gamma_k \left[\frac{\alpha}{k} n + \frac{\beta}{k}\right]}$$

Corollary 3.3. *When $p = q = 0$, equation* (49) *reduces to generalized k-Mittag-Leffler function defined by Saxena et al. [12]:*

$$\Im_\alpha \left[E_{k,\alpha,\beta}^{\gamma,\tau}(t) \right] (\omega)$$

$$= \frac{(k)^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{(k)^{\left(\tau-\frac{\alpha}{k}\right)n} \Gamma\left(\frac{\gamma}{k}+n\tau\right) (-1)^{-n} (i)^{-n-1} \omega^{\frac{-(n+1)}{\varsigma}}}{\Gamma\left(\frac{\alpha}{k} n + \frac{\beta}{k}\right)} \tag{50}$$

Acknowledgement

This work is supported by Post-Doctoral Fellowship of the National Board of Higher Mathematics (NBHM), Department of Atomic Energy, India to author (Jitendra Daiya).

REFERENCES

[1] R. Diaz - E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulgaciones Mathematicas 15 (2) (2007), 179–192.

[2] A. Erdélyi - W. Magnus - F. Oberhettinger - F. G. Tricomi, *Tables of Integral Transforms, Vol. 1*, McGraw-Hill, New York - Toronto - London 1953.

[3] A. Erdélyi - W. Magnus - F. Oberhettinger - F. G. Tricomi, *Tables of Integral Transforms, Vol. 2*, McGraw-Hill, New York - Toronto - London 1954.

[4] Y. Luchko - H. Martinez - J. Trujillo, *Fractional Fourier transform and some of its applications*, Fractional Calculus & Applied Analysis 11 (4) (2008), 457–470.

[5] A. M. Mathai - R. K. Saxena - H. J. Haubold, *The H-function, Theory and Applications*, Springer, New York, 2010.

[6] G. M. Mittag-Leffler, *Sur la nouvelle function* $E_\alpha(x)$, C. R. Acad. Sci., Paris 137 (1903), 554–558.

[7] G. M. Mittag-Leffler, *Sur la représentation analytique d'une branche uniforme une function monogène*, Acta. Math. 29 (1905), 101–181.

[8] T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohoma Math. J. 19 (1971), 7–15.

[9] L. Romero - R. Cerutti, *Fractional Fourier Transform and Special k-Function*, Intern. J. Contemp. Math. Sci. 7 (4) (2012), 693–704.

[10] L. Romero - R. Cerutti - L. Luque , *A new Fractional Fourier Transform and convolutions products*, International Journal of Pure and Applied Mathematics 66 (2011), 397–408.

[11] R. K. Saxena, *Certain properties of generalized Mittag-Leffler function*, Conference of the Society of Special functions & their applications, third annual Conference, 78–81, Chennai, India, 2002.

[12] R. K. Saxena - J. Daiya - A. Singh, *Integral transforms of the k-generalized Mittag-Leffler function* $E^{\gamma,\tau}_{k,\alpha,\beta}(z)$, Le Matematiche 69 (2) (2014), 7–16.

[13] K. Sharma, *Application of fractional calculus operators to related areas*, Gen. Math. Notes 7 (1) (2011), 33–40.

[14] K. Sharma - R. Jain, *A note on a generalized M-series as a special function of fractional calculus*, FCAA 12 (4) (2009), 449–452.

[15] H. M. Srivastava - Z. Tomovski, *Fractional calculus with an integral operator con-*

taining a generalized Mittag-Leffler function in the kernel, Appl. Math. Comput. 211 (2009), 198–210.

[16] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. London Math. Soc. 10 (1935), 286–293.

[17] E. M. Wright, *The asymptotic expansion of integral functions defined by Taylor series*. Philos. Trans. Roy. Soc. London, Ser. A 238 (1940), 423–451.

[18] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc. 46 (1940), 389–408.

RAM KISHORE SAXENA
Department of Mathematics and Statistics,
Jai Narain Vyas University, Jodhpur
e-mail: `ram.saxena@yahoo.com`

JITENDRA DAIYA
Department of Mathematics and Statistics,
Jai Narain Vyas University, Jodhpur
e-mail: `jitendra.daiya25@gmail.com`