SOME SUBORDINATION AND SUPERORDINATION RESULTS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTIONS ASSOCIATED WITH RUSCHEWEYH DERIVATIVE

ABDUL RAHMAN S. JUMA - FATEH S. AZIZ

Our purpose in this paper is to define a linear operator \( F_{p,q,s}[a_1,m] \), then applying it to obtain some results on subordination and superordination preserving properties of holomorphic multivalent functions in the open unit disc. And sandwich-type result for these holomorphic multivalent functions is also considered.

1. Introduction and definitions

Let \( A(U) \) be the class of functions analytic in \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( A[a,n] \) be the subclass of \( A(U) \) consisting of functions of the form \( f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \ldots \) with \( A_o = A[0,1] \) and \( A = A[1,1] \). Let \( A(p) \) denote the class of all analytic functions of the form

\[
f(z) = z^p + \sum_{n=1+p}^{\infty} a_nz^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots \}; z \in U).
\]

Let \( f \) and \( g \) be members of \( A(U) \). The function \( f(z) \) is said to be subordinate to \( g(z) \), or \( g(z) \) is said to be superordinate to \( f(z) \) if there exists a function

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$w(z)$ analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), such that $f(z) = g(w(z))$ ($z \in U$).

In such a case, we write

$$f \prec g \text{ or } f(z) \prec g(z) \quad (z \in U).$$

If the function $g(z)$ is univalent in $U$, then we have (see [8], [9])

$$f(z) \prec g(z) \quad (z \in U) \text{ if and only if } f(0) = g(0) \text{ and } f(U) \subset g(U).$$

**Definition 1.1** ([8]). Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the differential subordination:

$$\phi(p(z),zp'(z);z) \prec h(z) \quad (z \in U),$$

then $p(z)$ is called a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (2), or more simply a dominant, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (2) is said to be the best dominant.

**Definition 1.2** ([9]). Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let $h(z)$ be analytic in $U$. If $p(z)$ and $\phi(p(z),zp'(z);z)$ are univalent in $U$ and if $p(z)$ satisfies the first order differential superordination:

$$h(z) \prec \phi(p(z),zp'(z);z) \quad (z \in U),$$

then $p(z)$ is called a solution of the differential superordination (3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (3), or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of (3) is said to be the best subordinant.

**Definition 1.3** ([9]). Denote by $Q$ the class of functions $f$ that are analytic and injective on $U \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that

$$f'(\zeta) \neq 0 \quad (\zeta \in \partial U \setminus E(f)).$$

**Definition 1.4** ([9]). A function $L(z,t)$ ($z \in U$, $t \geq 0$) is said to be a subordination chain if $L(z,.)$ is analytic and univalent in $U$ for all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$, and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$. 

For analytic functions $f(z) \in A(p)$, given by (1) and $\phi(z) \in A(p)$ given by

$$\phi(z) = z^p + \sum_{n=1+p}^{\infty} b_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}; z \in U).$$

The Hadamard product (or convolution) of $f(z)$ and $\phi(z)$, is defined by

$$(f \ast \phi)(z) = z^p + \sum_{n=1+p}^{\infty} a_n b_n z^n = (\phi \ast f)(z). \quad (4)$$

For parameters $\alpha_j \in \mathbb{C} \ (j = 1, \ldots, q)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \ (j = 1, \ldots, s)$, the generalized hypergeometric function $qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is defined by the following infinite series (see [3, 4]):

$$qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k z^k}{(\beta_1)_k \ldots (\beta_s)_k k!} \quad (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(a)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & \text{for } k = 0 \\ a(a+1)(a+2) \ldots (a+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \ldots\}. \end{cases}$$

Corresponding to a function $h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^p qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z).$$

Liu-Srivastava [7] defined the operator $H_{p,q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) : A(p) \rightarrow A(p)$ by the following Hadamard product (or convolution)

$$H_{p,q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) f(z) = h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \ast f(z).$$

Recently, Miller and Mocanu [9] considered differential superordinations, as the dual problem of differential subordinations (see [1]). N.E. Cho [2], investigate the subordination and superordination preserving properties of the linear operator $H_{p,q,s}(\alpha_1)$ with the sandwich-type theorems.

For functions $f(z) \in A(p)$, in the form (1) using the $(m + p - 1)th$ order Ruscheweyh derivative $D^{m+p-1}$ for

$$D^{m+p-1} f(z) = \frac{z^p (z^{m-1} f(z))^{m+p-1}}{(m + p - 1)!}$$
and \( m \) is any integer such that \( m > -p \) (see Kumar and Shukla [5, 6]), where, it is easy to see that
\[
D^{m+p-1} f(z) = \frac{z^p}{(1-z)^{m+p}} * f(z).
\]
We define the linear operator \( F_{p,q,s}[\alpha_1, m] : A(p) \to A(p) \) as follows
\[
F_{p,q,s}[\alpha_1, m] f(z) = H_{p,q,s}[\alpha_1] * D^{m+p-1} f(z)
\]
\[
= z^p + \sum_{n=1+p}^{\infty} \Lambda \sigma_{n,p}(\alpha_1) \delta(m + p - 1, n) a_n z^n,
\]
where \( \Lambda = \frac{\Pi_{j=1}^q \Gamma(\beta_j)}{\Pi_{j=1}^q \Gamma(\alpha_j)} \), \( \sigma_{n,p}(\alpha_1) = \frac{\Pi_{j=1}^q \Gamma(\alpha_j + n - p)}{\Pi_{j=1}^q \Gamma(\beta_j + n - p)} \)
and finally \( \delta(m + p - 1, n) = \left( \begin{array}{c} m + p - 1 + n - 1 \\ m + p - 1 \end{array} \right) \).
\[
(5)
\]

The importance of this operator rests on the following relation
\[
z(F_{p,q,s}[\alpha_1, m] f(z)') = \alpha_1 F_{p,q,s}[\alpha_1 + 1, m] f(z) - (\alpha_1 - p) F_{p,q,s}[\alpha_1, m] f(z),
\]
that one can easily verify it by direct calculations and applying (5).

2. A Set of Lemmas

The following lemmas are needed in the proofs of our results.

**Lemma 2.1** ([10]). Let \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \) and let \( h \in A(U) \) with \( h(0) = c \). If
\[
\Re\{\beta h(z) + \gamma\} > 0 \quad (z \in U),
\]
then the solution of the following differential equation
\[
q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in U; \ q'(0) = c),
\]
is analytic in \( U \) and satisfies the inequality
\[
\Re\{\beta q(z) + \gamma\} > 0 \quad (z \in U).
\]

**Lemma 2.2** ([11]). Suppose that the function \( H : \mathbb{C}^2 \to \mathbb{C} \) satisfies the following condition
\[
\Re\{H(is, t)\} \leq 0,
\]
for all real \( s \) and
\[
t \leq -\frac{n(1+s^2)}{2} \quad (n \in \mathbb{N}).
\]
If the function \( p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots \) is analytic in \( U \) and
\[
\Re\{H(p(z),zp'(z))\} > 0 \quad (z \in U),
\]
then
\[
\Re\{p(z)\} > 0 \quad (z \in U).
\]

**Lemma 2.3 ([12]).** Let \( L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots \) with \( a_1(t) \neq 0 \) for all \( t \geq 0 \) and \( \lim_{t \to \infty} |a_1(t)| = \infty \). Suppose that \( L(z,:) \) is analytic in \( U \) for all \( t \geq 0 \), \( L(z,:) \) is continuously differentiable on \([0,\infty)\) for all \( z \in U \). If \( L(z,t) \) satisfies
\[
\Re\left\{\frac{z \partial L(z,t)}{\partial z} \frac{\partial L(z,t)}{\partial t}\right\} > 0 \quad (z \in U; t \geq 0).
\]
and
\[
|L(z,t)| \leq K_0 |a_1(t)|, |z| < r_o < 1, t \geq 0,
\]
for some positive constants \( K_0 \) and \( r_o \), then \( L(z,t) \) is a subordination chain.

**Lemma 2.4 ([11]).** Let \( p \in Q \) with \( p(0) = a \) and let
\[
q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,
\]
be analytic in \( U \) with
\[
q(z) \not\equiv a \quad \text{and} \quad n \geq 1.
\]
If \( q \) is not subordinate to \( p \), then there exists two points
\[
z_o = r_o e^{i\theta} \in U \quad \text{and} \quad \zeta_o \in \partial U \setminus E(q),
\]
such that
\[
q(U_{r_o}) \subset p(U); q(z_o) = p(\zeta_o) \quad \text{and} \quad z_o q'(z_o) = m \zeta_o p'(\zeta_o) \quad (m \geq n).
\]

**Lemma 2.5 ([9]).** Let \( p \in A[a,1] \) and \( \varphi : \mathbb{C}^2 \to \mathbb{C} \). Also set
\[
\varphi(q(z),zq'(z)) \equiv h(z) \quad (z \in U).
\]
If \( L(z,t) = \varphi(q(z),tzq'(z)) \) is a subordination chain and \( p \in A[a,1] \cap Q \), then
\[
h(z) \prec \varphi(q(z),zq'(z)) \quad (z \in U),
\]
implies that
\[
q(z) \prec p(z) \quad (z \in U).
\]
Furthermore, if \( \varphi(q(z),zq'(z)) = h(z) \) has a univalent solution \( q \in Q \), then \( q \) is the best subordinant.
3. Main Results

We shall assume in the remainder of this paper that the parameters $\eta, \alpha_j, j = 1, \ldots, q$ and $\beta_j, j = 1, \ldots, s$ ($q, s \in \mathbb{N}$) are positive real numbers and $(z \in U)$.

**Theorem 3.1.** Let the functions $f, g \in A(p)$ and

$$\Re \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\mu,$$

where

$$\phi(z) = \frac{F_{p,q,s}[\alpha_1 + 1, m]g(z)}{F_{p,q,s}[\alpha_1, m]g(z)} \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^\frac{1}{\eta},$$

and

$$\mu = \frac{1 + (\frac{\alpha_1}{\eta})^2 - |1 - (\frac{\alpha_1}{\eta})^2|}{4\frac{\alpha_1}{\eta}} (\alpha_1 > 0, \eta \geq \alpha_1; z \in U).$$

Then, the following subordination condition

$$\frac{F_{p,q,s}[\alpha_1 + 1, m]f(z)}{F_{p,q,s}[\alpha_1, m]f(z)} \left( \frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p} \right)^\frac{1}{\eta} \prec \frac{F_{p,q,s}[\alpha_1 + 1, m]g(z)}{F_{p,q,s}[\alpha_1, m]g(z)} \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^\frac{1}{\eta},$$

implies that

$$\left( \frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p} \right)^\frac{1}{\eta} \prec \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^\frac{1}{\eta}. $$

Moreover, the function $\left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^\frac{1}{\eta}$ is the best dominant.

**Proof.** Define the functions

$$F(z) = \left( \frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p} \right)^\frac{1}{\eta} \quad \text{and} \quad G(z) = \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^\frac{1}{\eta},$$

we assume here, without loss of generality, that $G(z)$ is analytic, univalent on $\overline{U}$ and

$$G'(\zeta) \neq 0 \quad (|\zeta| = 1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\overline{U}$, so we can use
them in the proof of our result and the results would follow by letting $\rho \to 1$. We first show that, if

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U), \quad (13)$$

then

$$\Re(q(z)) > 0 \quad (z \in U).$$

Now, differentiating the second equation in (12) we get

$$G'(z) = \frac{1}{\eta} \left( \frac{F_{p,q,s} [\alpha_1,m] g(z)}{z^p} \right)^{\frac{1}{\eta}} \frac{1}{F_{p,q,s} [\alpha_1,m] g(z)} \left\{ \alpha_1 F_{p,q,s} [\alpha_1 + 1,m] g(z) - (\alpha_1 - p) F_{p,q,s} [\alpha_1,m] g(z) - p F_{p,q,s} [\alpha_1,m] g(z) \right\}, \quad (15)$$

or

$$\frac{\eta}{\alpha_1} z G'(z) = \left( \frac{F_{p,q,s} [\alpha_1,m] g(z)}{z^p} \right)^{\frac{1}{\eta}} \frac{1}{F_{p,q,s} [\alpha_1,m] g(z)} \left\{ \alpha_1 F_{p,q,s} [\alpha_1 + 1,m] g(z) - F_{p,q,s} [\alpha_1,m] g(z) \right\}, \quad (16)$$

so

$$G(z) + \frac{\eta}{\alpha_1} z G'(z)$$

$$= \left( \frac{F_{p,q,s} [\alpha_1,m] g(z)}{z^p} \right)^{\frac{1}{\eta}} \left\{ 1 + \frac{F_{p,q,s} [\alpha_1 + 1,m] g(z) - F_{p,q,s} [\alpha_1,m] g(z)}{F_{p,q,s} [\alpha_1,m] g(z)} \right\}$$

$$= \frac{F_{p,q,s} [\alpha_1 + 1,m] g(z) - F_{p,q,s} [\alpha_1,m] g(z)}{F_{p,q,s} [\alpha_1,m] g(z)} \left( \frac{F_{p,q,s} [\alpha_1,m] g(z)}{z^p} \right)^{\frac{1}{\eta}} = \phi(z). \quad (17)$$

Differentiating both sides of (17) yields

$$\phi'(z) = G'(z) + \frac{\eta}{\alpha_1} \left\{ z G''(z) + G'(z) \right\}$$

$$= \left\{ \frac{\eta}{\alpha_1} \left( \frac{z G''(z)}{G'(z)} + 1 \right) \right\} G'(z), \quad (18)$$

using (13) in (18) gives

$$\phi'(z) = \left\{ \frac{\eta}{\alpha_1} q(z) + 1 \right\} G'(z). \quad (19)$$
Again differentiating both sides of (19) once more we obtain

\[
\phi''(z) = \left\{ \frac{\eta}{\alpha_1} q(z) + 1 \right\} G''(z) + \frac{\eta}{\alpha_1} q'(z) G'(z). \tag{20}
\]

Now

\[
1 + z\phi''(z) \phi'(z) = 1 + \frac{z\left( \frac{\eta}{\alpha_1} q(z) + 1 \right) G''(z) + \frac{\eta}{\alpha_1} q'(z) G'(z)}{\frac{\eta}{\alpha_1} q(z) + 1} G'(z)
\]

\[
= 1 + \frac{zG''(z)}{G'(z)} + \frac{\eta}{\alpha_1} \frac{zq'(z)}{q(z) + \alpha_1} = h(z). \tag{21}
\]

From (7) and (9) yields

\[
\Re\left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} + \frac{\alpha_1}{\eta} \right\} > 0, \tag{22}
\]

hence (21) and (22) give that

\[
\Re\left\{ h(z) + \frac{\alpha_1}{\eta} \right\} > 0.
\]

Moreover, by Lemma 2.1, we conclude that the differential equation (21) has a solution \( q(z) \in A(U) \) with

\[
q(0) = h(0) = 1.
\]

Let us put

\[
H(u, v) = u + \frac{v}{u + \alpha_1 \eta} + \mu, \tag{23}
\]

where \( \mu \) is given by (9).

From (21) and (23), we obtain

\[
H(q(z), zq'(z)) = q(z) + \frac{zq'(z)}{q(z) + \alpha_1} + \mu, \tag{24}
\]

hence

\[
\Re\{H(q(z), zq'(z))\} > 0. \tag{25}
\]

Now, we proceed to show that

\[
\Re\{H(is, t)\} \leq 0 \ (s \in \mathbb{R}; t \leq -\frac{1}{2}(1 + s^2)). \tag{26}
\]
Indeed, from (23), we have

$$\Re\{H(is, t)\} = \Re\left\{\frac{is + \frac{t}{is} + \alpha}{\eta} + \mu\right\} = \frac{\frac{t\alpha_1}{\eta} + \mu[s^2 + (\frac{\alpha_1}{\eta})^2]}{s^2 + (\frac{\alpha_1}{\eta})^2} \leq -\frac{h_\mu(s)}{2[s^2 + (\frac{\alpha_1}{\eta})^2]},$$

(27)

where

$$h_\mu(s) = (1 + s^2)\frac{\alpha_1}{\eta} - \frac{\alpha_1}{\eta}[s^2 + (\frac{\alpha_1}{\eta})^2] = \frac{\alpha_1}{\eta}[1 - (\frac{\alpha_1}{\eta})^2].$$

(28)

It is clear that $h_\mu(s) \geq 0$, so applying (27) we get that (26) holds true. Thus using Lemma 2.2, yields that

$$\Re\{q(z)\} > 0.$$

Moreover, we see that the condition

$$G'(0) \neq 0,$$

is satisfied. Hence the function $G(z)$ defined by (12) is convex (univalent) in $U$. To prove

$$F(z) \prec G(z),$$

(29)

for the functions $F$ and $G$ defined by (12). We consider the function $L(z, t)$ given by

$$L(z, t) = G(z) + \frac{\eta(1 + t)}{\alpha_1}zG'(z) \quad (0 \leq t < \infty; \quad \alpha_1 > 0, \quad \eta \geq \alpha_1; \quad z \in U).$$

(30)

We note that

$$\frac{\partial L(z,t)}{\partial z}|_{z=0} = G'(0)[\alpha_1 + \eta(1 + t)] \neq 0.$$

This shows that the function

$$L(z, t) = a_1(t)z + \ldots$$

satisfies the condition

$$a_1(t) \neq 0 \quad (0 \leq t < \infty).$$
Further, we have
\[ \Re \left\{ \frac{z \partial L(z,t)}{\partial z} \right\} = \Re \left\{ \frac{\alpha_1}{\eta} + (1+t)q(z) \right\} > 0. \]

Therefore by virtue of Lemma 2.3, \( L(z,t) \) is a subordination chain. It follows from the definition of subordination chain that
\[ \phi(z) = G(z) + \frac{\eta}{\alpha_1} z G'(z) = L(z,0), \]
and
\[ L(z,0) \prec L(z,t) \ (0 \leq t < \infty; z \in U), \]
which implies that
\[ L(\zeta,t) \notin L(U,0) = \phi(U) \ (0 \leq t < \infty; \zeta \in \partial U). \tag{31} \]

If \( F \) is not subordinate to \( G \), by using Lemma 2.4, we know that there exist two points \( z_o \in U \) and \( \zeta_o \in \partial U \), such that
\[ F(z_o) = G(\zeta_o) \text{ and } z_o F'(z_o) = (1+t)\zeta_o G'(\zeta_o) \ (0 \leq t < \infty). \tag{32} \]

Hence, by using (12), (6) (30), (32) and (10) we have
\[
L(\zeta_o,t) = G(\zeta_o) + \frac{\eta(1+t)}{\alpha_1} \zeta_o G'(\zeta_o) \\
= F(z_o) + \frac{\eta}{\alpha_1} z_o F'(z_o) \\
= \left( \frac{F_{p,q,s}[\alpha_1,m] f(z_o)}{z_o^p} \right)^{\frac{1}{\eta}} \left( \frac{F_{p,q,s}[\alpha_1+1,m] f(z_o)}{F_{p,q,s}[\alpha_1,m] f(z_o)} \right) \in \phi(U).
\]

By virtue of the subordination condition (10). This contradicts (31) \( L(\zeta_o,t) \notin \phi(U) \). Therefore, the subordination condition (10) must imply the subordination given by (29). Considering \( F(z) = G(z) \), we see that the function \( G \) is the best dominant.
This completes the proof of Theorem 3.1.

Next, we provide a dual problem of Theorem 3.1, in the sense that the subordinations are replaced by superordinations.

**Theorem 3.2.** Let the functions \( f, g \in A(p) \). Suppose that
\[ \Re \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\mu, \]
where
\[ \phi(z) = \frac{F_{p,q,s}[\alpha_1 + 1, m]g(z)}{F_{p,q,s}[\alpha_1, m]g(z)} \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^{\frac{1}{\eta}}, \]
and \( \mu \) is given by (9).
If the function
\[ \frac{F_{p,q,s}[\alpha_1 + 1, m]f(z)}{F_{p,q,s}[\alpha_1, m]f(z)} \left( \frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p} \right)^{\frac{1}{\eta}}, \]
is univalent in \( U \) and \( \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^{\frac{1}{\eta}} \in Q \). Then the following superordination condition
\[ \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^{\frac{1}{\eta}} \prec \left( \frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p} \right)^{\frac{1}{\eta}}, \]
implies that
\[ \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^{\frac{1}{\eta}} \prec \left( \frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p} \right)^{\frac{1}{\eta}}. \]

Moreover, the function \( \left( \frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p} \right)^{\frac{1}{\eta}} \) is the best subordinant.

**Proof:** Suppose that the functions \( F, G \) and \( q \) are defined by (12) and (13), respectively.
By applying similar method as in the proof of Theorem 3.1, we get
\[ \Re\{q(z)\} > 0 \ (z \in U). \]
So we get the desired result, that \( G \prec F \). To do this, we assume that the function \( L(z,t) \) be defined by (30). Since \( G \) is convex, then by applying a similar method as in Theorem 3.1, we deduce that \( L(z,t) \) is subordination chain. Hence, applying Lemma 2.5, we get that \( G \prec F \).
Moreover, since the differential equation
\[ \phi(z) = G(z) + \frac{\eta}{\alpha_1} z G'(z) = \varphi(G(z), z G'(z)), \]
has a univalent solution \( G \), it is the best subordinant.
This completes the proof of Theorem 3.2. \( \square \)
Combining Theorems 3.1 and 3.2, we obtain the following sandwich-type result.

**Theorem 3.3.** Let the functions $f, g_j \in A(p)$ ($j = 1, 2$), and

$$
\Re \left\{ 1 + \frac{z\phi_j''(z)}{\phi_j'(z)} \right\} > -\mu,
$$

where

$$
\phi_j(z) = \frac{F_{p,q,s}([\alpha_1 + 1, m]g_j(z))}{F_{p,q,s}([\alpha_1, m]g_j(z))} \left( \frac{F_{p,q,s}([\alpha_1, m]g_j(z))}{z^p} \right)^\frac{1}{\eta},
$$

and $\mu$ is given by (9).

If the function

$$
\frac{F_{p,q,s}([\alpha_1 + 1, m]f(z))}{F_{p,q,s}([\alpha_1, m]f(z))} \left( \frac{F_{p,q,s}([\alpha_1, m]f(z))}{z^p} \right)^\frac{1}{\eta},
$$

is univalent in $U$ and $\left( \frac{F_{p,q,s}([\alpha_1, m]f(z))}{z^p} \right)^\frac{1}{\eta} \in Q$. Then the condition

$$
\frac{F_{p,q,s}([\alpha_1 + 1, m]g_1(z))}{F_{p,q,s}([\alpha_1, m]g_1(z))} \left( \frac{F_{p,q,s}([\alpha_1, m]g_1(z))}{z^p} \right)^\frac{1}{\eta} \prec \frac{F_{p,q,s}([\alpha_1 + 1, m]f(z))}{F_{p,q,s}([\alpha_1, m]f(z))} \left( \frac{F_{p,q,s}([\alpha_1, m]f(z))}{z^p} \right)^\frac{1}{\eta} \prec \frac{F_{p,q,s}([\alpha_1 + 1, m]g_2(z))}{F_{p,q,s}([\alpha_1, m]g_2(z))} \left( \frac{F_{p,q,s}([\alpha_1, m]g_2(z))}{z^p} \right)^\frac{1}{\eta},
$$

implies that, for $z \in U$

$$
\left( \frac{F_{p,q,s}([\alpha_1, m]g_1(z))}{z^p} \right)^\frac{1}{\eta} \prec \left( \frac{F_{p,q,s}([\alpha_1, m]f(z))}{z^p} \right)^\frac{1}{\eta} \prec \left( \frac{F_{p,q,s}([\alpha_1, m]g_2(z))}{z^p} \right)^\frac{1}{\eta}.
$$

Moreover, the functions $\left( \frac{F_{p,q,s}([\alpha_1, m]g_1(z))}{z^p} \right)^\frac{1}{\eta}$ and $\left( \frac{F_{p,q,s}([\alpha_1, m]g_2(z))}{z^p} \right)^\frac{1}{\eta}$ are the best subordinant and the best dominant, respectively.

**Proof.** The proof of this theorem consists of the proofs of the Theorems 3.1 and 3.2. \qed
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ABDUL RAHMAN S. JUMA
Department of Mathematics
Alanbar University
Ramadi, Iraq
e-mail: dr_juma@hotmail.com

FATEH S. AZIZ
Department of Mathematics
Salahaddin University
Erbil, Region of Kurdistan, Iraq
e-mail: fatehsaber@gmail.com