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SOME SUBORDINATION AND SUPERORDINATION RESULTS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTIONS ASSOCIATED WITH RUSCHEWEYH DERIVATIVE

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Our purpose in this paper is to define a linear operator $F_{p,q,s}[\alpha_1, m]$, then applying it to obtain some results on subordination and superordination preserving properties of holomorphic multivalent functions in the open unit disc. And sandwich-type result for these holomorphic multivalent functions is also considered.

1. Introduction and definitions

Let $A(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $A[a,n]$ be the subclass of $A(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ with $A_o = A[0,1]$ and $A = A[1,1]$. Let $A(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1+p}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1,2,3,\dots\};\, z \in U). \tag{1}$$

Let f and g be members of $A(U)$. The function $f(z)$ is said to be subordinate to $g(z)$, or $g(z)$ is said to be superordinate to $f(z)$ if there exists a function

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$w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$.

In such a case, we write

$$f \prec g \ or \ f(z) \prec g(z) \ (z \in U).$$

If the function $g(z)$ is univalent in U, then we have (see [8], [9])

$$f(z) \prec g(z) \ (z \in U) \ if \ and \ only \ if \ f(0) = g(0) \ and \ f(U) \subset g(U).$$

Definition 1.1 ([8])**.** Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the differential subordination:

$$\phi(p(z), zp'(z); z) \prec h(z) \ (z \in U), \tag{2}$$

then $p(z)$ is called a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (2), or more simply a dominant, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (2) is said to be the best dominant.

Definition 1.2 ([9])**.** Let $\varphi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let $h(z)$ be analytic in U. If p and $\varphi(p(z), zp'(z); z)$ are univalent in U and if $p(z)$ satisfies the first order differential superordination:

$$h(z) \prec \varphi(p(z), zp'(z); z) \ (z \in U), \tag{3}$$

then $p(z)$ is called a solution of the differential superordination (3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (3), or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (3) is said to be the best subordinant.

Definition 1.3 ([9])**.** Denote by Q the class of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that

$$f'(\zeta) \neq 0 \ (\zeta \in \partial U \setminus E(f)).$$

Definition 1.4 ([9])**.** A function $L(z,t)$ $(z \in U, t \geq 0)$ is said to be a subordination chain if $L(.;t)$ is analytic and univalent in U for all $t \geq 0$, $L(z;.)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$, and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

For analytic functions $f(z) \in A(p)$, given by (1) and $\phi(z) \in A(p)$ given by

$$\phi(z) = z^p + \sum_{n=1+p}^{\infty} b_n z^n \quad (p \in \mathbb{N} = \{1,2,3,\dots\}; z \in U).$$

The Hadamard product (or convolution) of $f(z)$ and $\phi(z)$, is defined by

$$(f * \phi)(z) = z^p + \sum_{n=1+p}^{\infty} a_n b_n z^n = (\phi * f)(z). \tag{4}$$

For parameters $\alpha_j \in \mathbb{C}\,(j = 1,\dots,q)$ and $\beta_j \in \mathbb{C}\backslash\{0,-1,-2,\dots\}(j = 1,\dots,s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ is defined by the following infinite series (see [3, 4]):

$$ {}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k} \frac{z^k}{k!}$$
$$(q \le s+1; q,s \in \mathbb{N}_o = \mathbb{N} \cup \{0\}; z \in U),$$

where $(a)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & \text{for } k = 0 \\ a(a+1)(a+2)\dots(a+k-1) & \text{for } k \in \mathbb{N} = \{1,2,3,\dots\}. \end{cases}$$

Corresponding to a function $h_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ defined by

$$h_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) = z^p\,{}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z).$$

Liu-Srivastava [7] defined the operator $H_{p,q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) : A(p) \to A(p)$ by the following Hadamard product (or convolution)

$$H_{p,q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) f(z) = h_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) * f(z).$$

Recently, Miller and Mocanu [9] considered differential superordinations, as the dual problem of differential subordinations (see [1]). N.E. Cho [2], investigate the subordination and superordination preserving properties of the linear operator $H_{p,q,s}(\alpha_1)$ with the sandwich-type theorems.

For functions $f(z) \in A(p)$, in the form (1) using the $(m+p-1)th$ order Ruscheweyh derivative D^{m+p-1} for

$$D^{m+p-1} f(z) = \frac{z^p (z^{m-1} f(z))^{m+p-1}}{(m+p-1)!}$$

and m is any integer such that $m > -p$ (see Kumar and Shukla [5, 6]), where, it is easy to see that

$$D^{m+p-1}f(z) = \frac{z^p}{(1-z)^{m+p}} * f(z).$$

We define the linear operator $F_{p,q,s}[\alpha_1, m] : A(p) \to A(p)$ as follows

$$\begin{aligned} F_{p,q,s}[\alpha_1,m]f(z) &= Hp,q,s[\alpha_1] * D^{m+p-1}f(z) \\ &= z^p + \sum_{n=1+p}^{\infty} \Lambda\sigma_{n,p}(\alpha_1)\delta(m+p-1,n)a_n z^n, \\ \text{where } \Lambda &= \frac{\prod_{j=1}^{s}\Gamma(\beta_j)}{\prod_{j=1}^{q}\Gamma(\alpha_j)},\ \sigma_{n,p}(\alpha_1) = \frac{\prod_{j=1}^{q}\Gamma(\alpha_j+n-p)}{\prod_{j=1}^{s}\Gamma(\beta_j+n-p)} \\ \text{and finally } \delta(m+p-1,n) &= \binom{m+p-1+n-1}{m+p-1}. \end{aligned} \tag{5}$$

The importance of this operator rests on the following relation

$$z(F_{p,q,s}[\alpha_1,m]f(z))' = \alpha_1 F_{p,q,s}[\alpha_1+1,m]f(z) - (\alpha_1 - p)F_{p,q,s}[\alpha_1,m]f(z), \tag{6}$$

that one can easily verify it by direct calculations and applying (5).

2. A Set of Lemmas

The following lemmas are needed in the proofs of our results.

Lemma 2.1 ([10]). *Let $\beta,\gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in A(U)$ with $h(0) = c$. If*

$$\Re\{\beta h(z) + \gamma\} > 0 \ \ (z \in U),$$

then the solution of the following differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \ \ (z \in U;\ q(0) = c),$$

is analytic in U and satisfies the inequality

$$\Re\{\beta q(z) + \gamma\} > 0 \ \ (z \in U).$$

Lemma 2.2 ([11]). *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the following condition*

$$\Re\{H(is,t)\} \leq 0,$$

for all real s and

$$t \leq -\frac{n(1+s^2)}{2} \quad (n \in \mathbb{N}).$$

If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ *is analytic in* U *and*

$$\Re\{H(p(z), zp'(z))\} > 0 \quad (z \in U),$$

then

$$\Re\{p(z)\} > 0 \quad (z \in U).$$

Lemma 2.3 ([12])**.** *Let* $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ *with* $a_1(t) \neq 0$ *for all* $t \geq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$*. Suppose that* $L(.;t)$ *is analytic in* U *for all* $t \geq 0$*, L(z;.) is continuously differentiable on* $[0,\infty)$ *for all* $z \in U$*. If* $L(z,t)$ *satisfies*

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in U; t \geq 0).$$

and

$$|L(z,t)| \leq K_o |a_1(t)|, |z| < r_o < 1, t \geq 0,$$

for some positive constants K_o *and* r_o*, then* $L(z,t)$ *is a subordination chain.*

Lemma 2.4 ([11])**.** *Let* $p \in Q$ *with* $p(0) = a$ *and let*

$$q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,$$

be analytic in U *with*

$$q(z) \not\equiv a \quad \text{and} \quad n \geq 1.$$

If q is not subordinate to p, then there exists two points

$$z_o = r_o e^{i\theta} \in U \quad \text{and} \quad \zeta_o \in \partial U \backslash E(q),$$

such that

$$q(U_{r_o}) \subset p(U); q(z_o) = p(\zeta_o) \quad \text{and} \quad z_o q'(z_o) = m\zeta_o p'(\zeta_o) \quad (m \geq n).$$

Lemma 2.5 ([9])**.** *Let* $p \in A[a,1]$ *and* $\varphi : \mathbb{C}^2 \to \mathbb{C}$*. Also set*

$$\varphi(q(z), zq'(z)) \equiv h(z) \quad (z \in U).$$

If $L(z,t) = \varphi(q(z), tzq'(z))$ *is a subordination chain and* $p \in A[a,1] \cap Q$*, then*

$$h(z) \prec \varphi(q(z), zq'(z)) \quad (z \in U),$$

implies that

$$q(z) \prec p(z) \quad (z \in U).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in Q$*, then q is the best subordinant.*

3. Main Results

We shall assume in the remainder of this paper that the parameters $\eta, \alpha_j, j = 1, \ldots, q$ and $\beta_j, j = 1, \ldots, s$ $(q, s \in \mathbb{N})$ are positive real numbers and $(z \in U)$.

Theorem 3.1. *Let the functions $f, g \in A(p)$ and*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\mu, \tag{7}$$

where

$$\phi(z) = \frac{F_{p,q,s}[\alpha_1 + 1, m]g(z)}{F_{p,q,s}[\alpha_1, m]g(z)}\left(\frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p}\right)^{\frac{1}{\eta}}, \tag{8}$$

and

$$\mu = \frac{1 + (\frac{\alpha_1}{\eta})^2 - |1 - (\frac{\alpha_1}{\eta})^2|}{4\frac{\alpha_1}{\eta}} \quad (\alpha_1 > 0,\ \eta \geq \alpha_1;\ z \in U). \tag{9}$$

Then, the following subordination condition

$$\frac{F_{p,q,s}[\alpha_1 + 1, m]f(z)}{F_{p,q,s}[\alpha_1, m]f(z)}\left(\frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p}\right)^{\frac{1}{\eta}} \prec \frac{F_{p,q,s}[\alpha_1 + 1, m]g(z)}{F_{p,q,s}[\alpha_1, m]g(z)}\left(\frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p}\right)^{\frac{1}{\eta}}, \tag{10}$$

implies that

$$\left(\frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p}\right)^{\frac{1}{\eta}} \prec \left(\frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p}\right)^{\frac{1}{\eta}}. \tag{11}$$

Moreover, the function $(\frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p})^{\frac{1}{\eta}}$ is the best dominant.

Proof. Define the functions

$$F(z) = \left(\frac{F_{p,q,s}[\alpha_1, m]f(z)}{z^p}\right)^{\frac{1}{\eta}} \quad \text{and} \quad G(z) = \left(\frac{F_{p,q,s}[\alpha_1, m]g(z)}{z^p}\right)^{\frac{1}{\eta}}, \tag{12}$$

we assume here, without loss of generality, that $G(z)$ is analytic, univalent on $\overline{U}$ and

$$G'(\zeta) \neq 0 \ \ (|\zeta| = 1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\overline{U}$, so we can use them in the proof of our result and the results would follow by letting $\rho \to 1$. We first show that, if

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U), \tag{13}$$

then

$$\Re(q(z)) > 0 \quad (z \in U).$$

Now, differentiating the second equation in (12) we get $G'(z) =$

$$= \frac{1}{\eta}\left(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p}\right)^{\frac{1}{\eta}-1} \frac{z^p(F_{p,q,s}[\alpha_1,m]g(z))' - pz^{p-1}F_{p,q,s}[\alpha_1,m]g(z)}{z^{2p}}. \tag{14}$$

Applying (6) in (14) we obtain

$$zG'(z) = \frac{1}{\eta}\left(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p}\right)^{\frac{1}{\eta}} \frac{1}{F_{p,q,s}[\alpha_1,m]g(z)} \left\{\alpha_1 F_{p,q,s}[\alpha_1+1,m]g(z) - (\alpha_1 - p)F_{p,q,s}[\alpha_1,m]g(z) - pF_{p,q,s}[\alpha_1,m]g(z)\right\}, \tag{15}$$

or

$$\frac{\eta}{\alpha_1} zG'(z) = \left(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p}\right)^{\frac{1}{\eta}} \frac{F_{p,q,s}[\alpha_1+1,m]g(z) - F_{p,q,s}[\alpha_1,m]g(z)}{F_{p,q,s}[\alpha_1,m]g(z)}, \tag{16}$$

so

$$\begin{aligned} G(z) + \frac{\eta}{\alpha_1} zG'(z) &= \left(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p}\right)^{\frac{1}{\eta}} \left\{1 + \frac{F_{p,q,s}[\alpha_1+1,m]g(z) - F_{p,q,s}[\alpha_1,m]g(z)}{F_{p,q,s}[\alpha_1,m]g(z)}\right\} \\ &= \frac{F_{p,q,s}[\alpha_1+1,m]g(z)}{F_{p,q,s}[\alpha_1,m]g(z)} \left(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p}\right)^{\frac{1}{\eta}} = \phi(z). \end{aligned} \tag{17}$$

Differentiating both sides of (17) yields

$$\begin{aligned} \phi'(z) &= G'(z) + \frac{\eta}{\alpha_1}\{zG''(z) + G'(z)\} \\ &= \left\{\frac{\eta}{\alpha_1}\left(\frac{zG''(z)}{G'(z)} + 1\right) + 1\right\} G'(z), \end{aligned} \tag{18}$$

using (13) in (18) gives

$$\phi'(z) = \left\{\frac{\eta}{\alpha_1} q(z) + 1\right\} G'(z). \tag{19}$$

Again differentiating both sides of (19) once more we obtain

$$\phi''(z) = \left\{ \frac{\eta}{\alpha_1} q(z) + 1 \right\} G''(z) + \frac{\eta}{\alpha_1} q'(z) G'(z) \}. \tag{20}$$

Now

$$\begin{aligned} 1 + \frac{z\phi''(z)}{\phi'(z)} &= 1 + \frac{z\{(\frac{\eta}{\alpha_1} q(z) + 1) G''(z) + \frac{\eta}{\alpha_1} q'(z) G'(z)\}}{\{\frac{\eta}{\alpha_1} q(z) + 1\} G'(z)} \\ &= 1 + \frac{zG''(z)}{G'(z)} + \frac{\frac{\eta}{\alpha_1} z q'(z)}{\frac{\eta}{\alpha_1} q(z) + 1} \\ &= q(z) + \frac{zq'(z)}{q(z) + \frac{\alpha_1}{\eta}} = h(z). \end{aligned} \tag{21}$$

From (7) and (9) yields

$$\Re\left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} + \frac{\alpha_1}{\eta} \right\} > 0, \tag{22}$$

hence (21) and (22) give that

$$\Re\left\{ h(z) + \frac{\alpha_1}{\eta} \right\} > 0.$$

Moreover, by Lemma 2.1, we conclude that the differential equation (21) has a solution $q(z) \in A(U)$ with

$$q(0) = h(0) = 1.$$

Let us put

$$H(u,v) = u + \frac{v}{u + \frac{\alpha_1}{\eta}} + \mu, \tag{23}$$

where μ is given by (9).
From (21) and (23), we obtain

$$H(q(z), zq'(z)) = q(z) + \frac{zq'(z)}{q(z) + \frac{\alpha_1}{\eta}} + \mu, \tag{24}$$

hence

$$\Re\{H(q(z), zq'(z))\} > 0. \tag{25}$$

Now, we proceed to show that

$$\Re\{H(is,t)\} \leq 0 \quad (s \in \mathbb{R}; t \leq -\frac{1}{2}(1+s^2)). \tag{26}$$

Indeed, from (23), we have

$$\begin{aligned}\Re\{H(is,t)\} &= \Re\left\{is+\frac{t}{is+\frac{\alpha_1}{\eta}}+\mu\right\}\\ &= \frac{\frac{t\alpha_1}{\eta}+\mu[s^2+(\frac{\alpha_1}{\eta})^2]}{s^2+(\frac{\alpha_1}{\eta})^2} \\ &\le -\frac{h_\mu(s)}{2[s^2+(\frac{\alpha_1}{\eta})^2]},\end{aligned} \tag{27}$$

where

$$\begin{aligned}h_\mu(s) &= (1+s^2)\frac{\alpha_1}{\eta}-\frac{\alpha_1}{\eta}[s^2+(\frac{\alpha_1}{\eta})^2]\\ &= \frac{\alpha_1}{\eta}[1-(\frac{\alpha_1}{\eta})^2].\end{aligned} \tag{28}$$

It is clear that $h_\mu(s) \ge 0$, so applying (27) we get that (26) holds true.
Thus using Lemma 2.2, yields that

$$\Re\{q(z)\} > 0.$$

Moreover, we see that the condition

$$G'(0) \neq 0,$$

is satisfied. Hence the function $G(z)$ defined by (12) is convex (univalent) in U.
To prove

$$F(z) \prec G(z), \tag{29}$$

for the functions F and G defined by (12). We consider the function $L(z,t)$ given by

$$L(z,t) = G(z)+\frac{\eta(1+t)}{\alpha_1}zG'(z) \ \ (0 \le t < \infty;\ \alpha_1 > 0,\ \eta \ge \alpha_1;\ \ z \in U). \tag{30}$$

We note that

$$\frac{\partial L(z,t)}{\partial z}|_{z=0} = G'(0)[\alpha_1+\eta(1+t)] \neq 0.$$

This shows that the function

$$L(z,t) = a_1(t)z+\dots$$

satisfies the condition

$$a_1(t) \neq 0 \ \ (0 \le t < \infty).$$

Further, we have

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \Re\left\{\frac{\alpha_1}{\eta} + (1+t)q(z)\right\} > 0.$$

Therefore by virtue of Lemma 2.3, $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{\eta}{\alpha_1} zG'(z) = L(z,0),$$

and

$$L(z,0) \prec L(z,t) \quad (0 \le t < \infty; z \in U),$$

which implies that

$$L(\zeta,t) \notin L(U,0) = \phi(U) \quad (0 \le t < \infty; \zeta \in \partial U). \tag{31}$$

If F is not subordinate to G, by using Lemma 2.4, we know that there exist two points $z_o \in U$ and $\zeta_o \in \partial U$, such that

$$F(z_0) = G(\zeta_o) \text{ and } z_o F'(z_o) = (1+t)\zeta_o G'(\zeta_o) \quad (0 \le t < \infty). \tag{32}$$

Hence, by using (12), (6) (30), (32) and (10) we have

$$\begin{aligned} L(\zeta_o,t) &= G(\zeta_o) + \frac{\eta(1+t)}{\alpha_1}\zeta_o G'(\zeta_o) \\ &= F(z_o) + \frac{\eta}{\alpha_1} z_o F'(z_o) \\ &= \left(\frac{F_{p,q,s}[\alpha_1,m]f(z_o)}{z_o^p}\right)^{\frac{1}{\eta}} \left(\frac{F_{p,q,s}[\alpha_1+1,m]f(z_o)}{F_{p,q,s}[\alpha_1,m]f(z_o)}\right) \in \phi(U). \end{aligned}$$

By virtue of the subordination condition (10). This contradicts (31) $L(\zeta_o,t) \notin \phi(U)$. Therefore, the subordination condition (10) must imply the subordination given by (29). Considering $F(z) = G(z)$, we see that the function G is the best dominant.

This completes the proof of Theorem 3.1. □

Next, we provide a dual problem of Theorem 3.1, in the sense that the subordinations are replaced by superordinations.

Theorem 3.2. *Let the functions $f,g \in A(p)$. Suppose that*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\mu,$$

where

$$\phi(z) = \frac{F_{p,q,s}[\alpha_1+1,m]g(z)}{F_{p,q,s}[\alpha_1,m]g(z)} \left(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p}\right)^{\frac{1}{\eta}},$$

and μ is given by (9).
If the function

$$\frac{F_{p,q,s}[\alpha_1+1,m]f(z)}{F_{p,q,s}[\alpha_1,m]f(z)} \left(\frac{F_{p,q,s}[\alpha_1,m]f(z)}{z^p}\right)^{\frac{1}{\eta}},$$

is univalent in U and $(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p})^{\frac{1}{\eta}} \in Q$. Then the following superordination condition

$$\frac{F_{p,q,s}[\alpha_1+1,m]g(z)}{F_{p,q,s}[\alpha_1,m]g(z)} \left(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p}\right)^{\frac{1}{\eta}} \prec \frac{F_{p,q,s}[\alpha_1+1,m]f(z)}{F_{p,q,s}[\alpha_1,m]f(z)} \left(\frac{F_{p,q,s}[\alpha_1,m]f(z)}{z^p}\right)^{\frac{1}{\eta}},$$

implies that

$$\left(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p}\right)^{\frac{1}{\eta}} \prec \left(\frac{F_{p,q,s}[\alpha_1,m]f(z)}{z^p}\right)^{\frac{1}{\eta}}.$$

Moreover, the function $(\frac{F_{p,q,s}[\alpha_1,m]g(z)}{z^p})^{\frac{1}{\eta}}$ is the best subordinant.

Proof. Suppose that the functions F, G and q are defined by (12) and (13), respectively.
By applying similar method as in the proof of Theorem 3.1, we get

$$\Re\{q(z)\} > 0 \quad (z \in U).$$

So we get the desired result, that $G \prec F$. To do this, we assume that the function $L(z,t)$ be defined by (30). Since G is convex, then by applying a similar method as in Theorem 3.1, we deduce that $L(z,t)$ is subordination chain. Hence, applying Lemma 2.5, we get that $G \prec F$.
Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{\eta}{\alpha_1} zG'(z) = \varphi(G(z), zG'(z)),$$

has a univalent solution G, it is the best subordinant.
This completes the proof of Theorem 3.2. □

Combining Theorems 3.1 and 3.2, we obtain the following sandwich-type result.

Theorem 3.3. *Let the functions* $f, g_j \in A(p) \quad (j = 1,2)$, *and*

$$\Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\mu, \tag{33}$$

where

$$\phi_j(z) = \frac{F_{p,q,s}[\alpha_1+1,m]g_j(z)}{F_{p,q,s}[\alpha_1,m]g_j(z)}\left(\frac{F_{p,q,s}[\alpha_1,m]g_j(z)}{z^p}\right)^{\frac{1}{\eta}},$$

and μ *is given by (9).*
If the function

$$\frac{F_{p,q,s}[\alpha_1+1,m]f(z)}{F_{p,q,s}[\alpha_1,m]f(z)}\left(\frac{F_{p,q,s}[\alpha_1,m]f(z)}{z^p}\right)^{\frac{1}{\eta}},$$

is univalent in U *and* $(\frac{F_{p,q,s}[\alpha_1,m]f(z)}{z^p})^{\frac{1}{\eta}} \in Q$. *Then the condition*

$$\frac{F_{p,q,s}[\alpha_1+1,m]g_1(z)}{F_{p,q,s}[\alpha_1,m]g_1(z)}\left(\frac{F_{p,q,s}[\alpha_1,m]g_1(z)}{z^p}\right)^{\frac{1}{\eta}} \prec \frac{F_{p,q,s}[\alpha_1+1,m]f(z)}{F_{p,q,s}[\alpha_1,m]f(z)}\left(\frac{F_{p,q,s}[\alpha_1,m]f(z)}{z^p}\right)^{\frac{1}{\eta}} \prec \frac{F_{p,q,s}[\alpha_1+1,m]g_2(z)}{F_{p,q,s}[\alpha_1,m]g_2(z)}\left(\frac{F_{p,q,s}[\alpha_1,m]g_2(z)}{z^p}\right)^{\frac{1}{\eta}},$$

implies that, for $z \in U$

$$\left(\frac{F_{p,q,s}[\alpha_1,m]g_1(z)}{z^p}\right)^{\frac{1}{\eta}} \prec \left(\frac{F_{p,q,s}[\alpha_1,m]f(z)}{z^p}\right)^{\frac{1}{\eta}} \prec \left(\frac{F_{p,q,s}[\alpha_1,m]g_2(z)}{z^p}\right)^{\frac{1}{\eta}}.$$

Moreover, the functions $(\frac{F_{p,q,s}[\alpha_1,m]g_1(z)}{z^p})^{\frac{1}{\eta}}$ *and* $(\frac{F_{p,q,s}[\alpha_1,m]g_2(z)}{z^p})^{\frac{1}{\eta}}$ *are the best subordinant and the best dominant, respectively.*

Proof. The proof of this theorem consists of the proofs of the Theorems 3.1 and 3.2. □

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