LE MATEMATICHE
Vol. LXIX (2014) – Fasc. II, pp. 237–242
doi: 10.4418/2014.69.2.20

ON PTÁK FUNCTION FOR BOUNDED OPERATORS

ABDELLAH EL KINANI

The purpose of this paper is to prove that if the Pták function p is an operator norm, on $\mathcal{B}(E)$, associated to a norm $|.|$, then $(E,|.|)$ is a pseudo-Hilbert space. As a consequence, we obtain that if $\mathcal{B}(E)$ is a C^*-algebra, then E is a Hilbert space.

1. Introduction

A pseudo-Hibertizable normed space is a complex normed space $(E,\|.\|)$ on which is defined a scalar product $\langle .,. \rangle$ such that $\|x\|^2 = \langle x,x \rangle$, for every $x \in E$. Let A be a complex algebra. The spectrum and spectral radius of an element x of A will be denoted by Spx and $\rho(x)$, respectively. An algebra norm on A is a linear norm $\|.\|$ satisfying $\|xy\| \leq \|x\| \|y\|$, for every $x,y \in A$. A normed unitary algebra will be called Q-normed algebra if the group $G(A)$ if its invertible elements is open. Let A be a complex Banach algebra with the involution $x \longmapsto x^*$ and unit e. An element h of A is called hermitian if $h^* = h$. It is said to be positive if it is hermitian and $Sph \subset [0,+\infty[$. The set of all Hermitian (resp. positive) elements of A will be denoted by $H(A)$ (resp. $P(A)$). We say that the Banach algebra A is Hermitian if the spectrum of every element of $H(A)$ is real ([5]). For elements h and k of $H(A)$, we write $h \geq k$ to indicate that $h-k$ is

Entrato in redazione: 25 novembre 2013

AMS 2010 Subject Classification: 46K99, 46H30.
Keywords: Pseudo-Hilbertizable space, Hilbert space, Normed algebra, Q-algebra, Hermitian element, Hermitian Banach algebra, Pták function, Bounded linear operator, C^*-algebra, Mobius transformation.

positive, i. e., that $Sp(h-k) \subset [0,+\infty[$. Let x be an element of A. We denote by $p(x) = \rho(x^*x)^{\frac{1}{2}}$ the Ptàk function of an element $x \in A$. In [5], V. Pták proved the following result: if A is a Hermitian Banach algebra, then the function p is an algebra seminorm on A such that $\rho(x) \leq p(x)$, for every $x \in A$. A Banach algebra with involution $(A, \|.\|)$ is called a C^*-algebra if $\|x^*x\| = \|x\|^2$ for each $x \in A$.

Let $(A, \|.\|)$ be a complex unital Banach algebra. In the sequel, we will use the following result of Shirali-Ford ([6]):

$$A \text{ Hermitian } \Longleftrightarrow x^*x \geq 0, \text{ for every } x \in A. \tag{1}$$

Let $(E, \|.\|)$ be a normed space, we denote by $\mathcal{B}(E)$ the complex normed algebra of all bounded linear operators on E with respect to the operator norm associated to $\|.\|$. For scalars λ, we often write simply λ for the element λI of $\mathcal{B}(E)$, where I stands for the identity operator on E. In the case where the algebra $\mathcal{B}(E)$ is endowed with an involution $T \longmapsto T^*$, we say that Pták function p is an operator norm on $\mathcal{B}(E)$ if there exists a norm $|.|$ in E such that p is an operator norm associated to $|.|$, i. e.,

$$p(T) = \sup_{|x| \leq 1} |T(x)|, \text{ for every } T \in \mathcal{B}(E).$$

For $0 < |\lambda| < 1$, we consider the Möbius transformation Φ_λ defined by:

$$\Phi_\lambda(z) = \frac{z+\lambda}{1+\overline{\lambda}z}, \text{ for every } z \in \mathbb{C} \setminus \left\{-\frac{1}{\lambda}\right\}.$$

It is clear that Φ_λ is holomorphic in $\mathbb{C} \setminus \left\{-\frac{1}{\lambda}\right\}$. Therefore, in particular, in a neighborhood of the closed unit disk.

If E is a Hilbert space, then $(\mathcal{B}(E), \|.\|)$ is a C^*algebra. In this case

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\| = p(T), \text{ for every } T \in \mathcal{B}(E).$$

Therefore p is an operator norm associated to $\|.\|$. Whence the following question: Given a Banach space $(E, \|.\|)$ and an involution $T \longmapsto T^*$ on $\mathcal{B}(E)$ such that p is an operator norm, on $\mathcal{B}(E)$, associated to a norm $|.|$ on E. Does it follows that $(\mathcal{B}(E), \|.\|)$ is a C^*algebra? The purpose of this paper is to give the answer to this question. We prove (Theorem 2.1) that if there exists a norm $|.|$, in E, such that the Pták function p is an operator norm associated to $|.|$, then $(E, |.|)$ is a pseudo-Hibertizable space. As a consequence, we show (Corollary 2.2 that if $\mathcal{B}(E)$ is a C^*-algebra, then E is a Hilbert space. The method of the proof of Theorem 2.1 goes along the lines of [2] with suitable modifications.

The following lemmas will be needed later on.

Lemma 1.1. *Let A be a complex unital Hermitian Banach algebra. Then*

1. $P(A) = \{aa^* : a \in A\} = \left\{h^2 : h \in H(A)\right\}.$
2. *For every $x \in A$, we have*

$$p(x) \leq 1 \Longleftrightarrow e - x^*x \geq 0$$

Proof. 1. It ensues just from (1) and Ford's square root lemma ([4]).

2. First assume $p(x) \leq 1$. It is clear that $e - x^*x$ is a Hermitian element of A. By the spectral mapping theorem, we have:

$$Sp\,(e - x^*x) = \{1 - \lambda : \lambda \in Sp\,(x^*x)\} \tag{2}$$

It follows from 1) that $Sp\,(x^*x) \subset [0, r]$, where $r = p(x)^2 \leq 1$. Thus $\beta \geq 0$ for every $\beta \in Sp\,(e - x^*x)$. We prove the converse. By 1), we obtain $1 - \lambda \geq 0$ for every $\lambda \in Sp\,(x^*x)$. This implies that $\max\limits_{\lambda \in Sp(x^*x)} \lambda \leq 1$. Hence, $p(x) \leq 1$. □

Lemma 1.2. *Let $(E, \|.\|)$ be a complex Banach space and $|.|$ be a norm, in E, such that the Pták function p is an operator norm, on $\mathcal{B}(E)$, associated to $|.|$. Let $T \in \mathcal{B}(E)$ with $p(T) \leq 1$. Then $p\left(\Phi_\lambda\left(T\right)\right) \leq 1$.*

Proof. Observe first that the hypothesis imply that p is subadditive. Thus, by [4], $\mathcal{B}(E)$ is Hermitian algebra. Suppose now that $T \in \mathcal{B}(E)$ with $p(T) \leq 1$. Let $0 < |\lambda| < 1$ and consider r such that $0 < r < \left|\dfrac{1}{\lambda}\right| - 1$, i. e., $1 < 1 + r < \left|\dfrac{1}{\lambda}\right|$. Then the function Φ_λ is holomorphic in the open disk $D(0, 1+r) = \Omega$. Moreover $\Phi_\lambda\left(\underline{D(0,1)} \subset D\,(0,1)\right.$ and $\Phi_\lambda(C(0,1)) \subset C(0,1)$. Hence $|\Phi_\lambda(z)| \leq 1$, for every $z \in \overline{D(0,1)}$. On the other hand, we have

$$\Phi_\lambda\left(T\right) = \left(T + \lambda\right)\left(I + \overline{\lambda}T\right)^{-1}$$

and

$$\begin{aligned} I - \Phi_\lambda\left(T\right)^* \Phi_\lambda\left(T\right) &= I - \left(I + \lambda T^*\right)^{-1}\left(T^* + \overline{\lambda}\right)\left(T + \lambda\right)\left(I + \overline{\lambda}T\right)^{-1} \\ &= \left(1 - |\lambda|^2\right)\left(I + \lambda T^*\right)^{-1}\left(I - T^*T\right)\left(I + \overline{\lambda}T\right)^{-1}. \end{aligned}$$

Now, since $p(T) \leq 1$, we have $I - T^*T \geq 0$. So, by 1) of lemma 1.1, there exists a hermitian operator S on E so that $I - T^*T = S^2$. Hence

$$I - \Phi_\lambda\left(T\right)^* \Phi_\lambda\left(T\right) = \left(1 - |\lambda|^2\right)\left(I + \lambda T^*\right)^{-1} S^2 \left(I + \overline{\lambda}T\right)^{-1}.$$

Whence, by lemma 1.1, $I - \Phi_\lambda\left(T\right)^* \Phi_\lambda\left(T\right) \geq 0$ and $p\left(\Phi_\lambda\left(T\right)\right) \leq 1$. □

2. Main result

Theorem 2.1. *Let E be a complex Banach space. If there exists a norm* $|.|$*, in E, such that the Pták function p is an operator norm associated to* $|.|$*, then* $(E,|.|)$ *is pseudo-Hilbertizable space.*

Proof. By the same argument as in Lemma 1.2, the Banach algebra $\mathcal{B}(E)$ is Hermitian. Let $\varphi \in E^*$(the conjugate space of E) and $a \in E$ such that $|\varphi|\,|a| \leq 1$, and consider $T \in \mathcal{B}(E)$ defined by $T(x) = \varphi(x)a$. Then

$$|T| = \sup_{x\neq 0} \frac{|T(x)|}{|x|} \leq |\varphi|\,|a| \leq 1.$$

This implies that $p(T) \leq 1$ for $p(T) = |T|$. By Lemma 1.1, $p(\Phi_\lambda(T)) \leq 1$ and so

$$\left|(T+\lambda I)\left(I+\overline{\lambda}T\right)^{-1}(x)\right| \leq |x|, \text{ for every } x \in E.$$

This is equivalent to:

$$|(T+\lambda I)(x)| \leq \left|\left(I+\overline{\lambda}T\right)(x)\right|, \text{ for every } x \in E.$$

Since $T(x) = \varphi(x)a$, for every $x \in E$, we have

$$|\varphi(x)a+\lambda x| \leq \left|x+\overline{\lambda}\varphi(x)a\right|, \text{ for every } x \in E. \tag{3}$$

Consider $x,y \in E$ such that $|x| \geq |y| > 0$. It follows from the Hahn Banach theorem that there exists $\psi \in E^*$ such that

$$|\psi| = |x|^{-1} \text{ and } \psi(x) = 1.$$

Set $a = y$, we get

$$|\psi|\,|a| = \frac{|y|}{|x|} \leq 1$$

Then by (3), we have

$$|\psi(x)a+\lambda x| \leq \left|x+\overline{\lambda}\psi(x)a\right|$$

i. e.,

$$|y+\lambda x| \leq \left|x+\overline{\lambda}y\right|, \quad \text{for every } |\lambda| < 1.$$

Now using the same argument as in [2], the reader can prove that, for every $x,y \in E$ with $|x| = |y|$,

$$|\alpha x+\beta y| = |\beta x+\alpha y|, \text{ for every } (\alpha,\beta) \in \mathbb{R}^2.$$

Finally, by a result of Ficken ([3]), the norm $|.|$ is derivable from an inner product. So $(E,|.|)$ is a pseudo-Hilbertizable space. □

If $\mathcal{B}(E)$ is a C^*-algebra, then p is exactly the operator norm associated to $\|.\|$. Whence the preceding result forces the space E to be a Hilbert space, as the following result shows:

Corollary 2.2. *Let $(E, \|.\|)$ be a complex Banach space. The following assertions are equivalent:*

1. *$(E, \|.\|)$ is a Hilbert space.*
2. *$\mathcal{B}(E)$ is a C^*-algebra.*

Corollary 2.3. *Let $(A, \|.\|)$ be a complex semi-simple Q-normed algebra with unit e such that $\|e\| = 1$. If $\mathcal{B}(A)$ is a C^*-algebra, then A is isomorphic to the field of complex numbers.*

Proof. By Corollary 2.3, $(A, \|.\|)$ is a Hilbertizable algebra. So $(A, \|.\|)$ is a complex semi-simple Hilbertizable Q-normed algebra with unit e such that $\|e\| = 1$. If It follows, from Corollary 3.2 of [1], that A is isomorphic to the field of complex numbers. □

Acknowledgements

The author thanks the referee for his remarks and valuable suggestions.

REFERENCES

[1] A. El Kinani, *On pseudo-Hilbertizable Q-normed algebras*, Bull. Greek Math. Soc. 53 (2007), 101–105.

[2] C. Foias, *Sur certains théorèmes de J. von Neumann concernant les ensembles spectraux*, Acta Sci. Math (Szeged) 18 (1957), 15–20.

[3] F. A. Ficken, *Note on the existence of scalar products in normed linear spaces*, Annals of Math. 45 (1946), 362–366.

[4] J. W. M. Ford, *A square root lemma for Banach $*$-algebras*, J. London Math. Soc. 42 (1967), 521–522.

[5] V. Pták, *Banach algebras with involution*, Manuscripta Math. 6 (1972), 245–290.

[6] S. Shirali - J. W. M. Ford, *Symmetry in complex involutory Banach algebras II*, Duke Math. J. 37 (1970), 275–280.

ABDELLAH EL KINANI
Université Mohammed V-Agdal
École Normale Supérieure de Rabat
Department of Mathematics
B.P. 5118, 10105 Rabat (Morocco)
e-mail: abdellah_elkinani@yahoo.fr