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NEW APPROACH IN STABILITY TO THE COMPARISON METHOD APPLIED TO THE LIAPUNOV DIRECT METHOD

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In this paper we deal with the stability of the stationary solution of the Lagrange equations for holonomic-rehonomic mechanical systems, by applying the comparison method to the direct Liapunov method. The stability criteria are based on the lemma given in Section 2 , where we show that the eventual stability of the zero solution of the comparison equation can imply the stability of the stationary solution of the Lagrange equations.

1. Introduction.

As it is well known (e.g. see [1, 2, 3]) the classical approach to the stability theory when we apply the comparison method to the Liapunov direct method can be syntetically summarized in the following way.

Let us consider an ordinary differential system of the first order

$$\dot{\mathbf{z}} = \Phi(t, \mathbf{z}), \qquad \mathbf{z} \in \Re^m \tag{1.1}$$

where $\Phi : I \times \Gamma \to \Re^m$ is a continuous function such that

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$\Phi(t,0) \equiv 0$, $I = (\tau\ ,\ \infty)$ is an unbounded interval, and $\Gamma \subset \Re^m$ is an open connected subset containing the origin of coordinates. Moreover, we assume the *uniqueness* of the solutions for differential system (1.1).

Let us assume that there exists a Liapunov function $V = V(t, \mathbf{z})$, *positive definite* on $\mathbf{z}$, which along the solutions of system (1.1) satisfies the differential inequality

$$\dot{V} \leq g(t, V) \qquad \qquad on\ \ I \times S_{\mathbf{z}}$$

where $g : \Re^+ \times \Re^+ \to \Re$ is a continuous function, and $S_{\mathbf{z}} := \{\mathbf{z} \in \Gamma : \|\mathbf{z}\| < \rho\}$ is a suitable small neighbourhood of the origin (the symbol $\|\mathbf{z}\|$ denotes the Euclidean norm of the vector $\mathbf{z} \in \Re^m$).

If the zero solution of the comparison equation

$$\dot{u} = g(t, u) \tag{1.2}$$

is stable, then also the zero solution $\mathbf{z}(t) \equiv 0$ of system (1.1) is stable. Moreover, if the zero solution of the comparison equation $\dot{u} = g(t, u)$ is uniformly stable, and the Liapunov function $V(t, \mathbf{z})$ is uniformly small, then also the solution $\mathbf{z}(t) \equiv 0$ of system (1.1) is uniformly stable.

In Section 2 we improve these classical results by showing that the stationary solution of system (1.1) can be stable also when the zero solution of the comparison equation (1.2) is not stable, but only *eventually stable*.

In Section 3 we consider mechanical systems with holonomic-rheonomic ideal constraints, and we assume that the Lagrange equations posses a stationary solution, which in general corresponds to a relative equilibrium position. Then, by changing appropriately the Lagrange function, we show that the reduced potential energy satisfies the minimal condition to be positive definite, and thus it is right to construct the Liapunov function (3.2).

In Section 4 we apply the lemma established in Section 2 , and we provide stability criteria for the stationary solution of the Lagrange equation.

Finally, in the last section we indicate (by an example) an alternative way to construct the Liapunov function.

2. The Lemma.

Let us consider the differential system (1.1) satisfying al the conditions described in the Introduction. Let us assume that there exists a Liapunov function $V = V(t, \mathbf{z})$, *positive definite* on $\mathbf{z}$, which along the solutions of system (1.1) satisfies the differential inequality

$$\dot{V} \le g(t, V) \qquad \text{on} \quad I \times S_{\mathbf{z}} \tag{2.1}$$

where $g : \Re^+ \times \Re^+ \to \Re$ is a continuous function.

The following lemma holds.

Lemma 2.1 *If the zero solution of the comparison equation (1.2) is eventually stable, then the zero solution* $\mathbf{z}(t) \equiv 0$ *of system (1.1) is stable.*

Moreover, if the zero solution of the comparison equation (1.2) is eventually uniformly stable, and the Liapunov function $V(t, \mathbf{z})$ *is uniformly small, then the the zero solution* $\mathbf{z}(t) \equiv 0$ *of system (1.1) is uniformly stable.*

Proof. To prove the lemma, it is enough to notice that if the origin $u = 0$ of the comparison equation is eventually stable, by means of the comparison method it is easy to show that also the solution $\mathbf{z}(t) \equiv 0$ of system (1.1) is eventually stable. On the other hand, since $\mathbf{z}(t) \equiv 0$ is a solution of system (1.1), and since the conditions ensuring the uniqueness of the solutions are satisfied, it is well-known that *eventual stability implies stability*.

Besides, if the origin of the comparison equation (1.2) is eventually uniformly stable, and the Liapunov function V is uniformly small, it is simple to show that also the solution $\mathbf{z}(t) \equiv 0$ of system (1.1) is eventually uniformly stable. Then, by virtue of the uniqueness of the solutions, it is uniformly stable.

Example 2.1. Let $Oxyz$ be an Earth frame whose z axis is vertical. A material element P of mass m is constrained to move, without friction, on the *moving* parabola having Cartesian equations

$$y = \ell(t) + \frac{x^2}{2p}, \qquad z = 0$$

where p is a positive real constant, while $\ell : \Re^+ \to \Re$ is a given function of class $\mathcal{C}^2(\Re^+)$.

In addition to the weight, the element P is subject to a spring having coefficient $k \, (= cost. > 0)$, which attracts the element to the *moving* straight line whose Cartesian equations are : $y = \ell(t)$, $z = 0$, and to a dissipative force $\mathbf{f} = - h \, \mathbf{v}_P$ (where h is a positive constant and $\mathbf{v}_P$ is the velocity of the element) due to a viscous fluid.

The kinetic energy of the element, the potential energy of the spring and the Lagrange component of the dissipative force $\mathbf{f}$, are rispectively given by

$$T(t,x,\dot{x}) = \frac{1}{2} \, m \, \left(1 + \frac{x^2}{p^2} \right) \dot{x}^2 + m \, \dot{\ell}(t) \, \frac{x \, \dot{x}}{p} + \frac{1}{2} \, m \, \dot{\ell}^2(t)$$

$$\Pi(x) = \frac{1}{8} \, \frac{k}{p^2} \, x^4$$

$$Q(t,x,\dot{x}) = - \, h \, \left(1 + \frac{x^2}{p^2} \right) \dot{x} - h \, \dot{\ell}(t) \, \frac{x}{\ell(t)}$$

from wich we obtain the following Lagrange equation of the motion of the element

$$m \, \left(1 + \frac{x^2}{p^2} \right) \ddot{x} + m \, \frac{x \, \dot{x}^2}{p^2} +$$
$$h \, \left(1 + \frac{x^2}{p^2} \right) \dot{x} + h \, \dot{\ell}(t) \, \frac{x}{\ell(t)} + \frac{1}{2} \, \frac{k}{p^2} \, x^3 + m \, \ddot{\ell}(t) \, \frac{x}{p} = 0$$

The zero solution $x(t) \equiv 0$ is a stationary solution of the Lagrange equation. Choosing as Liapunov function

$$V(t,x,\dot{x}) \ := \ \frac{1}{2} \, m \, \left(1 + \frac{x^2}{p^2} \right) \dot{x}^2 + \frac{1}{8} \, \frac{k}{p^2} \, x^4$$

we see that it is positive definite on $\mathbf{z} = (x, \dot{x})$, and along every motion of the element we have

$$\dot{V} = - \, h \, \left(1 + \frac{x^2}{p^2} \right) \dot{x}^2 - m \, \ddot{\ell}(t) \, \frac{x \, \dot{x}}{p} - h \, \dot{\ell}(t) \, \frac{x \, \dot{x}}{\ell(t)}$$

The following differential inequality is then satisfied

$$\dot{V} \ \le \ \sqrt{\frac{2}{m}} \, \rho \, \left| m \frac{\ddot{\ell}(t)}{p} + h \frac{\dot{\ell}(t)}{\ell(t)} \right| \sqrt{V} \qquad on \quad I \times S_{\mathbf{Z}}$$

hence if

(2.2) $$\int_0^\infty \left| m \frac{\ddot{\ell}(t)}{p} + h \frac{\dot{\ell}(t)}{\ell(t)} \right| dt \ < \ \infty$$

the zero solution of the comparison equation

$$\dot{u} = \sqrt{\frac{2}{m}}\,\rho\,\left| m\,\frac{\ddot{\ell}(t)}{p} + h\,\frac{\dot{\ell}(t)}{\ell(t)} \right| \sqrt{u}$$

is *uniformly eventually stable*. Since the Liapunov function V is time independent, by virtue of Lemma 2.1 condition (2.2) ensures that the stationary solution $x(t) \equiv 0$ of the Lagrange equation is *uniformly stable*.

3. Rhenomic mechanical systems.

Let us consider a holonomic mechanical system S subject to time-dependent bilateral and frictionless constraints, with n degrees of freedom, and let $\mathbf{q}^T = (q_1, \ldots, q_n)$ be the column vector of the independent Lagrange coordinates.

The kinetic energy T of the system is the sum of three terms : $T_2(t, \mathbf{q}, \dot{\mathbf{q}}) + T_1(t, \mathbf{q}, \dot{\mathbf{q}}) + T_0(t, \mathbf{q})$. The first term $T_2(t, \mathbf{q}, \dot{\mathbf{q}}) = (1/2)\,\dot{\mathbf{q}}^T\,\mathbf{A}(t, \mathbf{q})\,\dot{\mathbf{q}}$ is a quadratic form on the generalized velocities $\dot{\mathbf{q}}$, while the second term $T_1(t, \mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^T\,\mathbf{h}(t, \mathbf{q})$ is linear on the generalized velocities. The symbols $\mathbf{A}$ and $\mathbf{h}$ denote a square matrix $n \times n$ and a $n \times 1$ vector respectively.

Throughout the paper, and without further mention, we assume that the square matrix $\mathbf{A}(t, 0)$ is *positive definite*, and so $T_2(t, 0, \dot{\mathbf{q}})$ is a quadratic form positive definite on the generalized velocities $\dot{\mathbf{q}}$.

Let $\Pi(t, \mathbf{q})$ be the (generalized) potential energy of the forces which derive from a potential, and let

$$\mathbf{Q}^T(t, \mathbf{q}, \dot{\mathbf{q}}) = (Q_1(t, \mathbf{q}, \dot{\mathbf{q}}),\ \ldots,\ Q_n(t, \mathbf{q}, \dot{\mathbf{q}}))$$

be the column vector which represents the Lagrange components of the non-potential forces.

Let us assume that all the functions above considered are defined and continuous on $I \times \Omega \times \Re^n$, where $\Omega \subseteq \Re^n$ is an open connected subset containing the origin of coordinates. Moreover, we assume the *uniqueness* of the solutions of the Lagrange equations of the motion

$$\frac{d}{dt}\,\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^T(t, \mathbf{q}, \dot{\mathbf{q}})$$

where as usual $\mathcal{L} := T - \Pi$.

Denoting by

$$P(t,\mathbf{q}) := \Pi(t,\mathbf{q}) - T_0(t,\mathbf{q}) - \Pi(t,0) + T_0(t,0)$$

the *reduced potential energy* of the system, the relative equilibrium position $\mathbf{q}(t) \equiv 0$ is a stationary solution of the Lagrange equations for the mechanical sistem S if and only if

$$\frac{\partial \mathbf{h}^T}{\partial t}(t,0) + \frac{\partial P}{\partial \mathbf{q}}(t,0) = \mathbf{Q}^T(t,0,0) \qquad \text{on } I \tag{3.1}$$

Notice that, as a general rule, the partial derivative $(\partial P/\partial \mathbf{q})\,(t,0)$ is not a zero and so, except special cases, the function P cannot be positive definite on $\mathbf{q}$. Hence, as a rule, the classical function

$$V(t,\mathbf{q},\dot{\mathbf{q}}) = T_2(t,\mathbf{q},\dot{\mathbf{q}}) + P(t,\mathbf{q}) \tag{3.2}$$

cannot be a Liapunov function.

On the other hand the following lemma holds.

Lemma 3.1 *A new equivalent Lagrange function* $\tilde{\mathcal{L}}$ *exists such that*

$$\frac{\partial \tilde{P}}{\partial \mathbf{q}}(t,0) = 0 \qquad \text{on } I$$

Proof. Denote by $w : I \times \Re^n \to \Re$ a scalar function of class $\mathcal{C}^1$ such that $w(t,0) \equiv 0$, and

$$\frac{\partial w}{\partial \mathbf{q}}(t,0) = \int_\tau^t \frac{\partial P}{\partial \mathbf{q}}(s,0)\,ds \qquad \textit{on } I \tag{3.3}$$

For example, we could simply choose $w(t,\mathbf{q}) = \left(\int_\tau^t (\partial P/\partial \mathbf{q})(s,0)ds\right)\mathbf{q}$.

Now, if we replace the Lagrange function $\mathcal{L} = T - \Pi$ with the *equivalent* new Lagrange function

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{d}{dt}\,w(t,\mathbf{q}) = T_2 + \left(T_1 + \frac{\partial w}{\partial \mathbf{q}}\dot{\mathbf{q}}\right) + \left(T_0 - \Pi + \frac{\partial w}{\partial t}\right)$$

the new reduced potential energy is given by $\tilde{P}(t,\mathbf{q}) = \Pi - T_0 - \dfrac{\partial w}{\partial t}$, and it is easy to show that

$$\frac{\partial \tilde{P}}{\partial \mathbf{q}}(t,0) = 0 \qquad \textit{on } I$$

For this reason, without loss of generality we can *always* assume that

$$(3.4) \qquad \frac{\partial P}{\partial \mathbf{q}}(t,0) = 0 \qquad\qquad on \quad I$$

and replace identity (3.1) by the following one

$$(3.5) \qquad \frac{\partial \mathbf{h}^T}{\partial t}(t,0) = \mathbf{Q}^T(t,0,0) \qquad\qquad on \quad I$$

Example 3.1 Let S be the mechanical system composed by two elements A and B, both of mass m, connected by a straight-lined rod having lenght ℓ and negligible mass.

This system moves on the vertical plane Oxy of an Earth frame $Oxyz$, whose y axis is vertical upwards, and during the motion, the end A is constrained to move without friction on the horizontal x axis.

In addition to the weight, there are a spring having coefficient $k\,(= cost. > 0)$, which connects the origin O with the end A, and the force $\mathbf{f} = -h\,(\mathbf{v}_B - \mathbf{v})$, where $h > 0$ is a real constant, and $\mathbf{v}_B$ denotes the velocity of the end B. The force $\mathbf{f}$ is applied to B, and it is produced by a viscous fluid, which moves with constant velocity $\mathbf{v}$ parallel to the x axis.

Finally, we assume that the lenght ℓ of the bar is not constant, namely

$$(3.6) \qquad \ell(t) = \ell_0 + \ell_1\, e^{-\frac{h}{m}t} \qquad\qquad \forall\, t \in \Re^+$$

where ℓ_0 e ℓ_1 are two given positive constants.

Let x and ϑ be the abscissa of A and the oriented angle formed by the rod with the vertical down rispectively. The kinetic energy of the system, the potential energy of the weight and of the spring, and the vector of the Lagrange components of the force $\mathbf{f}$, are then given by

$$T(t,x,\vartheta,\dot{x},\dot{\vartheta}) = m\dot{x}^2 + \frac{1}{2}m\,\ell^2(t)\dot{\vartheta}^2 + m\ell(t)\dot{x}\dot{\vartheta}\cos\vartheta$$
$$+ m\dot{\ell}(t)\dot{x}\sin\vartheta + \frac{1}{2}m\dot{\ell}^2(t)$$

$$\Pi(t,x,\vartheta) = \frac{1}{2}\,k\,x^2 - m\,g\,\ell(t)\,\cos\vartheta$$

$$\mathbf{Q}(t,x,\vartheta,\dot x,\dot\vartheta)=\begin{pmatrix} -h\,\dot x - h\,\dot\ell(t)\,\sin\vartheta - h\,\ell(t)\,\dot\vartheta\,\cos\vartheta + hv \\ \\ -h\,\ell(t)\,\dot x\,\cos\vartheta - h\,\ell^2(t)\,\dot\vartheta + h\,\ell(t)\,v\,\cos\vartheta \end{pmatrix}$$

In view of (3.6), we verify that the Lagrange equations possess the following stationary solution

$$x(t)\equiv\frac{hv}{k},\qquad \vartheta(t)\equiv\bar\vartheta$$

being $\bar\vartheta\in(0,\pi/2)$ and $\mathrm{tg}\bar\vartheta=hv/mg$.

Putting $\xi=x-hv/k$ and $\varphi=\vartheta-\bar\vartheta$, we see that the function

$$P(t,\xi,\varphi)=\Pi(t,\xi,\varphi)-\Pi(t,0,0)=$$
$$=\frac{1}{2}k\left(\xi+\frac{hv}{k}\right)^2-mg\ell(t)\cos(\varphi+\bar\vartheta)-\frac{1}{2}\frac{h^2v^2}{k}+mg\ell(t)\cos\bar\vartheta$$

don't satisfy condition (3.4).

We could change the Lagrange function by putting

$$w(t,\xi,\varphi)=hvt\xi-mgL(t)\cos(\varphi+\bar\vartheta)+mgL(t)\cos\bar\vartheta\qquad\left(L(t):=\int_0^t\ell(s)ds\right)$$

but we obtain

$$\tilde P(t,\xi,\varphi)=P(t,\xi,\varphi)-\frac{\partial w}{\partial t}=\frac{1}{2}k\xi^2$$

which satisfies (3.4), but is not positive definite on $\mathbf{q}^T=(\xi,\varphi)$.

Another function satisfying (3.3) is given by

$$w(t,\xi,\varphi)=hvt\xi+hvL(t)\sin(\varphi+\bar\vartheta)-hvL(t)\sin\bar\vartheta\qquad\left(L(t):=\int_0^t\ell(s)ds\right)$$

The corresponding new reduced potential energy

$$\begin{aligned}\tilde P(t,\xi,\varphi)=P(t,\xi,\varphi)-\frac{\partial w}{\partial t}&=\frac{1}{2}k\xi^2\\ &-mg\ell(t)\Big(\cos(\varphi+\bar\vartheta)-\cos\bar\vartheta\Big)\\ &-h\ell(t)v\Big(\sin(\varphi+\bar\vartheta)-\sin\bar\vartheta\Big)\end{aligned}\tag{3.7}$$

is positive definite on $\mathbf{q}^T=(\xi,\varphi)$, and so it is appropriate to construct a Liapunov function.

4. Stability of the stationary solution $\mathbf{q}(t) \equiv \mathbf{0}$.

Assume that conditions (3.3) and (3.4) hold. Since matrix $\mathbf{A}(t,0)$ is positive definite it easily to see that two scalar continuous functions $\alpha_1, \beta_1 : I \to \Re^+$ exist such that

$$(4.1) \qquad -\frac{\partial T_2}{\partial t}(t,\mathbf{q},\dot{\mathbf{q}}) = -\frac{1}{2}\dot{\mathbf{q}}^T \frac{\partial \mathbf{A}(t,\mathbf{q})}{\partial t}\dot{\mathbf{q}} \le \alpha_1(t) T_2(t,\mathbf{q},\dot{\mathbf{q}}) \text{ on } I \times S_{\mathbf{q}} \times \Re^n$$

$$(4.2) \qquad -\frac{\partial T_1}{\partial t}(t,\mathbf{q},\dot{\mathbf{q}}) = -\dot{\mathbf{q}}^T \frac{\partial \mathbf{h}(t,\mathbf{q})}{\partial t} \le \beta_1(t)\sqrt{T_2(t,\mathbf{q},\dot{\mathbf{q}})} \text{ on } I \times S_{\mathbf{q}} \times \Re^n$$

where $S_{\mathbf{q}} = \{\mathbf{q} \in \Omega : \|\mathbf{q}\| < \rho\}$, being $\|\mathbf{q}\|$ the Euclidean norm of the n-vector $\mathbf{q}$, and $\rho > 0$ a suitable small real number.

Moreover, as the constraints are time-dependent, the inner product $\mathbf{Q}^T(t,\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$, which represents the *virtual* power of the non-potential forces, as a rule *is not negative, not even when these forces are dissipative*.

In the particular case (very interesting from the physical point of view) where the vector $\mathbf{Q}$ represents dissipative forces linear respect to the velocities of the elements of the mechanical system, we notice that the power $\mathbf{Q}^T(t,\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$ is the sum of *quadratic* terms on the generalized velocities $\dot{\mathbf{q}}$, and of *linear* terms on $\dot{\mathbf{q}}$. Therefore, we can assume that there exist two continuous functions $\alpha_2, \beta_2 : I \to \Re^+$ such that

$$(4.3) \qquad \mathbf{Q}^T(t,\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} \le -\alpha_2(t) T_2(t,\mathbf{q},\dot{\mathbf{q}}) + \beta_2(t)\sqrt{T_2(t,\mathbf{q},\dot{\mathbf{q}})} \text{ on } I \times S_{\mathbf{q}} \times S_{\dot{\mathbf{q}}}$$

where $S_{\dot{\mathbf{q}}} = \{\dot{\mathbf{q}} \in \Re^n : \|\dot{\mathbf{q}}\| < \rho\}$.

By virtue of inequalities (4.1), (4.2) and (4.3), it makes physically sense to assume that two continuous functions $\alpha : I \to \Re$ and $\beta : I \to \Re^+$ exist such that

$$(4.4) \qquad \mathbf{Q}^T\dot{\mathbf{q}} - \frac{\partial T_2}{\partial t} - \frac{\partial T_1}{\partial t} + \frac{\partial P}{\partial t} \le \alpha(t)\left(T_2 + P\right) + \beta(t)\sqrt{T_2} \text{ on } I \times S_{\mathbf{q}} \times S_{\dot{\mathbf{q}}}$$

The following theorem hold.

Theorem 4.1. *Suppose that function* $P(t,\mathbf{q})$ *is positive definite on* $\mathbf{q}$ *, and that condition (4.4) is satisfied. Moreover, putting* $A(t) :=$

$\exp\left\{-\int_\tau^t \alpha(s)ds\right\}$, *assume that*

$$(i) \qquad \frac{1}{\sqrt{A(t)}}\int_\tau^t \beta(r)\sqrt{A(r)}dr \longrightarrow 0 \ as \ t \longrightarrow \infty$$

Then the stationary solution $\mathbf{q}(t) \equiv 0$ *of the Lagrange equations is stable.*

If we replace condition (i) with the following one

$$(i)' \qquad A_0 \le A(t) \le A_1, \quad \int_\tau^t \beta(r)dr < \infty$$

where $0 < A_0 \le A_1 < \infty$ *are two real constants, and if the Liapunov function* $V(t, \mathbf{q}, \dot{\mathbf{q}})$ *is uniformly small, then the stationary solution* $\mathbf{q}(t) \equiv 0$ *of the Lagrange equations is uniformly stable.*

Proof. Let us consider the Liapunov function $V(t, \mathbf{q}, \dot{\mathbf{q}})$ defined by (3.2). This function is *positive definite* on $\mathbf{q}$ and $\dot{\mathbf{q}}$, and its time derivative calculated along the solutions of the Lagrange equation is

$$\dot{V} = \mathbf{Q}^T\dot{\mathbf{q}} - \frac{\partial T_2}{\partial t} - \frac{\partial T_1}{\partial t} + \frac{\partial P}{\partial t}$$

Thus, taking into account condition (4.4) we obtain the following inequalitiy

$$\dot{V} \le \alpha(t)V + \beta(t)\sqrt{T_2} \le \alpha(t)V + \beta(t)\sqrt{V} \qquad on I \times S_{\mathbf{q}} \times S_{\dot{\mathbf{q}}}$$

Condition (i) ensures that the zero solution of the differential comparison equation

$$(4.5) \qquad \dot{u} = \alpha(t)u + \beta(t)\sqrt{u}$$

is *eventually stable* ([7] , Corollary 3.2), and so, by virtue of Lemma 1.1 , the stationary solution $\mathbf{q}(t) \equiv 0$ is *stable*.

Condition $(i)'$ ensures that the zero solution of the comparison equation (4.5) is *eventually uniformly stable* ([7] , Corollary 3.3). Moreover, the Liapunov function is uniformly small, and so inview of Lemma 1.1 we get that the stationary solution $\mathbf{q}(t) \equiv 0$ is *uniformly stable*.

This completes the proof.

Remark 4.1. Using the classical approach described in the introduction, instead of (4.4) we should assume that a scalar function $g(t, u)$ exists such that $\dot{V} \leq g(t, V)$, i.e.

$$\text{(4.6)} \qquad \mathbf{Q}^T \dot{\mathbf{q}} - \frac{\partial T_2}{\partial t} - \frac{\partial T_1}{\partial t} + \frac{\partial P}{\partial t} \leq g\left(t, T_2 + P\right) \quad \text{on} \quad I \times S_{\mathbf{q}} \times S_{\dot{\mathbf{q}}}$$

and the zero solution of the comparison equation $\dot{u} = g(t, u)$ has to be stable or uniformly stable (e.g. see [8]).

Unfortunately, in general the left-hand side of (4.6) contains linear terms on the velocities $\dot{\mathbf{q}}$, that appears whether in the power $\mathbf{Q}^T \dot{\mathbf{q}}$ or in the partial derivative $(\partial T_1 / \partial t)$. Consequently, as a rule, in the applications the inequality (4.6) can't be satisfied.

We emphasize that our approach allows us to overcome this difficulty, thanks to the term $\beta(t)\sqrt{T_2}$ that in (4.4) plays a fundamental role.

Example 4.1 We could consider again the mechanical system described in the Example 3.1 . Function $P(t, \mathbf{q})$ given in (3.7) is positive definite on $\mathbf{q}^T = (\xi, \varphi)$, and we see that condition (4.4) and condition (i)' of the theorem are satisfied by choosing

$$\alpha(t) = c|\dot{\ell}(t)|, \qquad \beta(t) \equiv 0$$

being c an appropriate real constant. Since the Liapunov function $V = T_2 + P$ is uniformly small, we can conclude that the stationary solution $x(t) = hv/k, \vartheta(t) \equiv \bar{\vartheta}$ is uniformly stable.

We obtain another example making the following changes in mechanical system of Example 3.1 :

(a) the lenght ℓ of the rod is constant ;

(b) the end A is constrained to move without friction on the moving straight line whose Cartesian equation are $y = s(t), z = 0$, being $s : \Re^+ \to \Re$ a given function of class $\mathcal{C}^2(\Re^+)$;

(c) the spring connects the end A with the point $C \equiv (0, s(t), 0)$;

(d) the constant velocity $\mathbf{v}$ of the viscous fluid is parallel to the y axis .

The kinetic energy of the system, the potential energy of the weight and of the spring, and the vector of the Lagrange components of the force $\mathbf{f}$, are now given by

$$T(t, x, \vartheta, \dot{x}, \dot{\vartheta}) = m\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\vartheta}^2 + m\ell\dot{x}\dot{\vartheta}\cos\vartheta + m\ell\dot{s}(t)\dot{\vartheta}\sin\vartheta + \frac{1}{2}m\dot{s}^2(t)$$

$$\Pi(x,\vartheta) = \frac{1}{2}kx^2 - mg\ell\cos\vartheta$$

$$\mathbf{Q}(t,x,\vartheta,\dot{x},\dot{\vartheta}) = \begin{pmatrix} -h\dot{x} - h\ell\dot{\vartheta}\cos\vartheta \\ -h\ell\dot{x}\cos\vartheta - h\ell^2\dot{\vartheta} + h\ell v\sin\vartheta - h\ell\dot{s}(t)\sin\vartheta \end{pmatrix}$$

The Lagrange equations possess the stationary solution $x = 0, \vartheta = 0$, and if we assume

$$hv < mg \quad \text{and} \quad \int_0^\infty |h\dot{s}(t) + m\ddot{s}(t)|\, dt < \infty$$

we see that the function

$$P(x,\vartheta) = \frac{1}{2}kx^2 + \ell\,(mg - hv)\,(1 - \cos\vartheta)$$

is positive definite on (x,ϑ) , and condition (4.4) and condition (i)' of the theorem are satisfied by choosing

$$\alpha(t) \equiv 0, \qquad \beta(t) = c\,|h\dot{s}(t) + m\ddot{s}(t)|$$

being c an appropriate real constant.

The Liapunov function $V = T_2 + P$ is uniformly small, and so we can conclude that the stationary solution $x(t) = 0, \vartheta(t) = 0$ is uniformly stable.

5. Remarks.

Remark 5.1. When $(\partial P/\partial\mathbf{q})\,(t,0)$ is not a zero on I , instead of change the Lagrange function, under appropriate hypotheses we can follow a different way.

Namely, let us consider the n-vector $\mathbf{Q}$ of the Lagrange componets of the non-potential forces, and decompose this vector in the sum of two vectors

$$\mathbf{Q}(t,\mathbf{q},\dot{\mathbf{q}}) = \mathbf{Q}_1(t,\mathbf{q},\dot{\mathbf{q}}) + \mathbf{Q}_2(t,\mathbf{q})$$

the second of which does not depend on the generalized velocities.

Suppose that

$$\text{(5.1)} \qquad \mathbf{Q}_1(t,0,0) = 0 \qquad \textit{on} \quad I$$

and there exists a scalar function of class $\mathcal{C}^1$, $R : I \times \Re^n \to \Re$, with

$R(t, 0) \equiv 0 on I$, such that

$$\text{(5.2)} \quad \mathbf{Q}_2(t, \mathbf{q}) = \frac{\partial \mathbf{h}}{\partial t}(t, \mathbf{q}) + \frac{\partial R}{\partial \mathbf{q}}(t, \mathbf{q}) \qquad \textit{on} \quad I \times S_{\mathbf{q}} \times S_{\dot{\mathbf{q}}}$$

Then putting

$$\text{(5.3)} \qquad F(t, \mathbf{q}) := P(t, \mathbf{q}) + R(t, \mathbf{q})$$

we see that

$$\frac{\partial P}{\partial \mathbf{q}}(t, 0) = \mathbf{Q}_2^T(t, 0) - \frac{\partial \mathbf{h}}{\partial t}(t, 0)$$

and

$$\frac{\partial R}{\partial \mathbf{q}}(t, 0) = \mathbf{Q}_2^T(t, 0) - \frac{\partial \mathbf{h}}{\partial t}(t, 0) \quad \text{on } I$$

hence it results

$$\frac{\partial F}{\partial \mathbf{q}}(t, 0) = 0 \quad \text{on } I$$

The "necessary" condition so that $F(t, \mathbf{q})$ is positive definite on $\mathbf{q}$ is thus satisfied, and so we can try to apply the comparison method by usin the Liapunov function

$$V(t, \mathbf{q}, \dot{\mathbf{q}}) = T_2(t, \mathbf{q}, \dot{\mathbf{q}}) + F(t, \mathbf{q})$$

We limit ourselves to illustrate this method by the following example.

Example 5.1 Consider again the mechanical system described in Example 3.1 . Conditions (5.1) and (5.2) are satisfied by choosing

$$\mathbf{Q}_1 = \begin{pmatrix} -h\dot{\xi} - h\ell(t)\dot{\varphi}\cos(\varphi + \bar{\vartheta}) \\ -h\ell(t)\dot{\xi}\cos(\varphi + \bar{\vartheta}) - h\ell^2(t)\dot{\varphi} \end{pmatrix},$$

$$\mathbf{Q}_2 = \begin{pmatrix} -h\dot{\ell}(t)\sin(\varphi + \bar{\vartheta}) + hv \\ h\ell(t)v\cos(\varphi + \bar{\vartheta}) \end{pmatrix}$$

hence function R is given by

$$R(t, \xi, \varphi) = hv\xi + h\ell(t)v\sin(\varphi + \bar{\vartheta}) - h\ell(t)v\sin\bar{\vartheta}$$

The function F defined in (5.3) has then the following expression

$$F(t, \xi, \varphi) = \frac{1}{2}k\xi^2 - mg\ell(t)\Big(\cos(\varphi + \bar{\vartheta}) - \cos\bar{\vartheta}\Big) - h\ell(t)v\Big(\sin(\varphi + \bar{\vartheta}) - \sin\bar{\vartheta}\Big)$$

which coincides with function $\tilde{P}$ obtained in (3.7) by changing the Lagrange equation, and used in the previuous section to construct the Liapunov function.

Remark 5.2. Theorem 4.1 could be extended to the quasi-variational ordinary differential systems of the type

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}^T(t, \mathbf{q}, \dot{\mathbf{q}})$$

where $\mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}}) = G(t, \mathbf{q}, \dot{\mathbf{q}}) + F(t, \mathbf{q})$, and $G : I \times \Omega \times \Re^n \to \Re^+$ is a function of class $\mathfrak{C}^1$, stictly convex in the variable $\dot{\mathbf{q}}$ for every $(t, \mathbf{q}) \in I \times \Omega$, with $G(t, \mathbf{q}, 0) \equiv 0$ and $(\partial G/\partial \dot{\mathbf{q}})(t, \mathbf{q}, 0) \equiv 0$ (for further details see [4 , 6] where, however, G not depends on the time, and [**5**] in the general case).

This could be argument of a next work.

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