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SUBCLASS OF HARMONIC STARLIKE FUNCTIONS ASSOCIATED WITH SALAGEAN DERIVATIVE

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The purpose of the present paper is to establish some results involving coefficient conditions, distortion bounds, extreme points, convolution, and convex combinations for a new class of harmonic univalent functions in the open unit disc associated with the Salagean operator. We also discuss a class preserving integral operator. Relevant connections of the results presented here with various known results are briefly indicated.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that $u = \Re(U)$ and $v = Im(V)$. Then

$$f(z) = h(z) + \overline{g(z)},$$

where h and g are respectively, the analytic functions $(U+V)/2$ and $(U-V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by

$$J_f = \left|h'(z)\right|^2 - \left|g'(z)\right|^2.$$

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The mapping $z \to f(z)$ is orientation preserving and locally one-to-one in D if and only if $J_f > 0$ in D. The function $f = h + \overline{g}$ is said to be harmonic univalent in D if the mapping $z \to f(z)$ is orientation preserving, harmonic and one-to-one in D. We call h the analytic part and g the co-analytic part of f. See Clunie and Sheil-Small [2].

Denote by H the class of functions $f = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in H$, we may express the analytic functions f and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \ |b_1| < 1. \tag{1}$$

Note that H reduces to the class S of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

A function $f = h + \overline{g}$ with h and g be given by (1) is said to be harmonic starlike of order $\beta, (0 \leq \beta < 1)$ for $|z| = r < 1$, if

$$\frac{\partial}{\partial \theta}(\arg f(re^{i\theta})) = \Re \left\{ \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} \geq \beta.$$

The class of all harmonic starlike functions of order β is denoted by $S_H^*(\beta)$ and extensively studied by Jahangiri [6]. The case $\beta = 0$ and $b_1 = 0$ were studied by Silverman and Silvia [19] and Silverman [18], for other related works of the class H, (see also [1], [3], [4], [5], [7], [8], [9], [12], [15], [16]).

Definition 1.1. Let $D^n f = D^n h + \overline{D^n g}$ with h and g be given by (1). Where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad \overline{D^n g(z)} = \sum_{k=1}^{\infty} k^n \overline{b}_k \overline{z}^k, \ |b_1| < 1. \tag{3}$$

Then $f \in S_{H,n}^*(\alpha, \beta)$ if and only if, for $\beta, (0 \leq \beta < 1)$ and $\alpha \geq 0$,

$$\Re \left\{ \frac{\alpha D^{n+2} h(z) + (1-\alpha) D^{n+1} h(z) + \overline{\alpha D^{n+2} g(z)} + (\alpha - 1)\overline{D^{n+1} g(z)}}{D^n h(z) + \overline{D^n g(z)}} \right\} \geq \beta.$$

We note that for $\alpha = 0$ and $n = 0$, the class $S^*_{H,n}(\alpha,\beta)$ reduces to the class $S^*_H(\beta)$. Also we note that

$$\begin{aligned} S^*_{H,0}(\alpha,\beta) &= S^*_H(\alpha,\beta) \\ &= \left\{ f(z) \in H : \Re\left\{ \frac{\alpha z^2 h''(z) + z h'(z) + \overline{\alpha z^2 g''(z) + (2\alpha-1) z g'(z)}}{h(z)+\overline{g(z)}} \right\} \geq \beta \right\}. \end{aligned}$$

Further, if the co-analytic part $g(z)$ is zero the class $S^*_{H,n}(\alpha,\beta)$ reduces to the class $P_n(\alpha,\beta)$ of function $f \in S$ which satisfy the condition

$$\Re\left\{ \frac{az^2 (D^n f)'' + z (D^n f)'}{D^n f(z)} \right\} \geq \beta$$

for some $\beta (0 \leq \beta < 1), \alpha \geq 0, \frac{f(z)}{z} \neq 0$ and $z \in U$.

Observe that the classes $P_0(\alpha,\beta)$ and $P_0(\alpha,0)$ were introduced and studied by many authors such as Obradovic and Joshi [13], Padmanabhan [14], Liu and Owa [11], Xu and Yang [21], Singh and Gupta [20] and Lashin [10]. We also note that for $\alpha = 0$, the class $P_0(0,\beta)$ was studied by Silverman [17].

2. Main Results

We begin with the statement of the following lemma due to Jahangiri [6].

Lemma 2.1. *Let $f = h + \overline{g}$ with h and g of the form (1). Furthermore, let*

$$\sum_{k=2}^{\infty} \frac{k-\beta}{1-\beta} |a_k| + \sum_{k=1}^{\infty} \frac{k+\beta}{1-\beta} |b_k| \leq 1$$

where $0 \leq \beta < 1$. Then f is harmonic, orientation preserving, univalent in U, and $f \in S^*_H(\beta)$.

Theorem 2.2. *Let $f = h + \overline{g}$ with h and g of the form (1). If*

$$\sum_{k=2}^{\infty} k^n \frac{\alpha k(k-1) + k - \beta}{1-\beta} |a_k| + \sum_{k=1}^{\infty} k^n \frac{\alpha k(k+1) + k + \beta}{1-\beta} |b_k| \leq 1 \tag{4}$$

for some $\beta, (0 \leq \beta < 1)$ and $\alpha \geq 0$, then f is harmonic, orientation preserving, univalent in U and $f \in S^*_{H,n}(\alpha,\beta)$.

Proof. Since

$$\frac{k-\beta}{1-\beta} \leq k^n \frac{\alpha k(k-1)+k-\beta}{1-\beta} \quad \text{and} \quad \frac{k+\beta}{1-\beta} \leq k^n \frac{\alpha k(k+1)+k+\beta}{1-\beta}, (k \geq 1),$$

it follows from Lemma 2.1 , that $f \in S_H^*(\beta)$ and hence f is harmonic, orientation preserving and univalent in U. Now, we only need to show that if (4) holds then

$$\Re\left\{\frac{\alpha D^{n+2}h(z)+(1-\alpha)D^{n+1}h(z)+\overline{\alpha D^{n+2}g(z)}+(\alpha-1)\overline{D^{n+1}g(z)}}{D^n h(z)+\overline{D^n g(z)}}\right\}$$
$$=\Re\frac{A(z)}{B(z)}\geq\beta.$$

Using the fact that $\Re(w)\geq\beta$ if and only if $|1-\beta+w|\geq|1+\beta-w|$, it suffices to show that

$$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|\geq 0, \tag{5}$$

where

$$A(z)=\alpha D^{n+2}h(z)+(1-\alpha)D^{n+1}h(z)+\overline{\alpha D^{n+2}g(z)}+(\alpha-1)\overline{D^{n+1}g(z)}$$

and

$$B(z)=D^n h(z)+\overline{D^n g(z)}$$

Substituting for $A(z)$ and $B(z)$ in (5), we obtain

$$\begin{aligned}
&|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|\\
&=\Bigg|(2-\beta)z+\sum_{k=2}^{\infty}k^n\left[(\alpha k+1)(k-1)+2-\beta\right]a_k z^k\\
&\quad+\sum_{k=1}^{\infty}k^n\left[\alpha k(k+1)-(k-1+\beta)\right]\overline{b_k}\overline{z^k}\Bigg|\\
&\quad-\Bigg|-\beta z+\sum_{k=2}^{\infty}k^n\left[(\alpha k+1)(k-1)-\beta\right]a_k z^k\\
&\quad+\sum_{k=1}^{\infty}k^n\left[\alpha k(k+1)-(k+1+\beta)\right]\overline{b_k}\overline{z^k}\Bigg|\\
&\geq(2-\beta)|z|-\sum_{k=2}^{\infty}k^n\left[(\alpha k+1)(k-1)+2-\beta\right]|a_k||z|^k\\
&\quad-\sum_{k=1}^{\infty}k^n\left|\alpha k(k+1)-(k-1+\beta)\right||b_k||z|^k-\beta|z|\\
&\quad-\sum_{k=2}^{\infty}k^n\left[(\alpha k+1)(k-1)-\beta\right]|a_k||z|^k-\sum_{k=1}^{\infty}k^n\left|\alpha k(k+1)-(k+1+\beta)\right||b_k||z|^k
\end{aligned}$$

$$\geq 2(1-\beta)|z| - 2\sum_{k=2}^{\infty} k^n [\alpha k(k-1)+k-\beta] |a_k| \left|z^k\right|$$
$$- 2\sum_{k=1}^{\infty} k^n [\alpha k(k+1)+(k+\beta)] |b_k| |z|^k$$

$$\geq 2(1-\beta)|z| \left\{ 1 - \sum_{k=2}^{\infty} k^n \tfrac{[\alpha k(k-1)+k-\beta]}{1-\beta} |a_k| - \sum_{k=1}^{\infty} k^n \tfrac{[\alpha k(k+1)+k+\beta]}{1-\beta} |b_k| \right\} \geq 0,$$

by the given condition (4). □

The Harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1-\beta}{k^n [\alpha k(k-1)+k-\beta]} x_k z^k + \sum_{k=1}^{\infty} \frac{1-\beta}{k^n [\alpha k(k+1)+k+\beta]} \overline{y}_k \overline{z}^k \quad (6)$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given in Theorem 2.2 is sharp.

The functions of the form (6) are in $S^*_{H.n}(\alpha,\beta)$ since

$$\sum_{k=2}^{\infty} k^n \frac{[\alpha k(k-1)+k-\beta]}{1-\beta} |a_k| + \sum_{k=1}^{\infty} k^n \frac{[\alpha k(k+1)+k+\beta]}{1-\beta} |b_k|$$
$$= \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1.$$

We note that:

(i) when $n=0$ and $\alpha=0$ in Theorem 2.2, we obtain the result obtained by Jahangiri [[6], Theorem 1];

(ii) when $n=0$ and $g(z)=0$ in Theorem 2.2, we obtain the result obtained by Lashin [[10], Theorem 2.1].

We denote by $\overline{S}^*_{H,n}(\alpha,\beta)$ to the class of functions $f \in S^*_{H,n}(\alpha,\beta)$ whose coefficients satisfy the condition (4).

Theorem 2.3. *Let* $0 \leq \alpha_1 < \alpha_2$ *and* $0 \leq \beta < 1$. *Then* $\overline{S}^*_{H,n}(\alpha_2,\beta) \subset \overline{S}^*_{H,n}(\alpha_1,\beta)$.

Proof. It follows from (4) that

$$\sum_{k=2}^{\infty} k^n \frac{[\alpha_1 k(k-1)+k-\beta]}{1-\beta} |a_k| + \sum_{k=1}^{\infty} k^n \frac{[\alpha_1 k(k+1)+k+\beta]}{1-\beta} |b_k|$$
$$< \sum_{k=2}^{\infty} k^n \frac{[\alpha_2 k(k-1)+k-\beta]}{1-\beta} |a_k| + \sum_{k=1}^{\infty} k^n \frac{[\alpha_2 k(k+1)+k+\beta]}{1-\beta} |b_k| \leq 1$$

for $f \in \overline{S}^*_{H,n}(\alpha_2,\beta)$. Hence $f \in \overline{S}^*_{H,n}(\alpha_1,\beta)$. □

Theorem 2.4. *Let $\alpha > 0,\ 0 \leq \beta < 1$. Then*

$$\overline{S}^*_{H,n+1}(\alpha,\beta) \subset \overline{S}^*_{H,n}(\alpha,\beta)$$

The proof follows immediately from Theorem 2.2.

In the following Corollary we show that the functions in $\overline{S}^*_{H,n}(\alpha,\beta)$ are starlike harmonic in U.

Corollary 2.5. *For $0 \leq \alpha$ and $0 \leq \beta < 1$,*

$$\overline{S}^*_{H,n}(\alpha,\beta) \subset S^*_H(\beta).$$

The proof is now immediate from Theorems 2.3 and 2.4.

3. Distortion Bounds and Extreme Points

Now we obtain the distortion bounds for functions in the class $\overline{S}^*_{H,n}(\alpha,\beta)$.

Theorem 3.1. *Let $f = h + \overline{g}$ with h and g of the form (1) and $f \in \overline{S}^*_{H,n}(\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{2^n}\left[\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right]r^2 \tag{7}$$

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{2^n}\left[\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right]r^2 \tag{8}$$

where

$$|b_1| \leq \frac{1-\beta}{2\alpha+1+\beta}.$$

The result is sharp.

Proof. We shall prove the first inequality. Let $f \in \overline{S}^*_{H,n}(\alpha,\beta)$. Then we have

$$|f(z)| \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)$$

$$= (1+|b_1|)r + \frac{1-\beta}{2^n[2\alpha+2-\beta]}\sum_{k=2}^{\infty}\frac{2^n[2\alpha+2-\beta]}{1-\beta}(|a_k|+|b_k|)r^2$$

and so

$$|f(z)| \leq (1+|b_1|)r$$

$$+\frac{1-\beta}{2^n[2\alpha+2-\beta]}\sum_{k=2}^{\infty}k^n\left[\frac{\alpha k(k-1)+k-\beta}{1-\beta}|a_k| + \frac{\alpha k(k+1)+k+\beta}{1-\beta}|b_k|\right]r^2$$

$$\leq (1+|b_1|)r + \frac{1-\beta}{2^n[2\alpha+2-\beta]}\left[1 - \frac{2\alpha+1+\beta}{1-\beta}|b_1|\right]r^2$$

$$= (1+|b_1|)r + \frac{1}{2^n}\left[\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right]r^2.$$

The proof of the inequality (8) is similar, thus we omit it.

The upper bound given for $f \in \overline{S}^*_{H,n}(\alpha,\beta))$ is sharp and the equality occurs for the function

$$f(z) = z + |b_1|\overline{z} + \frac{1}{2^n}\left[\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right]\overline{z}^2 (z=r),$$

$$|b_1| \leq \frac{1-\beta}{2\alpha+1+\beta}.$$

This completes the proof of Theorem 3.1. □

Now we determine the extreme points of the closed convex hull of the class $\overline{S}^*_{H,n}(\alpha,\beta)$ denoted by $clcoH\overline{S}^*_{H,n}(\alpha,\beta)$.

Theorem 3.2. *Let $f = h + \overline{g}$, where h and g are given by (1). Then $f \in clcoH\overline{S}^*_{H,n}(\alpha,\beta)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k) \tag{9}$$

where

$$h_1(z) = z$$
$$h_k(z) = z + \frac{1-\beta}{k^n[\alpha k(k-1)+k-\beta]}z^k \qquad (k=2,3,\dots),$$
$$g_k(z) = z + \frac{1-\beta}{k^n[\alpha k(k+1)+k+\beta]}\overline{z}^k \qquad (k=1,2,3,\dots),$$

$\sum_{k=1}^{\infty}(X_k+Y_k) = 1$, $X_k \geq 0$ *and* $Y_k \geq 0$. *In particular, the extreme points of the class $\overline{S}^*_{H,n}(\alpha,\beta)$ are $\{h_k\}$ and $\{g_k\}$ respectively.*

Proof. For a function f of the form (9), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k) \\ &= \sum_{k=1}^{\infty}(X_k+Y_k)z + \sum_{k=2}^{\infty}\frac{1-\beta}{k^n[\alpha k(k-1)+k-\beta]}X_k z^k \\ &\qquad + \sum_{k=1}^{\infty}\frac{1-\beta}{k^n[\alpha k(k+1)+k+\beta]}Y_k\overline{z}^k \\ &= z + \sum_{k=2}^{\infty}\frac{1-\beta}{k^n[\alpha k(k-1)+k-\beta]}X_k z^k + \sum_{k=1}^{\infty}\frac{1-\beta}{k^n[\alpha k(k+1)+k+\beta]}Y_k\overline{z}^k. \end{aligned}$$

But

$$\sum_{k=2}^{\infty} \frac{k^n[\alpha k(k-1)+k-\beta]}{1-\beta}\left[\frac{1-\beta}{k^n[\alpha k(k-1)+k-\beta]}X_k\right] + \sum_{k=2}^{\infty} \frac{k^n[\alpha k(k+1)+k+\beta]}{1-\beta}\left[\frac{1-\beta}{k^n[\alpha k(k+1)+k+\beta]}X_k\right]$$
$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1.$$

Thus $f \in clcoH\overline{S}^{*}_{H,n}(\alpha,\beta)$.

Conversely, suppose that $f \in clcoH\overline{S}^{*}_{H,n}(\alpha,\beta)$. Set

$$X_k = \frac{k^n[\alpha k(k-1)+k-\beta]}{1-\beta}|a_k| \qquad (k=2,3,\dots), \tag{10}$$

and

$$Y_k = \frac{k^n[\alpha k(k+1)+k+\beta]}{1-\beta}|b_k| \qquad (k=1,2,3,\dots). \tag{11}$$

Then by the inequality (4), we have $0 \leq X_k \leq 1 (k=2,3,\dots)$ and $0 \leq Y_k \leq 1 (k=1,2,\dots)$. Define $X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$ and note that $X_1 \geq 0$. Thus we obtain $f(z) = \sum_{k=1}^{\infty} X_k h_k + Y_k g_k$. This completes the proof of Theorem 3.2. □

4. Convolution and Convex Combinations

For two harmonic functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k}\overline{z}^k$$

and

$$F(z) = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k}\overline{z}^k$$

we define their convolution

$$(f*F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} \overline{b_k B_k}\overline{z}^k,$$

using this definition, we show that the class $\overline{S}^{*}_{H,n}(\alpha,\beta)$ is closed under convolution.

Theorem 4.1. *For some* $0 \leq \beta < 1$ *and* $\alpha \geq 0$, *let* $f, F \in \overline{S}^{*}_{H,n}(\alpha,\beta)$. *Then* $f * F \in \overline{S}^{*}_{H,n}(\alpha,\beta)$.

Proof. We note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now for the convolution $(f * F)$ we have

$$\begin{aligned} &\sum_{k=2}^{\infty} k^n \frac{[\alpha k(k-1)+k-\beta]}{1-\beta} |A_k a_k| + \sum_{k=1}^{\infty} k^n \frac{[\alpha k(k+1)+k+\beta]}{1-\beta} |B_k b_k| \\ \leq\ &\sum_{k=2}^{\infty} k^n \frac{[\alpha k(k-1)+k-\beta]}{1-\beta} |a_k| + \sum_{k=1}^{\infty} k^n \frac{[\alpha k(k+1)+k+\beta]}{1-\beta} |b_k| \leq 1. \end{aligned}$$

Therefore $f * F \in \overline{S}^*_{H,n}(\alpha,\beta)$. □

We show that the class $\overline{S}^*_{H,n}(\alpha,\beta)$ is closed under convex combination of its members.

Theorem 4.2. *The class $\overline{S}^*_{H,n}(\alpha,\beta)$ is closed under convex combination.*

Proof. For $(i = 1,2,3\ldots)$ let $f_i \in \overline{S}^*_{H,n}(\alpha,\beta)$ where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{ki} z^k + \sum_{k=1}^{\infty} \overline{b_{ki}}\,\overline{z}^k.$$

Then by (4), we have

$$\sum_{k=2}^{\infty} k^n \frac{[\alpha k(k-1)+k-\beta]}{1-\beta} |a_{ki}| + \sum_{k=1}^{\infty} k^n \frac{[\alpha k(k+1)+k+\beta]}{1-\beta} |b_{ki}| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i = z + \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{ki} \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \overline{b_{ki}} \right) \overline{z}^k.$$

Then by (4), we have

$$\begin{aligned} &\sum_{k=2}^{\infty} k^n \frac{[\alpha k(k-1)+k-\beta]}{1-\beta} \left| \sum_{i=1}^{\infty} t_i a_{ki} \right| + \sum_{k=1}^{\infty} k^n \frac{[\alpha k(k+1)+k+\beta]}{1-\beta} \left| \sum_{i=1}^{\infty} t_i \overline{b_{ki}} \right| \\ &\leq \sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} k^n \frac{[\alpha k(k-1)+k-\beta]}{1-\beta} |a_{ki}| + \sum_{k=1}^{\infty} k^n \frac{[\alpha k(k+1)+k+\beta]}{1-\beta} |b_{ki}| \right) \\ &\leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

Therefore $\sum_{i=1}^{\infty} t_i f_i \in \overline{S}^*_{H,n}(\alpha,\beta)$. □

5. A Family of Class Preserving Integral Operator

Let $f(z) = h(z) + \overline{g(z)} \in H$ be given by(1) then $F(z)$ defined by relation

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt}\,, \quad (c > -1) \tag{12}$$

Theorem 5.1. *Let $f = h + \overline{g}$, where h and g are given by (1) be in the class $\overline{S}^*_{H,n}(\alpha,\beta)$, then $F(z)$ defined by (12) also belong to the class $\overline{S}^*_{H,n}(\alpha,\beta)$.*

Proof. From the representation of F, it follows that

$$F(z) = z + \sum_{k=2}^{\infty}\frac{c+1}{c+k}a_k z^k + \sum_{k=1}^{\infty}\frac{c+1}{c+k}\overline{b_k}\overline{z}^k.$$

Now

$$\sum_{k=2}^{\infty} k^n \frac{[\alpha k(k-1)+k-\beta]}{1-\beta}\left(\frac{c+1}{c+k}|a_k|\right) + \sum_{k=1}^{\infty} k^n \frac{[\alpha k(k+1)+k+\beta]}{1-\beta}\left(\frac{c+1}{c+k}|b_k|\right)$$
$$\le \sum_{k=2}^{\infty} k^n \frac{[\alpha k(k-1)+k-\beta]}{1-\beta}|a_k| + \sum_{k=1}^{\infty} k^n \frac{[\alpha k(k+1)+k+\beta]}{1-\beta}|b_k| \le 1$$

by (4). Thus $F(z) \in \overline{S}^*_{H,n}(\alpha,\beta)$. □

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