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NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING ONE OR TWO VALUES WITH FINITE WEIGHT

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The purpose of the paper is to study the uniqueness of meromorphic functions sharing a small function with finite weight. The results of the paper improve and generalize the recent results due to X. B. Zhang and J. F. Xu [20]. We also solve an open problem as posed in the last section of [20].

1. Introduction, Definitions and Results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [6], [15] and [17]. For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities). A finite value z_0 is said to be a

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fixed point of $f(z)$ if $f(z_0) = z_0$. Throughout this paper, we need the following definition.

$$\Theta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

In 1959, W.K. Hayman (see [5], Corollary of Theorem 9) proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, C. C. Yang and X. H. Hua obtained the following uniqueness result corresponding to Theorem A.

Theorem B ([14])**.** *Let f and g be two non-constant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2002, using the idea of sharing fixed points, M. L. Fang and H. L. Qiu further generalized and improved Theorem B in the following manner.

Theorem C ([3])**.** *Let f and g be two non-constant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

For the last couple of years a handful numbers of astonishing results have been obtained regarding the value sharing of nonlinear differential polynomials which are mainly the k-th derivative of some linear expression of f and g.

In 2010, J. F. Xu, F. Lu and H. X. Yi studied the analogous problem corresponding to Theorem C where in addition to the fixed point sharing problem sharing of poles are also taken under supposition. Thus the research has somehow been shifted towards the following direction.

Theorem D ([12])**.** *Let f and g be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1 c_2)^n c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem E ([12])**.** *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let n, k be two positive integers with $n \geq 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

Recently Xiao-Bin Zhang and Jun-Feng Xu [20] further generalized as well as improved the results of [12] as follows.

Theorem F ([20]). *Let f and g be two non-constant meromorphic functions, and $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Let n, k and m be three positive integers with $n > 3k+m+8$ and let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$ or $P(w) \equiv c_0$ where $a_0(\neq 0), a_1, \ldots, a_{m-1}, a_m(\neq 0), c_0(\neq 0)$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM, then*

(I) when $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$, one of the following three cases holds:

(I1) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m,\ldots,n+m-i,\ldots,n)$, $a_{m-i} \neq 0$ for some $i = 1,2,\ldots,m$,

(I2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$,

(I3) $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv a^2$;

(II) when $P(w) \equiv c_0$, one of the following two cases holds:

(II1) $f \equiv tg$ for some constant t such that $t^n = 1$,

(II2) $c_0^2[f^n]^{(k)}[g^n]^{(k)} \equiv a^2$.

Theorem G ([20]). *Let f and g be two non-constant meromorphic functions, and $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f with finitely many zeros and poles. Let n, k and m be three positive integers with $n > 3k+m+7$ and let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$ where $a_0(\neq 0), a_1, \ldots, a_{m-1}, a_m(\neq 0)$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM, f and g share ∞ IM then one of the following two cases holds:*

(1) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m,\ldots,n+m-i,\ldots,n)$, $a_{m-i} \neq 0$ for some $i = 1,2,\ldots,m$,

(2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$.

Theorem H ([20]). *Let f and g be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial with $deg(p) = l \leq 5$, n, k and m be three positive integers with $n > 3k+m+7$. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$ be a nonzero polynomial. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share p CM, f and g share ∞ IM then one of the following three cases hold:*

(1) $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^d = 1$*, where* $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$*,* $a_{m-i} \neq 0$ *for some* $i = 1, 2, \ldots, m$*,*

(2) f *and* g *satisfy the algebraic equation* $R(f,g) \equiv 0$*, where* $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$*;*

(3) $P(z)$ *reduces to a nonzero monomial, namely* $P(z) = a_i z^i \not\equiv 0$ *for some* $i \in \{0, 1, \ldots, m\}$*;*
if $p(z)$ *is not a constant, then* $f = c_1 e^{cQ(z)}$*,* $g = c_2 e^{-cQ(z)}$*, where* $Q(z) = \int_0^z p(z)dz$*,* c_1*,* c_2 *and* c *are constants such that* $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$*,*
if $p(z)$ *is a nonzero constant* b*, then* $f = c_3 e^{cz}$*,* $g = c_4 e^{-cz}$*, where* c_3*,* c_4 *and* c *are constants such that* $(-1)^k a_i^2 (c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

Theorem I ([20]). *Let* f *and* g *be two non-constant meromorphic functions, and* $a(z)(\not\equiv 0, \infty)$ *be a small function of* f*. Let* n *and* m *be two positive integers such that* $n > \max\{m+10, 3m+3\}$ *and let* $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$*, where* $a_0(\neq 0)$*,* a_1*,* $\ldots$*,* $a_m(\neq 0)$ *are complex constants. If* $f^n P(f) f'$ *and* $g^n P(g) g'$ *share* a *CM then either* $f(z) = tg(z)$ *for a constant* t *such that* $t^d = 1$*, where* $d = GCD(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$*,* $a_{m-i} \neq 0$ *for some* $i \in \{0, 1, 2, \ldots, m\}$ *or* f *and* g *satisfy the algebraic equation* $R(f,g) \equiv 0$*, where* $R(\omega_1, \omega_2) = \omega_1^{n+1}(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1}) - \omega_2^{n+1}(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1})$.

In [[20], Remark 1.2] Zhang and Xu pointed out that the computation will be very complicated when $deg(p)$ becomes large, in Theorem H. They expressed their inability to study the same for general polynomial $p(z)$ as well. Naturally at the end of the paper the following open problem was posed by the authors in [20].

Problem 1.1. What happens to Theorem H if the condition "$l \leq 5$" is removed?

One of our objectives in writing this paper is to solve this open problem. Now observing the above results it is quite natural to place the following question.

Question 1.2. Is it possible to relax the nature of sharing and at the same time further reduce the lower bound of n in Theorems F, G, H, I?

In this paper, taking the possible answer of the above questions into background we obtain the following results.

Theorem 1.3. *Let* f *and* g *be two non-constant meromorphic functions, and* $\alpha(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f *and* g*. Let* n*,* k *and* m *be three positive integers such that* $n > 3k + m + 8 - 2\{\Theta(\infty; f) + \Theta(\infty; g)\} - \{\Theta(0; f) + \Theta(0; g)\} - k\min\{\Theta(\infty; f), \Theta(\infty; g)\}$ *and* $P(w)$ *be defined as in Theorem F. If* $[f^n P(f)]^{(k)}$ *and* $[g^n P(g)]^{(k)}$ *share* $(\alpha, 2)$ *then the conclusion of Theorem F holds.*

Theorem 1.4. *Let f and g be two non-constant meromorphic functions, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f with finitely many zeros and poles. Let n, k and m be three positive integers such that $n > 3k+m+6$ and $P(w)$ be defined as in Theorem G. If $[f^nP(f)]^{(k)}$, $[g^nP(g)]^{(k)}$ share (α,k_1) where $k_1 = \left[\frac{3+k}{n+m-k-1}\right]+3$ and f, g share $(\infty,0)$ then the conclusion of Theorem G holds.*

Theorem 1.5. *Let f and g be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial such that either $deg(p) \leq n-1$ or all the zeros of $p(z)$ are of multiplicities atmost $n-k-1$, where $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers such that $n > 3k+m+6$ and $P(w)$ be defined as in Theorem H. If $[f^nP(f)]^{(k)}$, $[g^nP(g)]^{(k)}$ share (p,k_1) where $k_1 = \left[\frac{3+k}{n+m-k-1}\right]+3$ and f, g share $(\infty,0)$ then the conclusion of Theorem H holds.*

Let b_i, $i = 1,2,\ldots,s$ be the distinct roots of the equation $P(w) = 0$, where $P(w)$ be defined as in Theorem I. Also we suppose that $\Theta_f = \Theta(0;f)+\Theta(\infty;f)$ $+\sum_{i=1}^{s}\Theta(b_i;f)$. Θ_g can be similarly defined.
Again we define k_2 and k_3 respectively by

$$k_2 = \Theta(\infty;f)+\Theta(\infty;g)+2\{\Theta(0;f)+\Theta(0;g)\}+\min\{\Theta_f,\Theta_g\} \tag{1}$$

$$k_3 = \frac{4m}{s}-(m-1). \tag{2}$$

where s and m are two positive integers such that $s \leq m$.

Theorem 1.6. *Let f and g be two non-constant meromorphic functions, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f and g. Let n and m be two positive integers such that $n > \max\{m+10-k_2,k_3\}$, where k_2, k_3 are respectively defined by (1) and (2) and s denotes the number of distinct roots of the equation $P(w) = 0$ and $P(w)$ be defined as in Theorem I. If $f^nP(f)f'$, $g^nP(g)g'$ share $(\alpha,2)$ then the conclusion of Theorem I holds.*

We now explain the following definitions and notations which are used in the paper.

Definition 1.7 ([7]). Let $a \in \mathbb{C}\cup\{\infty\}$. For a positive integer p we denote by $N(r,a;f \mid\leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we can define $N(r,a;f \mid\geq p)$ and $\overline{N}(r,a;f \mid\geq p)$.

Definition 1.8 ([9]). Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \cdots + \overline{N}(r,a;f \mid\geq k).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 1.9 ([8, 9]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.10 ([1]). Let f and g be two non-constant meromorphic functions such that f and g share the value a IM for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p and also an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ ($\overline{N}_L(r,a;g)$) the reduced counting function of those a-points of f and g, where $p > q \geq 1$ ($q > p \geq 1$). Also we denote by $\overline{N}_E^{(1}(r,a;f)$ the reduced counting function of those a-points of f and g, where $p = q \geq 1$.

Definition 1.11 ([8, 9]). Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 1.12 ([10]). Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H and V the functions as follows:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right), \tag{3}$$

$$V=\left(\frac{F'}{F-1}-\frac{F'}{F}\right)-\left(\frac{G'}{G-1}-\frac{G'}{G}\right). \tag{4}$$

Lemma 2.1 ([13]). *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ..., $a_0(z)$ be meromorphic functions such that $T(r,a_i(z))=S(r,f)$ for $i=0,1,2,\dots,n$. Then*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2.2 ([19]). *Let f be a non-constant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right)\le T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}(r,0;f)+S(r,f), \tag{5}$$

$$N_p\left(r,0;f^{(k)}\right)\le k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f). \tag{6}$$

Lemma 2.3 ([6], Theorem 3.10). *Suppose that f is a non-constant meromorphic function, $k\ge 2$ is an integer. If*

$$N(r,\infty,f)+N(r,0;f)+N(r,0;f^{(k)})=S(r,\frac{f'}{f}),$$

then $f=e^{az+b}$, where $a\neq 0$, b are constants.

Lemma 2.4 ([4]). *Let $f(z)$ be a non-constant entire function and let $k\ge 2$ be a positive integer. If $f(z)f^{(k)}(z)\neq 0$, then $f(z)=e^{az+b}$, where $a\neq 0, b$ are constant.*

Lemma 2.5 ([17], Theorem 1.24). *Let f be a non-constant meromorphic function and let k be a positive integer. Suppose that $f^{(k)}\not\equiv 0$, then*

$$N(r,0;f^{(k)})\le N(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.6 ([11]). *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}\mid f\neq 0)\le k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\ge k)+S(r,f).$$

Lemma 2.7. *Suppose that f and g be two non-constant meromorphic functions. Let $F=[f^nP(f)]^{(k)}$, $G=[g^nP(g)]^{(k)}$, where $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ are positive integers and $P(w)$ be defined as in Theorem F. If f, g share ∞ IM and $V\equiv 0$, then $F\equiv G$.*

Proof. Suppose $V\equiv 0$. Then by integration we obtain

$$1-\frac{1}{F}\equiv A(1-\frac{1}{G}).$$

It is that if z_0 is a pole of f then it is a pole of g. Hence from the definition of F and G we have $\frac{1}{F(z_0)}=0$ and $\frac{1}{G(z_0)}=0$. So $A=1$ and hence $F\equiv G$. □

Lemma 2.8. *Suppose that f and g be two non-constant meromorphic functions. F, G be defined as in Lemma 2.7 and $H\not\equiv 0$. If f, g share $(\infty,0)$ and F, G share $(1,k_1)$, where $0\leq k_1\leq\infty$ then*

$$\begin{aligned}&\big(n+m-k-1\big)\overline{N}(r,\infty;f)\\&\qquad\leq(k+m+1)\{T(r,f)+T(r,g)\}+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Similar result holds for g also.

Proof. Suppose ∞ is an e.v.P of f and g then the lemma follows immediately. Next suppose ∞ is not an e.v.P of f and g. Since $H\not\equiv 0$ from Lemma 2.7 we have $V\not\equiv 0$. We suppose that z_0 is a pole of f with multiplicity q and a pole of g with multiplicity r. Clearly z_0 is a pole of F with multiplicity $(n+m)q+k$ and a pole of G with multiplicity $(n+m)r+k$. Noting that f, g share $(\infty,0)$ from the definition of V it is clear that z_0 is a zero of V with multiplicity at least $n+m+k-1$. Now using the Milloux theorem [6], p. 55, and Lemma 2.1, we obtain from the definition of V that

$$m(r,V)=S(r,f)+S(r,g).$$

Thus using Lemmas 2.1 and 2.6 we get

$$\begin{aligned}&\big(n+m+k-1\big)\overline{N}(r,\infty;f)\\&\leq N(r,0;V)\\&\leq T(r,V)+O(1)\\&\leq N(r,\infty;V)+m(r,V)+O(1)\\&\leq\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\&\leq N_{k+1}(r,0;f^nP(f))+N_{k+1}(r,0;g^nP(g))+k\overline{N}(r,\infty;f)+k\overline{N}(r,\infty;g)\\&\quad+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\&\leq N_{k+1}(r,0;f^n)+N_{k+1}(r,0;P(f))+N_{K+1}(r,0;g^n)+N_{k+1}(r,0;P(g))\\&\quad+2k\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\end{aligned}$$

$$\leq (k+1)\overline{N}(r,0;f)+N(r,0;P(f))+(k+1)\overline{N}(r,0;g)+N(r,0;P(g))$$
$$+2k\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).$$

This gives

$$\big(n+m-k-1\big)\overline{N}(r,\infty;f) \leq (k+m+1)\{T(r,f)+T(r,g)\}+\overline{N}_*(r,1;F,G)$$
$$+S(r,f)+S(r,g).$$

This completes the proof of the lemma. □

Lemma 2.9. *Let f be a non-constant meromorphic function and $F=f^nP(f)f^{(k)}$, where $n(\geq 1)$, $m(\geq 0)$, $k(\geq 1)$ are positive integers and $P(w)$ be defined as in the Theorem F. Then*

$$(n+m-1)T(r,f) \leq T(r,F)-N(r,\infty;f)-N(r,0;f^{(k)})+S(r,f).$$

Proof. Note that

$$N(r,\infty;F)=N(r,\infty;f^nP(f))+N(r,\infty;f^{(k)})$$
$$=N(r,\infty;f^nP(f))+N(r,\infty;f)+k\overline{N}(r,\infty;f).$$

That is

$$N(r,\infty;f^nP(f))=N(r,\infty,F)-N(r,\infty;f)-k\overline{N}(r,\infty,f)+S(r,f).$$

Also

$$m(r,f^nP(f))=m(r,\frac{F}{f^{(k)}})$$
$$\leq m(r,F)+m(r,\frac{1}{f^{(k)}})+S(r,f)$$
$$=m(r,F)+T(r,f^{(k)})-N(r,0;f^{(k)})+S(r,f)$$
$$=m(r,F)+N(r,\infty;f^{(k)})+m(r,f^{(k)})-N(r,0;f^{(k)})+S(r,f)$$
$$\leq m(r,F)+N(r,\infty;f)+k\overline{N}(r,\infty;f)+m(r,\frac{f^{(k)}}{f})+m(r,f)$$
$$-N(r,0;f^{(k)})+S(r,f)$$
$$=m(r,F)+T(r,f)+k\overline{N}(r,\infty;f)-N(r,0;f^{(k)})+S(r,f).$$

Now

$$(n+m)T(r,f)=N(r,\infty;f^nP(f))+m(r,f^nP(f))$$
$$\leq T(r,F)+T(r,f)-N(r,\infty;f)-N(r,0;f^{(k)})+S(r,f).$$

i.e

$$(n+m-1)\,T(r,f) \leq T(r,F)-N(r,\infty;f)-N(r,0;f^{(k)})+S(r,f). \quad \square$$

Lemma 2.10. *Let f be a non-constant meromorphic function and a_i, $i = 1, 2, \ldots, n$ be finite distinct complex numbers, where $n \geq 2$. Then*

$$N(r,0;f^{'}) \leq T(r,f) + \overline{N}(r,\infty;f) - \sum_{i=1}^{n} m(r,a_i;f) + S(r,f)$$

Proof. Let $F = \sum\limits_{i=1}^{n} \frac{1}{f-a_i}$. Then $\sum\limits_{i}^{n} m(r,a_i;f) = m(r,F) + O(1)$. Note that

$$m(r,F) \leq m(r,0;f^{'}) + m(r,\sum_{i=1}^{n} \frac{f^{'}}{f-a_i}) = T(r,f^{'}) - N(r,0;f^{'}) + S(r,f).$$

Also we observe that

$$T(r,f^{'}) \leq m(r,f) + m(r,\frac{f^{'}}{f}) + N(r,f) + \overline{N}(r,f) = T(r,f) + \overline{N}(r,f) + S(r,f).$$

Hence the Lemma follows. □

Lemma 2.11 ([20])**.** *Let f and g be two non-constant meromorphic functions,let $P(w)$ be defined as in Theorem H and k, m, $n > 2k+m+1$ be three positive integers. If $[f^nP(f)]^{(k)} \equiv [g^nP(g)]^{(k)}$, then $f^nP(f) \equiv g^nP(g)$.*

Lemma 2.12 ([16], Lemma 6)**.** *If $H \equiv 0$, then F, G share 1 CM. If further F, G share ∞ IM then F, G share ∞ CM.*

Lemma 2.13. *Let f, g be two non-constant meromorphic functions and $F = \frac{[f^nP(f)]^{(k)}}{\alpha}$, $G = \frac{[g^nP(g)]^{(k)}}{\alpha}$, where $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f, $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ are positive integers such that $n > 3k+m+3$ and $P(w)$ be defined as in Theorem F. If f, g share $(\infty,0)$ and $H \equiv 0$ then either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv \alpha^2$ or $f^nP(f) \equiv g^nP(g)$.*

Proof. Since $H \equiv 0$, by Lemma 2.12 we get F and G share 1 CM.
On integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{7}$$

where a, b are constants and $a \neq 0$. From (7) it is clear that F and G share $(1,\infty)$. We now consider the following cases.
Case 1. Let $b \neq 0$ and $a \neq b$.

If $b = -1$, then from (7) we have

$$F \equiv \frac{-a}{G-a-1}.$$

Therefore
$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f).$$
So in view of Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned}
&(n+m)\,T(r,g)\\
&\le T(r,G)+N_{k+1}(r,0;g^nP(g))-\overline{N}(r,0;G)+S(r,g)\\
&\le \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,a+1;G)+N_{k+1}(r,0;g^nP(g))\\
&\quad -\overline{N}(r,0;G)+S(r,g)\\
&\le \overline{N}(r,\infty;g)+N_{k+1}(r,0;g^nP(g))+\overline{N}(r,\infty;f)+S(r,g)\\
&\le \overline{N}(r,\infty;g)+N_{k+1}(r,0;g^n)+N_{k+1}(r,0;P(g))+\overline{N}(r,\infty;f)+S(r,g)\\
&\le 2\overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+S(r,g)\\
&\le (k+m+3)\,T(r,g)+S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f)\le T(r,g)$ for $r\in I$. So for $r\in I$, $S(r,f)$ can be replaced by $S(r,g)$. So for $r\in I$, we get a contradiction from above since $n>3k+m+3$. If $b\neq -1$, from (7) we obtain that

$$F-(1+\frac{1}{b})\equiv \frac{-a}{b^2[G+\frac{a-b}{b}]}.$$

So
$$\overline{N}(r,\frac{(b-a)}{b};G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)$$
Using Lemma 2.2 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b\neq 0$ and $a=b$.

If $b=-1$, then from (7) we have

$$FG\equiv 1,$$

i.e.,
$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv \alpha^2,$$
where $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share α CM.

If $b\neq -1$, from (7) we have

$$\frac{1}{F}\equiv\frac{bG}{(1+b)G-1}.$$

Therefore
$$\overline{N}(r,\frac{1}{1+b};G)=\overline{N}(r,0;F).$$

So in view of Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned}
&(n+m)\,T(r,g)\\
&\le T(r,G)+N_{k+1}(r,0;g^nP(g))-\overline{N}(r,0;G)+S(r,g)\\
&\le \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{1}{1+b};G)+N_{k+1}(r,0;g^nP(g))\\
&\quad -\overline{N}(r,0;G)+S(r,g)\\
&\le \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+\overline{N}(r,0;F)+S(r,g)\\
&\le \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+(k+1)\overline{N}(r,0;f)\\
&\quad +T(r,P(f))+k\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\le k+m+2)\,T(r,g)+(2k+m+1)\,T(r,f)+S(r,f)+S(r,g).
\end{aligned}$$

So for $r\in I$ we have

$$(n+m)\,T(r,g)\le(3k+2m+3)\,T(r,g)+S(r,g),$$

which is a contradiction since $n>3k+m+3$.
Case 3. Let $b=0$. From (7) we obtain

$$F\equiv\frac{G+a-1}{a}. \tag{8}$$

If $a\neq 1$ then from (8) we obtain

$$\overline{N}(r,1-a;G)=\overline{N}(r,0;F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $a=1$ and from (8) we obtain

$$F\equiv G.$$

Then by Lemma 2.11 we have

$$f^nP(f)\equiv g^nP(g).\quad\square$$

Lemma 2.14. *Let f, g be two non-constant meromorphic functions and $F=\frac{[f^nP(f)]^{(k)}}{\alpha}$, $G=\frac{[g^nP(g)]^{(k)}}{\alpha}$, where $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f, $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ are positive integers such that $n>3k+m+8-2\{\Theta(\infty;f)+\Theta(\infty;g)\}-\{\Theta(0;f)+\Theta(0;g)\}-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}$ and $P(w)$ be defined as in Theorem F. If $H\equiv 0$ then either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv\alpha^2$ or $f^nP(f)\equiv g^nP(g)$.*

Proof. Since $H \equiv 0$, on integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{9}$$

where a,b are constants and $a \neq 0$. From (9) it is clear that F and G share $(1,\infty)$ and hence they share $(1,2)$. So in this case we always have $n > 3k+m+8-2\{\Theta(\infty;f)+\Theta(\infty;g)\}-\{\Theta(0;f)+\Theta(0;g)\}-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}$. Now we consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$.

If $b=-1$, then from (9) we have

$$F \equiv \frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f).$$

So in view of Lemma 2.2 and following the same argument as used in the proof of the Case 1 of Lemma 2.13 we get

$$\begin{aligned}
&(n+m)\,T(r,g)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+S(r,g)\\
&\leq (1-\Theta(\infty;f)+\varepsilon)T(r,f)+(k+m+2-\Theta(\infty;g)-\Theta(0;g)+\varepsilon)T(r,g)\\
&\quad +S(r,f)+S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f)\leq T(r,g)$ for $r \in I$.

So for $r \in I$ we have

$$(n-k-3+\Theta(\infty;f)+\Theta(\infty;g)+\Theta(0;g)-2\varepsilon)T(r,g)\leq S(r,g),$$

which is a contradiction for arbitrary $\varepsilon>0$.

If $b \neq -1$, from (9) we obtain that

$$F-(1+\frac{1}{b}) \equiv \frac{-a}{b^2[G+\frac{a-b}{b}]}.$$

So

$$\overline{N}(r,\frac{(b-a)}{b};G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)$$

Using Lemma 2.2 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a=b$.

If $b=-1$, then from (9) we have

$$FG \equiv 1,$$

i.e.,

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv \alpha^2.$$

If $b \neq -1$, from (9) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r,0;F).$$

So in view of Lemma 2.2 and following the procedure as adopted in the proof of Case 2 of Lemma 2.13 we get

$$\begin{aligned}
&(n+m)\,T(r,g)\\
&\leq (2k+m+1-k\Theta(\infty;f)-\Theta(0;f)+\varepsilon)\,T(r,f)\\
&\quad +(k+m+2-\Theta(\infty;g)-\Theta(0;g)+\varepsilon)\,T(r,g)+S(r,f)+S(r,g)\\
&\leq (2k+m+1-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}-\Theta(0;f)+\varepsilon)\,T(r,f)\\
&\quad +(k+m+2-\Theta(\infty;g)-\Theta(0;g)+\varepsilon)\,T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

So for $r \in I$ we have

$$\begin{aligned}
&(n-3k-m-3+\Theta(\infty;g)+\Theta(0;f)+\Theta(0;g)\\
&+k\min\{\Theta(\infty;f),\Theta(\infty;g)\}+2\varepsilon)\,T(r,g) \leq S(r,g),
\end{aligned}$$

which is a contradiction for arbitrary $\varepsilon > 0$.
Case 3. Let $b=0$. Following the proof of Case 3 of Lemma 2.13 we obtain

$$F \equiv G.$$

Note that

$$\begin{aligned}
&n > 3k+m+8-2\{\Theta(\infty;f)+\Theta(\infty;g)\}-\{\Theta(0;f)+\Theta(0;g)\}\\
&-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}
\end{aligned}$$

always implies that

$$n > 2k+m+1.$$

Then by Lemma 2.11 we have

$$f^n P(f) \equiv g^n P(g). \quad \square$$

Lemma 2.15. *Let f and g be two non-constant meromorphic functions and $\alpha(z)(\not\equiv 0,\infty)$ be small function of f and g. Let n and m be two positive integers such that $n > k_3$, where k_3 be defined by (2), s denotes the number of distinct roots of the equation $P(w) = 0$ and $P(w)$ is defined as in Theorem I. Then*

$$f^n P(f) f^{'} g^n P(g) g^{'} \not\equiv \alpha^2,$$

Proof. First suppose that

$$f^n P(f) f^{'} g^n P(g) g^{'} \equiv \alpha^2(z). \tag{10}$$

Let d_i be the distinct zeros of $P(w) = 0$ with multiplicity p_i, where $i = 1, 2, \ldots, s$, $1 \le s \le m$ and $\sum\limits_{i=1}^{s} p_i = m$.
Now by the second fundamental theorem for f and g we get respectively

$$sT(r,f) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \sum_{i=1}^{s} \overline{N}(r,d_i;f) - \overline{N}_0(r,0;f^{'}) + S(r,f), \tag{11}$$

and

$$sT(r,g) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \sum_{i=1}^{s} \overline{N}(r,d_i;g) - \overline{N}_0(r,0;g^{'}) + S(r,g), \tag{12}$$

where $\overline{N}_0(r,0;f^{'})$ denotes the reduced counting function of those zeros of $f^{'}$ which are not the zeros f and $f - d_i$, $i = 1, 2, \ldots, s$ and $\overline{N}_0(r,0;g^{'})$ can be similarly defined.
Let z_0 be a zero of f with multiplicity p but $a(z_0) \ne 0, \infty$. Clearly z_0 must be a pole of g with multiplicity q. Then from (10) we get $np + p - 1 = nq + mq + q + 1$. This gives

$$mq + 2 = (n+1)(p-q). \tag{13}$$

From (13) we get $p - q \ge 1$ and so $q \ge \frac{n-1}{m}$. Now $np + p - 1 = nq + mq + q + 1$ gives
$p \ge \frac{n+m-1}{m}$. Thus we have

$$\overline{N}(r,0;f) \le \frac{m}{n+m-1} N(r,0;f) \le \frac{m}{n+m-1} T(r,f). \tag{14}$$

Let $z_1(a(z_1) \ne 0, \infty)$ be a zero of $f - d_i$ with multiplicity q_i, $i = 1, 2, \ldots, s$. obviously z_1 must be a pole of g with multiplicity r. Then from (10) we get $q_i p_i + q_i - 1 = (n+m+1)r + 1 \ge n + m + 2$. This gives $q_i \ge \frac{n+m+3}{p_i+1}$ for $i = 1, 2, \ldots, s$ and so we get

$$\overline{N}(r,d_i;f) \le \frac{p_i+1}{n+m+3} N(r,d_i;f) \le \frac{p_i+1}{n+m+3} T(r,f).$$

Clearly

$$\sum_{i=1}^{s} \overline{N}(r,d_i;f) \le \frac{m+s}{n+m+3}\, T(r,f). \tag{15}$$

Similarly we have

$$\overline{N}(r,0;g) \le \frac{m}{n+m-1}\, T(r,g), \tag{16}$$

and

$$\sum_{i=1}^{s} \overline{N}(r,d_i;g) \le \frac{m+s}{n+m+3}\, T(r,g). \tag{17}$$

Also it is clear that

$$\begin{aligned} &\overline{N}(r,\infty;f) \\ &\le \overline{N}(r,0;g) + \sum_{i=1}^{s} \overline{N}(r,d_i;g) + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g) \\ &\le \left(\frac{m}{n+m-1} + \frac{m+s}{n+m+3}\right) T(r,g) + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g), \end{aligned} \tag{18}$$

by (16) and (17).
Then by (11), (14), (15) and (18) we get

$$\begin{aligned} &s\,T(r,f) \\ &\le \left(\frac{m}{n+m-1} + \frac{m+s}{n+m+3}\right) \{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;g^{'}) \\ &\quad - \overline{N}_0(r,0;f^{'}) + S(r,f) + S(r,g). \end{aligned} \tag{19}$$

Similarly we have

$$\begin{aligned} &s\,T(r,g) \\ &\le \left(\frac{m}{n+m-1} + \frac{m+s}{n+m+3}\right) \{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;f^{'}) \\ &\quad - \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g). \end{aligned} \tag{20}$$

Then from (19) and (20) we get

$$\begin{aligned} &s\{T(r,f)+T(r,g)\} \le 2\left(\frac{m}{n+m-1} + \frac{m+s}{n+m+3}\right) \{T(r,f)+T(r,g)\} \\ &+ S(r,f) + S(r,g), \end{aligned}$$

i.e.,

$$\left(s - \frac{2m}{n+m-1} - \frac{2(m+s)}{n+m+3}\right) \{T(r,f)+T(r,g)\} \le S(r,f) + S(r,g). \tag{21}$$

Since

$$\left(s-\frac{2m}{n+m-1}-\frac{2(m+s)}{n+m+3}\right)$$
$$=\frac{(n+m-1)^2s+2(n+m-1)(s-2m)-8m}{(n+m-1)(n+m+3)},$$

we note that when $n+m-1>\frac{4m}{s}$, i.e., when $n>\frac{4m}{s}-(m-1)=k_2$, then clearly $s-\frac{2m}{n+m-1}-\frac{2(m+s)}{n+m+3}>0$ and so (21) leads to a contradiction. This completes the proof. □

Lemma 2.16 ([20])**.** *Let f ,g be non-constant meromorphic functions, let n, k be two positive integers with $n>k+2$, and let $P(w)$ be defined as in Theorem H. Let $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f with finitely many zeros and poles . If $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv\alpha^2$, f and g share ∞ IM, then $P(w)$ is reduced to a nonzero monomial, namely $P(w)=a_iw^i\not\equiv 0$ for some $i\in\{0,1,\ldots,m\}$.*

Lemma 2.17 ([17])**.** *Let f_j $(j=1,2,3)$ be a meromorphic and f_1 be non-constant. Suppose that*

$$\sum_{j=1}^{3}f_j\equiv 1$$

and

$$\sum_{j=1}^{3}N(r,0;f_j)+2\sum_{j=1}^{3}\overline{N}(r,\infty;f_j)<(\lambda+o(1))T(r),$$

as $r\longrightarrow+\infty$, $r\in I$, $\lambda<1$ and $T(r)=\max_{1\leq j\leq 3}T(r,f_j)$. Then $f_2\equiv 1$ or $f_3\equiv 1$.

Lemma 2.18. *Let f, g be two transcendental meromorphic functions, $p(z)$ be a nonzero polynomial such that either $deg(p)\leq n-1$ or all the zeros of $p(z)$ are of multiplicities atmost $n-k-1$, where n and k be two positive integers such that $n>k$. Let $[f^n]^{(k)}$, $[g^n]^{(k)}$ share p CM and f, g share ∞ IM. Now when $[f^n]^{(k)}[g^n]^{(k)}\equiv p^2$,*
(i) if $p(z)$ is not a constant, then $f=c_1e^{cQ(z)}$, $g=c_2e^{-cQ(z)}$, where $Q(z)=\int_0^z p(z)dz$, c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n=-1$,
(ii) if $p(z)$ is a nonzero constant b, then $f=c_3e^{dz}$, $g=c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k}=b^2$.

Proof. Suppose

$$[f^n]^{(k)}[g^n]^{(k)}\equiv p^2. \tag{22}$$

Since f and g share ∞ IM, (22) one can easily say that f and g are transcendental entire functions.

We consider the following cases:

Case 1: Let $deg(p(z)) = l(\geq 1)$.
Since $n > k$, it follows that $N(r,0;f) = O(logr)$ and $N(r,0;g) = O(logr)$.
Let

$$F_1 = \frac{[f^n]^{(k)}}{p} \quad and \quad G_1 = \frac{[g^n]^{(k)}}{p}. \tag{23}$$

From (22) we get

$$F_1 G_1 \equiv 1. \tag{24}$$

If $F_1 \equiv cG_1$, where c is a nonzero constant, then by (24), F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv G_1$.
Let

$$\Phi = \frac{[f^n]^{(k)} - p}{[g^n]^{(k)} - p}. \tag{25}$$

We deduce from(25) that

$$\Phi \equiv e^{\beta}, \tag{26}$$

where β is an entire function.
Let $f_1 = F_1$, $f_2 = -e^{\beta} G_1$ and $f_3 = e^{\beta}$. Here f_1 is transcendental. Now from (26), we have

$$f_1 + f_2 + f_3 \equiv 1.$$

Hence by Lemma 2.5 we get

$$\begin{aligned} \sum_{j=1}^{3} N(r,0;f_j) + 2\sum_{j=1}^{3} \overline{N}(r,\infty;f_j) &\leq N(r,0;F_1) + N(r,0;e^{\beta}G_1) + O(logr) \\ &\leq (\lambda + o(1))T(r), \end{aligned}$$

as $r \longrightarrow +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1\leq j\leq 3} T(r,f_j)$.
So by Lemma 2.17, we get either $e^{\beta}G_1 \equiv -1$ or $e^{\beta} \equiv 1$. But here the only possibility is that $e^{\beta}G_1 \equiv -1$, i.e., $[g^n]^{(k)} \equiv -e^{-\beta}p(z)$ and so from (22) we obtain

$$F_1 \equiv e^{\gamma_1} G_1,$$

i.e.,

$$[f^n]^{(k)} \equiv e^{\gamma_1}[g^n]^{(k)},$$

where γ_1 is a non-constant entire function. Now from (22) we get

$$(f^n)^{(k)} \equiv ce^{\frac{1}{2}\gamma_1}p(z), \quad (g^n)^{(k)} \equiv ce^{-\frac{1}{2}\gamma_1}p(z), \tag{27}$$

where $c = \pm 1$. This shows that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 0 CM.

Since $N(r,0;f) = O(logr)$ and $N(r,0;g) = O(logr)$, so we can take

$$f(z) = h_1(z)e^{\alpha(z)}, \quad g(z) = h_2(z)e^{\beta(z)}, \tag{28}$$

where h_1 and h_2 are nonzero polynomials and α, β are two non-constant entire functions.

We deduce from (22) and (28) that either both α and β are transcendental entire functions or both are polynomials.
We consider the following cases:
Subcase 1.1: Let $k \geq 2$.
First we suppose both α and β are transcendental entire functions.
Let $\alpha_1 = \alpha^{'} + \frac{h_1^{'}}{h_1}$ and $\beta_1 = \beta^{'} + \frac{h_2^{'}}{h_2}$. Clearly both α_1 and β_1 are transcendental entire functions.
Note that

$$S(r, n\alpha_1) = S(r, \frac{[f^n]^{'}}{f^n}), \quad S(r, n\beta_1) = S(r, \frac{[g^n]^{'}}{g^n}).$$

Moreover we see that

$$N(r, 0; [f^n]^{(k)}) \leq N(r, 0; p^2) = O(logr).$$

$$N(r, 0; [g^n]^{(k)}) \leq N(r, 0; p^2) = O(logr).$$

From these and using (28) we have

$$N(r, \infty; f^n) + N(r, 0; f^n) + N(r, 0; [f^n]^{(k)}) = S(r, n\alpha_1) = S(r, \frac{[f^n]^{'}}{f^n}) \tag{29}$$

and

$$N(r, \infty; g^n) + N(r, 0; g^n) + N(r, 0; [g^n]^{(k)}) = S(r, n\beta_1) = S(r, \frac{[g^n]^{'}}{g^n}). \tag{30}$$

Then from (29), (30) and Lemma 2.3 we must have

$$f = e^{az+b}, \quad g = e^{cz+d}, \tag{31}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But these types of f and g do not agree with the relation (22).

Next we suppose α and β are both non-constant polynomials, since otherwise f, g reduces to a polynomials contradicting that they are transcendental.
Also from (22) we get $\alpha + \beta \equiv C$ i.e., $\alpha^{'} \equiv -\beta^{'}$. Therefore $deg(\alpha) = deg(\beta)$.

Suppose h_i's $i = 1, 2$ are non-constant polynomials. We deduce from (28) that

$$[f^n]^{(k)} \equiv A h_1^{n-k} [h_1^k (\alpha^{'})^k + P_{k-1}(\alpha^{'}, h_1^{'})] e^{n\alpha} \equiv p(z) e^{n\alpha}, \tag{32}$$

and

$$[g^n]^{(k)} \equiv B h_2^{n-k} [h_2^k (\beta^{'})^k + Q_{k-1}(\beta^{'}, h_2^{'})] e^{n\beta} \equiv p(z) e^{n\beta}, \tag{33}$$

where A, B are nonzero constants, $P_{k-1}(\alpha^{'},h_1^{'})$ and $Q_{k-1}(\beta^{'},h_2^{'})$ are differential polynomials in $\alpha^{'},h_1^{'}$ and $\beta^{'},h_2^{'}$ respectively.

By virtue of the polynomial $p(z)$, from (32) and (33) we conclude that both h_1 and h_2 are nonzero constants.
So we can rewrite f and g as follows:

$$f = e^{\gamma}, \quad g = e^{\delta}, \tag{34}$$

where $\gamma + \delta \equiv C$ and $deg(\gamma) = deg(\delta)$. Clearly $\gamma^{'} \equiv -\delta^{'}$.
If $deg(\gamma) = deg(\delta) = 1$, then we again get a contradiction from (22).
Next we suppose $deg(\gamma) = deg(\delta) \geq 2$.
We deduce from (34) that

$$(f^n)^{'} = n\gamma^{'} e^{n\gamma}$$

$$(f^n)^{''} = [n^2(\gamma^{'})^2 + n\gamma^{''}]e^{n\gamma}$$

$$(f^n)^{'''} = [n^3(\gamma^{'})^3 + 3n^2\gamma^{'}\gamma^{''} + n\gamma^{'''}]e^{n\gamma}$$

$$(f^n)^{(iv)} = [n^4(\gamma^{'})^4 + 6n^3(\gamma^{'})^2\gamma^{''} + 3n^2(\gamma^{''})^2 + 4n^2\gamma^{'}\gamma^{'''} + n\gamma^{(iv)}]e^{n\gamma}$$

$$(f^n)^{(v)} = [n^5(\gamma^{'})^5 + 10n^4(\gamma^{'})^3\gamma^{''} + 15n^3\gamma^{'}(\gamma^{''})^2 + 10n^3(\gamma^{'})^2\gamma^{'''} + 10n^2\gamma^{''}\gamma^{'''} \\ + 5n^2\gamma^{'}\gamma^{(iv)} + n\gamma^{(v)}]e^{n\gamma}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$[f^n]^{(k)} = [n^k(\gamma^{'})^k + K(\gamma^{'})^{k-2}\gamma^{''} + P_{k-2}(\gamma^{'})]e^{n\gamma}.$$

Similarly we get

$$\begin{aligned}[g^n]^{(k)} &= [n^k(\delta^{'})^k + K(\delta^{'})^{k-2}\delta^{''} + P_{k-2}(\delta^{'})]e^{n\delta} \\ &= [(-1)^k n^k(\gamma^{'})^k - K(-1)^{k-2}(\gamma^{'})^{k-2}\gamma^{''} + P_{k-2}(-\gamma^{'})]e^{n\delta},\end{aligned}$$

where K is a suitably positive integer and $P_{k-2}(\gamma^{'})$ is a differential polynomial in $\gamma^{'}$.
Since $deg(\gamma) \geq 2$, we observe that $deg((\gamma^{'})^k) \geq k\ deg(\gamma^{'})$ and so $(\gamma^{'})^{k-2}\gamma^{''}$ is either a nonzero constant or $deg((\gamma^{'})^{k-2}\gamma^{''}) \geq (k-1)\ deg(\gamma^{'}) - 1$. Also we see that

$$deg\left((\gamma^{'})^k\right) > deg\left((\gamma^{'})^{k-2}\gamma^{''}\right) > deg\left(P_{k-2}(\gamma^{'})\right)\ (or\ \ deg\left(P_{k-2}(-\gamma^{'})\right)).$$

Now from (27) we see that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 0 CM and so the polynomials

$$n^k(\gamma^{'})^k + K(\gamma^{'})^{k-2}\gamma^{''} + P_{k-2}(\gamma^{'})$$

and

$$(-1)^k n^k (\gamma^{'})^k - K(-1)^{k-2}(\gamma^{'})^{k-2}\gamma^{''} + P_{k-2}(-\gamma^{'})$$

must be identical but this is impossible for $k \geq 2$.
Actually the terms $n^k(\gamma^{'})^k + K(\gamma^{'})^{k-2}\gamma^{''}$ and $(-1)^k n^k(\gamma^{'})^k - K(-1)^{k-2}(\gamma^{'})^{k-2}\gamma^{''}$ can not be identical for $k \geq 2$.

Subcase 1.2: Let $k = 1$.
Now from (22) we get

$$f^{n-1}f^{'}g^{n-1}g^{'} \equiv p_1^2, \tag{35}$$

where $p_1^2 = \frac{1}{n^2}p^2$.
First we suppose both α and β are transcendental entire functions.
Let $h = fg$. Clearly h is a transcendental entire function. Then from (35) we get

$$\left(\frac{g^{'}}{g} - \frac{1}{2}\frac{h^{'}}{h}\right)^2 \equiv \frac{1}{4}\left(\frac{h^{'}}{h}\right)^2 - h^{-n}p_1^2. \tag{36}$$

Let

$$\alpha_2 = \frac{g^{'}}{g} - \frac{1}{2}\frac{h^{'}}{h}.$$

From (36) we get

$$\alpha_2^2 \equiv \frac{1}{4}\left(\frac{h^{'}}{h}\right)^2 - h^{-n}p_1^2. \tag{37}$$

First we suppose $\alpha_2 \equiv 0$. Then we get $h^{-n}p_1^2 \equiv \frac{1}{4}\left(\frac{h^{'}}{h}\right)^2$ and so $T(r,h) = S(r,h)$, which is impossible. Next we suppose that $\alpha_2 \not\equiv 0$. Differentiating (37) we get

$$2\alpha_2\alpha_2^{'} \equiv \frac{1}{2}\frac{h^{'}}{h}\left(\frac{h^{'}}{h}\right)^{'} + n\,h^{'}h^{-n-1}p_1^2 - 2h^{-n}p_1p_1^{'}.$$

Applying (37) we obtain

$$h^{-n}\left(-n\frac{h^{'}}{h}p_1^2 + 2p_1p_1^{'} - 2\frac{\alpha_2^{'}}{\alpha_2}p_1^2\right) \equiv \frac{1}{2}\frac{h^{'}}{h}\left(\left(\frac{h^{'}}{h}\right)^{'} - \frac{h^{'}}{h}\frac{\alpha_2^{'}}{\alpha_2}\right). \tag{38}$$

First we suppose

$$-n\frac{h^{'}}{h}p_1^2 + 2p_1p_1^{'} - 2\frac{\alpha_2^{'}}{\alpha_2}p_1^2 \equiv 0.$$

Then there exist a non-zero constant c such that $\alpha_2^2 \equiv ch^{-n}p_1^2$ and so from (37) we get

$$(c+1)h^{-n}p_1^2 \equiv \frac{1}{4}\left(\frac{h^{'}}{h}\right)^2.$$

If $c=-1$, then h will be a constant. If $c\neq -1$, then we have $T(r,h)=S(r,h)$, which is impossible. Next we suppose that

$$-n\frac{h^{'}}{h}p_1^2+2p_1p_1^{'}-2\frac{\alpha_2^{'}}{\alpha_2}p_1^2\not\equiv 0.$$

Then by (38) we have

$$\begin{aligned}
n\,T(r,h) &= n\,m(r,h)\\
&\leq m\left(r, h^n\frac{1}{2}\frac{h^{'}}{h}\left(\left(\frac{h^{'}}{h}\right)^{'}-\frac{h^{'}}{h}\frac{\alpha_2^{'}}{\alpha_2}\right)\right)+m\left(r,\frac{1}{\frac{1}{2}\frac{h^{'}}{h}\left(\left(\frac{h^{'}}{h}\right)^{'}-\frac{h^{'}}{h}\frac{\alpha_2^{'}}{\alpha_2}\right)}\right)+O(1)\\
&\leq T\left(r,\frac{1}{2}\frac{h^{'}}{h}\left(\left(\frac{h^{'}}{h}\right)^{'}-\frac{h^{'}}{h}\frac{\alpha_2^{'}}{\alpha_2}\right)\right)+m\left(r, n\frac{h^{'}}{h}p_1^2+2p_1p_1^{'}-2\frac{\alpha_2^{'}}{\alpha_2}p_1^2\right)\\
&\leq \overline{N}(r,0;\alpha_2)+S(r,h)+S(r,\alpha_2). \qquad (39)
\end{aligned}$$

From (37) we get

$$T(r,\alpha_2)\leq \frac{1}{2}n\,T(r,h)+S(r,h).$$

Now from (39) we get

$$\frac{1}{2}n\,T(r,h)\leq S(r,h),$$

which is impossible .

Thus α and β are both polynomials. Also from (22) we can conclude that $\alpha(z)+\beta(z)\equiv C$ for a constant C and so $\alpha^{'}(z)+\beta^{'}(z)\equiv 0$. We deduce from (22) that

$$[f^n]^{'}\equiv n[h_1^n\alpha^{'}+h_1^{n-1}h_1^{'}]e^{n\alpha}\equiv p(z)e^{n\alpha}, \qquad (40)$$

and

$$[g^n]^{'}=n[h_2^n\beta^{'}+h_2^{n-1}h_2^{'}]e^{n\beta}\equiv p(z)e^{n\beta}. \qquad (41)$$

By the virtue of the polynomial $p(z)$, from (40) and (41) we conclude that both h_1 and h_2 are nonzero constant.

So we can rewrite f and g as follows:

$$f=e^{\gamma_2},\quad g=e^{\delta_2}. \qquad (42)$$

Now from (22) we get

$$n^2\gamma_2^{'}\delta_2^{'}e^{n(\gamma_2+\delta_2)}\equiv p^2. \qquad (43)$$

Also from (43) we can conclude that $\gamma_2(z)+\delta_2(z)\equiv C$ for a constant C and so $\gamma_2^{'}(z)+\delta_2^{'}(z)\equiv 0$. Thus from (43) we get $n^2e^{nC}\gamma_2^{'}\delta_2^{'}\equiv p^2(z)$. By computation we get

$$\gamma_2^{'}=cp(z),\quad \delta_2^{'}=-cp(z). \tag{44}$$

Hence

$$\gamma_2=cQ(z)+b_1,\quad \delta_2=-cQ(z)+b_2, \tag{45}$$

where $Q(z)=\int_0^z p(z)dz$ and b_1, b_2 are constants. Finally we take f and g as

$$f(z)=c_1e^{cQ(z)},\quad g(z)=c_2e^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n=-1$.

Case 2: Let $p(z)$ be a nonzero constant b.

In this case we see that f and g have no zeros and so we can take f and g as follows:

$$f=e^{\alpha},\quad g=e^{\beta}, \tag{46}$$

where $\alpha(z)$, $\beta(z)$ are two non-constant entire functions.

We now consider the following two subcases:

Subcase 2.1: Let $k\geq 2$.

We see that

$$N(r,0;[f^n]^{(k)})=0.$$

From this and using (46) we have

$$f^n(z)[f^n(z)]^{(k)}\neq 0. \tag{47}$$

Similarly we have

$$g^n(z)[g^n(z)]^{(k)}\neq 0. \tag{48}$$

Then from (47), (48) and Lemma 2.4 we must have

$$f=e^{az+b},\quad g=e^{cz+d}, \tag{49}$$

where $a\neq 0$, b, $c\neq 0$ and d are constants.

Subcase 2.1: Let $k=1$.

Considering **Subcase 1.2** one can easily get

$$f=e^{az+b},\quad g=e^{cz+d}, \tag{50}$$

where $a\neq 0$, b, $c\neq 0$ and d are constants.

Finally we can take f and g as

$$f=c_3e^{dz},\quad g=c_4e^{-dz},$$

where c_3, c_4 and d are nonzero constants such that $(-1)^k(c_3c_4)^n(nd)^{2k}=b^2$.

This completes the proof. □

Lemma 2.19. *Let f and g be two transcendental meromorphic functions, $p(z)$ be a nonzero polynomial such that either $deg(p) \leq n-1$ or zeros of $p(z)$ are of multiplicities atmost $n-k-1$, where n, k be two positive integers such that $n > k+2$ and let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$ be a nonzero polynomial. Let $[f^n P(f)]^{(k)}$, $[g^n P(g)]^{(k)}$ share p CM and f, g share ∞ IM. If $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv p^2$, then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$; if $p(z)$ is not a constant, then*
$f = c_1 e^{cQ(z)}$, $g = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)dz$, c_1, c_2 and c are constants such that $a_i^2 (c_1 c_2)^{n+i} [(n+i)c]^2 = -1$,
if $p(z)$ is a nonzero constant b, then
$f = c_3 e^{cz}$, $g = c_4 e^{-cz}$, where c_3, c_4 and c are constants such that $(-1)^k a_i^2 (c_3 c_4)^{n+i} [(n+i)c]^{2k} = b^2$.

Proof. Suppose that

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv p^2. \tag{51}$$

Note that $n > k+2$. By Lemma 2.16, $P(w)$ reduces to a nonzero monomial, namely $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. Then we have

$$a_i^2 [f^{n+i}]^{(k)} [g^{n+i}]^{(k)} \equiv p^2. \tag{52}$$

Let $s = n+i$. Since f and g share ∞ IM we get that f and g have no poles, hence f and g are both transcendental entire functions. Remaining part of the Lemma follows from Lemma 2.18. $\square$

Lemma 2.20 ([2]). *Let f and g be two non-constant meromorphic functions sharing $(1,k_1)$, where $2 \leq k_1 \leq \infty$. Then*

$$\overline{N}(r,1;f|=2) + 2\overline{N}(r,1;f|=3) + \ldots + (k_1-1)\overline{N}(r,1;f|=k_1) + k_1 \overline{N}_L(r,1;f)$$
$$+ (k_1+1)\overline{N}_L(r,1;g) + k_1 \overline{N}_E^{(k_1+1}(r,1;g) \leq N(r,1;g) - \overline{N}(r,1;g).$$

3. Proofs of the Theorem

Proof of Theorem 1.3. Let $F = \frac{[f^n P(f)]^{(k)}}{\alpha}$ and $G = \frac{[g^n P(g)]^{(k)}}{\alpha}$. Also F, G share $(1,2)$ except the zeros and poles of $\alpha(z)$.
Case 1. Let $H \not\equiv 0$.

From (3) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned}
& N(r,\infty;H) \qquad (53)\\
\le\ & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)\\
& +\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g),
\end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $\alpha(z_0)\neq 0,\infty$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1)\le N(r,0;H)\le N(r,\infty;H)+S(r,f)+S(r,g). \qquad (54)$$

Using (53) and (54) we get

$$\begin{aligned}
& \overline{N}(r,1;F) \qquad (55)\\
\le\ & N(r,1;F|=1)+\overline{N}(r,1;F|\ge 2)\\
\le\ & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+\overline{N}_*(r,1;F,G)\\
& +\overline{N}(r,1;F|\ge 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Now in view of Lemma 2.6 we get

$$\begin{aligned}
& \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\ |\ge 2)+\overline{N}_*(r,1;F,G) \qquad (56)\\
\le\ & \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F|\ge 2)+\overline{N}(r,1;F|\ge 3)\\
=\ & \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;G|\ge 2)+\overline{N}(r,1;G|\ge 3)\\
\le\ & \overline{N}_0(r,0;G^{'})+N(r,1;G)-\overline{N}(r,1;G)\\
\le\ & N(r,0;G^{'}\mid G\neq 0)\\
\le\ & \overline{N}(r,0;G)+\overline{N}(r,\infty;g)+S(r,g),
\end{aligned}$$

Hence using (55), (56), Lemmas 2.1 and 2.2 we get from the second fundamental theorem that

$$\begin{aligned}
& (n+m)T(r,f) \qquad (57)\\
\le\ & T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
\le\ & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)
\end{aligned}$$

$$
\begin{aligned}
& -N_0(r,0;F^{'}) \\
& \leq 2\,\overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))+\overline{N}(r,0;F|\geq 2) \\
& \quad +\overline{N}(r,0;G|\geq 2)+\overline{N}(r,1;F|\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G^{'}) \\
& \quad -N_2(r,0;F)+S(r,f)+S(r,g) \\
& \leq 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_{k+2}(r,0;f^nP(f)))+N_2(r,0;G)+S(r,f) \\
& \quad +S(r,g) \\
& \leq 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g) \\
& \quad +\ N_{k+2}(r,0;g^nP(g))+S(r,f)+S(r,g) \\
& \leq 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g\}+(k+2)\,\overline{N}(r,0;f)+T(r,P(f))+(k+2)\,\overline{N}(r,0;g) \\
& \quad +T(r,P(g))+k\,\overline{N}(r,\infty;g)+S(r,f)+S(r,g) \\
& \leq (k+m+4-2\Theta(\infty;f)-\Theta(0;f)+\varepsilon)\,T(r,f)+(2k+m+4-2\Theta(\infty;g) \\
& \quad -\Theta(0;g)-k\,\Theta(\infty;g)+\varepsilon)\,T(r,g)+S(r,f)+S(r,g) \\
& \leq (3k+2m+8-2\Theta(\infty;f)-2\Theta(\infty;g)-\Theta(0;f)-\Theta(0;g)-k\Theta(\infty;g) \\
& \quad +2\varepsilon)\,T(r)+S(r),
\end{aligned}
$$

where $T(r)=\max\{T(r,f),T(r,g)\}$ and $S(r)=o(T(r))$, outside a set of finite linear measure. In a similar way we can obtain

$$
\begin{aligned}
& (n+m)\,T(r,g) \\
\leq\ & (3k+2m+8-2\Theta(\infty;f)-2\Theta(\infty;g)-\Theta(0;f)-\Theta(0;g) \\
& -k\Theta(\infty;f)+\varepsilon)\ T(r)+S(r).
\end{aligned} \tag{58}
$$

Combining (57) and (58) we see that

$$
\begin{aligned}
& (n+m)\,T(r) \\
\leq\ & (3k+2m+8-2\{\Theta(\infty;f)+\Theta(\infty;g)\}-\{\Theta(0;f)+\Theta(0;g)\} \\
& -k\min\{\Theta(\infty;f),\Theta(\infty;g)\}+2\varepsilon)\ T(r)+S(r),
\end{aligned}
$$

i.e

$$
\begin{aligned}
& (n-3k-m-8+2\{\Theta(\infty;f)+\Theta(\infty;g)\}+\{\Theta(0;f)+\Theta(0;g)\} \\
& +k\min\{\Theta(\infty;f),\Theta(\infty;g)\}-2\varepsilon]\ T(r)\leq S(r).
\end{aligned} \tag{59}
$$

Since $\varepsilon>0$ is arbitrary, (59) leads to a contradiction.
Case 2. Let $H\equiv 0$. Then by the Lemma 2.14 we have either

$$
[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv\alpha^2, \tag{60}
$$

or

$$
f^nP(f)\equiv g^nP(g). \tag{61}
$$

Now when
$$P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$$
we have from (61) that
$$f^n(a_m f^m + a_{m-1} f^{m-1} + \ldots + a_0) \equiv g^n(a_m g^m + a_{m-1} g^{m-1} + \ldots + a_0). \quad (62)$$
Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ into (62) we deduce that
$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \ldots + a_0 g^n(h^n - 1) \equiv 0,$$
which implies $h^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is not a constant, then we know by (62) that f and g satisfying the algebraic equation $R(f,g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \ldots + a_0)$.

When $P(w) \equiv c_0$ then from (60) we get $c_0^2 [f^n]^{(k)} [g^n]^{(k)} \equiv \alpha^2$ and from (61) we get $f \equiv tg$ for a constant t such that $t^n = 1$. This completes the the proof of the theorem. □

Proof of Theorem 1.5. Let $F = \frac{[f^n P(f)]^{(k)}}{p(z)}$ and $G = \frac{[g^n P(g)]^{(k)}}{p(z)}$. Note that since f and g are transcendental meromorphic functions, $p(z)$ is a small function with respect to both $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$. Also F, G share $(1, k_1)$ except the zeros of $p(z)$ and f, g share $(\infty, 0)$.

Case 1. Let $H \not\equiv 0$.

Proceeding in the same way as (53) we get
$$\begin{aligned} &N(r,\infty;H) \quad (63)\\ &\leq \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2)\\ &\quad + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g). \end{aligned}$$
Let z_0 be a simple zero of $F-1$ but $p(z_0) \neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So
$$N(r,1;F| = 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,f) + S(r,g). \quad (64)$$
Using (63) and (64) we get
$$\begin{aligned} &\overline{N}(r,1;F) \quad (65)\\ &\leq N(r,1;F| = 1) + \overline{N}(r,1;F| \geq 2)\\ &\leq \overline{N}_*(r,\infty;f,g) + \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_*(r,1;F,G)\\ &\quad + \overline{N}(r,1;F| \geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_*(r,1;F,G)\\ &\quad + \overline{N}(r,1;F| \geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g). \end{aligned}$$

Now in view of Lemmas 2.20 and 2.6 we get

$$\begin{aligned}
&\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_*(r,1;F,G) &(66)\\
&\leq \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\mid=2)+\overline{N}(r,1;F\mid=3)+\ldots+\overline{N}(r,1;F\mid=k_1)\\
&\quad+\overline{N}_E^{(k_1+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G)\\
&\leq \overline{N}_0(r,0;G^{'})-\overline{N}(r,1;F\mid=3)-\ldots-(k_1-2)\overline{N}(r,1;F\mid=k_1)\\
&\quad-(k_1-1)\overline{N}_L(r,1;F)-k_1\overline{N}_L(r,1;G)-(k_1-1)\overline{N}_E^{(k_1+1}(r,1;F)\\
&\quad+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G)\\
&\leq \overline{N}_0(r,0;G^{'})+N(r,1;G)-\overline{N}(r,1;G)-(k_1-2)\overline{N}_L(r,1;F)-\\
&\quad(k_1-1)\overline{N}_L(r,1;G)\\
&\leq N(r,0;G^{'}\mid G\neq 0)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\\
&\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G),
\end{aligned}$$

Hence using (65), (66), Lemmas 2.2 and 2.8 we get from the second fundamental theorem that

$$\begin{aligned}
&(n+m)T(r,f) &(67)\\
&\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))\\
&\quad-N_2(r,0;F)-N_0(r,0;F^{'})\\
&\leq \overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))\\
&\quad+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_0(r,0;G^{'})-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leq 3\,\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)-(k_1-2)\,\overline{N}_*(r,1;F,G)\\
&\quad-\overline{N}_L(r,1;G)+S(r,f)+S(r,g)\\
&\leq 3\,\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))\\
&\quad-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leq (3+k)\,\overline{N}(r,\infty;f)+(k+2)\,\overline{N}(r,0;f)+T(r,P(f))+(k+2)\,\overline{N}(r,0;g)\\
&\quad+T(r,P(g))-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leq (k+m+2)\,\{T(r,f)+T(r,g)\}+(3+k)\overline{N}(r,\infty;f)\\
&\quad-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leq (k+m+2)\,\{T(r,f)+T(r,g)\}+\frac{(3+k)(k+m+1)}{n+m-k-1}\,\{T(r,f)+T(r,g)\}\\
&\quad+\frac{3+k}{n+m-k-1}\,\overline{N}_*(r,1;F,G)-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)
\end{aligned}$$

$$\leq \left[k+m+2+\frac{(3+k)(k+m+1)}{n+m-k-1}\right]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

In a similar way we can obtain

$$(n+m)T(r,g) \tag{68}$$
$$\leq \left[k+m+2+\frac{(3+k)(k+m+1)}{n+m-k-1}\right]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

Adding (67) and (68) we get

$$\left[n-m-2k-4-\frac{(6+2k)(k+m+1)}{n+m-k-1}\right]\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

Since the quantity in the third bracket can be written as

$$\left[\frac{(n+m-k-1)^2-(2m+k+3)(n+m-k-1)-2(k+3)(k+m+1)}{n+m-k-1}\right], \tag{69}$$

by a simple computation one can easily verify that when

$$n+m-k-1>2m+2k+5$$
$$>\frac{2m+k+3+\sqrt{(2m+k+3)^2+8(k+3)(k+m+1)}}{2},$$

i.e., when $n>3k+m+6$ we obtain a contradiction from (69).
Case 2. Let $H\equiv 0$. Then by the Lemma 2.13 we have either

$$f^nP(f)\equiv g^nP(g), \tag{70}$$

or

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv p^2. \tag{71}$$

Now from (70) and following the method of proof of **Case 2** in Theorem 1.3 we get the first two conclusions of Theorem H. Also with the help the Lemma 2.19 we get from (71) the last conclusion of the Theorem H. This completes the proof of the Theorem. □

Proof of Theorem 1.4. We omit the proof since it can be carried out in the line of proof of Theorem 1.5. □

Proof of Theorem 1.6. Let $F=\frac{f^nP(f)f'}{a(z)}$ and $G=\frac{g^nP(g)g'}{a(z)}$. Also F, G share $(1,2)$.
Case 1. Let $H\not\equiv 0$.

Here with the same argument as used in the proof of Theorem 1.3 we can obtain (55) and (56).

Also using Lemma 2.10 with $n=s+1$, $a_i=b_i (i=1,2,\ldots,s)$ and $a_{s+1}=0$, where b_i are the distinct roots of the equation $P(z)=0$, $s\leq m$ we note that

$$\begin{aligned}
& N(r,0;g^{'}|g\neq 0,b_1,b_2,\ldots,b_s) \qquad (72)\\
\leq\; & N(r,0;g^{'})+\sum_{i=1}^{s+1}\overline{N}(r,a_i;g)-\sum_{i=1}^{s+1}N(r,a_i;g)\\
\leq\; & T(r,g)+\overline{N}(r,\infty;g)-\sum_{i=1}^{s+1}\{m(r,a_i;g)+N(r,a_i;g)\}+\sum_{i=1}^{s+1}\overline{N}(r,a_i;g)\\
\leq\; & \overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\sum_{i=1}^{s}\overline{N}(r,b_i;g)-s\,T(r,g).
\end{aligned}$$

Hence using (55), (56) and (72) we get from the second fundamental theorem that

$$\begin{aligned}
& T(r,F) \qquad (73)\\
\leq\; & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N_0(r,0;F^{'})\\
\leq\; & 2\,\overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;G|\geq 2)\\
& +\overline{N}(r,1;F|\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
\leq\; & 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_2(r,0;F)+N_2(r,0;G)+S(r,f)+S(r,g)\\
\leq\; & 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+2\,\overline{N}(r,0;f)+N(r,0;P(f))\\
& +N_2(r,0;f^{'})+2\overline{N}(r,0;g)+N(r,0;P(g))+N_2(r,0;g^{'}|g\neq 0,b_1,\ldots,b_s)\\
& +S(r,f)+S(r,g)\\
\leq;\; & 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+2\,\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\
& +m\,\{T(r,f)+T(r,g)\}+N(r,0;f^{'})+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\\
& +\sum_{i=1}^{s}\overline{N}(r,b_i;g)-s\,T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

Now using Lemma 2.9 for $k=1$ we get from (73)

$$\begin{aligned}
& (n+m-1)\,T(r,f) \qquad (74)\\
\leq\; & T(r,F)-N(r,\infty;f)-N(r,0;f^{'})+S(r,f)\\
\leq\; & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+2\,\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\
& +m\,T(r,f)+(m+1)\,T(r,g)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\sum_{i=1}^{s}\overline{N}(r,b_i;g)\\
& -s\,T(r,g)+S(r,f)+S(r,g)
\end{aligned}$$

$$\leq (m+3-\Theta(\infty;f)-2\Theta(0;f)+\varepsilon)T(r,f)+(m+6-\Theta(\infty;g)$$
$$-2\Theta(0;g)-\Theta_g+\varepsilon)T(r,g)$$
$$+S(r,f)+S(r,g)$$
$$\leq (2m+9-\Theta(\infty;f)-\Theta(\infty;g)-2\Theta(0;f)-2\Theta(0;g)-\Theta_g+2\varepsilon)T(r)$$
$$+S(r).$$

In a similar way we can obtain

$$(n+m-1)\,T(r,g) \tag{75}$$
$$\leq (2m+9-\Theta(\infty;f)-\Theta(\infty;g)-2\Theta(0;f)-2\Theta(0;g)-\Theta_f+2\varepsilon)\;T(r)$$
$$+S(r).$$

Combining (74) and (75) we see that

$$(n+m-1)\,T(r)$$
$$\leq (2m+9-\Theta(\infty;f)-\Theta(\infty;g)-2\Theta(0;f)-2\Theta(0;g)$$
$$-\min\{\Theta_f,\Theta_g\}+2\varepsilon)\;T(r)+S(r),$$

i.e

$$(n-m-10+\Theta(\infty;f)+\Theta(\infty;g)+2\Theta(0;f)+2\Theta(0;g) \tag{76}$$
$$+\min\{\Theta_f,\Theta_g\}-2\varepsilon)\;T(r)\leq S(r).$$

Since $\varepsilon>0$ was arbitrary, (76) leads to a contradiction.

Case 2. Let $H\equiv 0$. Proceeding in the same way as done in the proof of the Lemma 2.14 and using (73), Lemma 2.9 instead of Lemma 2.2 we can obtain either $f^nP(f)f^{'}g^nP(g)g^{'}\equiv a^2$ or $f^nP(f)f^{'}\equiv g^nP(g)g^{'}$. Since $n>k_3$, it follows by the Lemma 2.15 that $f^nP(f)f^{'}g^nP(g)g^{'}\not\equiv a^2$. Then

$$f^nP(f)f^{'}\equiv g^nP(g)g^{'}. \tag{77}$$

Let $h=\frac{f}{g}$. If h is a constant, by putting $f=hg$ in (77) we get

$$a_mg^m(h^{n+m+1}-1)+a_{m-1}g^{m-1}(h^{n+m}-1)+\ldots+a_1g(h^{n+2}-1)+a_0(h^{n+1}-1)\equiv 0,$$

which implies that $h^d=1$, where $d=GCD(n+m+1,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i}\neq 0$ for some $i\in\{0,1,\ldots,m\}$. Thus $f\equiv tg$ for a constant t such that $t^d=1$,where $d=GCD(n+m+1,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i}\neq 0$ for some $i\in\{0,1,\ldots,m\}$.

If h is not constant then f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where

$$R(\omega_1,\omega_2)=\omega_1^{n+1}(\frac{a_m\omega_1^m}{n+m+1}+\frac{a_{m-1}\omega_1^{m-1}}{n+m}+\ldots+\frac{a_0}{n+1})$$
$$-\omega_2^{n+1}(\frac{a_m\omega_2^m}{n+m+1}+\frac{a_{m-1}\omega_2^{m-1}}{n+m}+\ldots+\frac{a_0}{n+1}). \quad \square$$

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