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# SOME NOTES ON BORNOLOGICAL AND NONBORNOLOGICAL LOCALLY CONVEX CONES

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Bornological and $b$-bornological locally convex cones have been studied in [D. Ayaseh and A. Ranjbari, Bornological Locally Convex Cones, Le Matematiche, Vol. LXIX (2014) Fasc. II, pp. 267-284]. In this paper, we obtain some new results on bornological locally convex cones and present an example of a nonbornological locally convex cone. We show that the projective limit of bornological cones is not bornological in general. Also, we present a bornological locally convex cone which is not $b$-bornological.

## 1. Introduction

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a+\beta a$, $\alpha(a+b) = \alpha a+\alpha b$, $1a = a$ and $0a = 0$ for all $a,b \in \mathcal{P}$ and $\alpha,\beta \geq 0$.

The theory of locally convex cones as developed in [3] and [8] uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use the latter for recent researches [1, 2, 5, 7].

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Let $\mathcal{P}$ be a cone. A collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a convex quasiuniform structure on $\mathcal{P}$, if the following properties hold:
$(U_1)$ $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ ( $\Delta = \{(a,a) : a \in \mathcal{P}\}$);
$(U_2)$ for all $U,V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
$(U_3)$ $\lambda U o \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
$(U_4)$ $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

Here, for $U,V \subseteq \mathcal{P}^2$, by $UoV$ we mean the set of all $(a,b) \in \mathcal{P}^2$ such that there is some $c \in \mathcal{P}$ with $(a,c) \in U$ and $(c,b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P},\mathfrak{U})$ is a locally convex cone if
$(U_5)$ for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\rho > 0$ such that $(0,a) \in \rho U$.

With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies: the neighborhood bases for an element $a$ in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b,a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in \mathcal{P} : (a,b) \in U\}, \quad U \in \mathfrak{U}.$$

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

Let $\mathfrak{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathfrak{U}$ is finer than $\mathcal{W}$ if $\mathcal{W} \subseteq \mathfrak{U}$.

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \mathbb{R}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0.(+\infty) = 0$. We set $\mathcal{V} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \{(a,b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\}.$$

Then $\tilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a+\varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a,b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex cones, the operator $T$ is called *(uniformly) continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. We denote the set of all linear functionals on $\mathcal{P}$ by $L(\mathcal{P})$ (the algebraic dual of $\mathcal{P}$). For a subset $F$ of $\mathcal{P}^2$ we define *polar* $F^{\circ}$ as follows

$$F^\circ = \{\mu \in L(\mathcal{P}) : \mu(a) \le \mu(b) + 1, \forall (a,b) \in F\}.$$

A linear functional $\mu$ on $(\mathcal{P},\mathfrak{U})$ is (uniformly) continuous if there is $U \in \mathfrak{U}$ such that $\mu \in U^\circ$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P},\mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars $U^\circ$ of neighborhoods $U \in \mathfrak{U}$.

## 2. Some Notes on Bornological and Nonbornological locally convex cones

Suppose that $(\mathcal{P},\mathfrak{U})$ is a locally convex cone. We shall say that $F \subseteq \mathcal{P}^2$ is *u-bounded (uniformly-bounded)* if it is absorbed by each $U \in \mathfrak{U}$. A subset $A$ of $\mathcal{P}$ is called *bounded above(below)* whenever $A \times \{0\}$(res. $\{0\} \times A$) is u-bounded (see [1]).

Suppose that $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex cones and $T : \mathcal{P} \to \mathcal{Q}$ is a linear operator. We shall say $T$ is *u-bounded* if $(T \times T)(F)$ is $u$-bounded in $\mathcal{Q}^2$ for every $u$-bounded subset $F$ of $\mathcal{P}^2$. We shall say $(\mathcal{P},\mathfrak{U})$ is a *bornological cone* if every $u$-bounded linear operator from $(\mathcal{P},\mathfrak{U})$ into any locally convex cone is continuous (see [1]).

Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones. The linear operator $T : \mathcal{P} \to \mathcal{Q}$ is called *bounded below* whenever $T$ maps bounded below subsets of $\mathcal{P}$ into bounded below subsets of $\mathcal{Q}$. The locally convex cone $(\mathcal{P},\mathfrak{U})$ is called *b-bornological* whenever every bounded below linear operator from $(\mathcal{P},\mathfrak{U})$ into other locally convex cone is continuous (see [1]).

Let $\mathcal{P}$ be a cone. A subset $B$ of $\mathcal{P}^2$ is called *uniformly convex* whenever it has the properties $(U_1)$ and $(U_3)$. The locally convex cone $(\mathcal{P},\mathfrak{U})$ is called a *uc-cone* whenever $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ for some $U \in \mathfrak{U}$ (see [1]). For a subset $F$ of $\mathcal{P}^2$ we denote by $uch(F)$, the smallest uniformly convex subset of $\mathcal{P}^2$, which contains $F$ and call it the uniform convex hull of $F$(such a subset of $\mathcal{P}^2$ obviously exists, since the properties $(U_1)$ and $(U_3)$ are preserved by arbitrary intersections).

Bornological and $b$-bornological locally convex cones are studied in [1]. Firstly, we review the construction of this structure briefly: Let $\mathcal{P}$ be a cone and $U$ be a uniformly convex subset of $\mathcal{P}$. We set $\mathcal{P}_U = \{a \in \mathcal{P} : \exists \lambda > 0,\ (0,a) \in \lambda U\}$ and $\mathfrak{U}_U = \{\alpha U : \alpha > 0\}$. Then $(\mathcal{P}_U,\mathfrak{U}_U)$ is a locally convex cone (a $uc$-cone). In [1] we proved that there is the finest convex quasiuniform structure $\mathfrak{U}_\tau$ (or $\mathfrak{U}_{b\tau}$) on locally convex cone $(\mathcal{P},\mathfrak{U})$ such that $\mathcal{P}^2$ (or $\mathcal{P}$) has the same $u$-bounded (or bounded below) subsets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$(or $\mathfrak{U}_{b\tau}$). The locally convex cone $(\mathcal{P},\mathfrak{U}_\tau)$ is the inductive limit of the $uc$-cones $(\mathcal{P}_U,\mathfrak{U}_U)_{U\in\mathfrak{B}}$, where $\mathfrak{B}$ is the collection of all uniformly convex and $u$-bounded subsets of $\mathcal{P}^2$. Also $(\mathcal{P},\mathfrak{U}_{b\tau})$ is the inductive limit of the $uc$-cones $(\mathcal{P}_U,\mathfrak{U}_U)_{U\in\mathfrak{B}}$, where $\mathfrak{B} = \{uch(\{0\} \times B) : B$ is bounded below$\}$. The locally convex cone $(\mathcal{P},\mathfrak{U})$ is bornological or $b$-bornological if and only if $\mathfrak{U}$ is equivalent to $\mathfrak{U}_\tau$ or $\mathfrak{U}_{b\tau}$, respectively. It is well known that there are locally convex topological vector spaces which are not bornological. For example in [4], Nachbin presented an example of nonbornological spaces. Suppose $E$ is a completely regular topological space and $\mathbb{R}^E$ is the vector space of all real-valued continuous functions on $E$ endowed with the compact-open topology. Then we have:

**Theorem 2.1** ([4], Theorem 2)**.** *$\mathbb{R}^E$ is bornological if and only if $E$ is complete under the weakest uniform structure on it with respect to which all $f \in \mathbb{R}^E$ are uniformly continuous.*

Theorem 2.1 gives an example of nonbornological space. If $E$ is not complete under the weakest uniform structure on it with respect to which all $f \in \mathbb{R}^E$ are uniformly continuous, then $\mathbb{R}^E$ endowed with the compact-open topology, is nonbornological.

In [1], we have characterized bornological locally convex cones as follows:

**Theorem 2.2** ([1], Theorem 3.10)**.** *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. The followings are equivalent:*

*(a)* *$(\mathcal{P},\mathfrak{U})$ is bornological.*

*(b)* *Every uniformly convex subset of $\mathcal{P}^2$ that absorbs all u-bounded subsets of $\mathcal{P}^2$ contains an element of $\mathfrak{U}$.*

*(c)* *Every u-bounded linear mapping of $(\mathcal{P},\mathfrak{U})$ into each uc-cone is continuous.*

*(d)* *$(\mathcal{P},\mathfrak{U})$ is an inductive limit of a family of uc- subcones of $\mathcal{P}$.*

Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and $\mathcal{P}^*$ be its dual. We introduced the convex quasiuniform structure $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$ on $\mathcal{P}$ in [1]. The convex quasiuniform structure $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$ is the weakest convex quasiuniform structure on $\mathcal{P}$ that makes all $\mu \in \mathcal{P}^*$ continuous. In fact $(\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*))$ is the projective limit of $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$ under all $\mu \in \mathcal{P}^*$ (the projective and inductive limits in locally convex cones were defined in [6]). For the locally convex cone $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$, obviously, $\tilde{\mathcal{V}}_\sigma(\overline{\mathbb{R}},\overline{\mathbb{R}}^*)$ is equivalent to $\tilde{\mathcal{V}}$.

It is proved in [1], that the inductive limit of bornological locally convex cones is bornological. The following example shows that the projective limit of bornological locally convex cones is not bornological, generally. Also, an example of nonbornological locally convex cone is presented.

**Example 2.3.** Let $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$. We set $U = \{(a,b) \in \overline{\mathbb{R}}_+^2 : a \leq b\}$ and $\mathfrak{U} = \{U\}$. Then $\mathfrak{U}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}_+$ and $(\overline{\mathbb{R}}_+,\mathfrak{U})$ is a locally convex cone (a *uc*-cone). The dual cone of $(\overline{\mathbb{R}}_+,\mathfrak{U})$ consists of all nonnegative reals and functionals $\bar{0}$ and $\overline{+\infty}$ acting as:

$$\bar{0}(a)=\begin{cases} +\infty & a=\infty,\\ 0 & \text{else}\end{cases} \quad \text{and} \quad \overline{+\infty}(a)=\begin{cases} 0 & a=0,\\ \infty & \text{else}\end{cases}$$

respectively. We claim that $\mathfrak{U}$ is strictly finer than $\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+)$. If it is not true, then there are $n\in\mathbb{N}$ and $\mu_1,\ldots,\mu_n\in\overline{\mathbb{R}}^*_+$ such that $\bigcap_{i=1}^n(\mu_i\times\mu_i)^{-1}(\tilde{1})\subset U$. Without loss of generality we can suppose there exist $\alpha_1,\ldots,\alpha_n>0$ such that $\mu_i(a)=\alpha_i a$ for $a\in\overline{\mathbb{R}}_+$ and $i=1,\ldots,n$. We set $\alpha=\max_{1\leqslant i\leqslant n}\alpha_i$. Then for $b>0$, we have

$$(b+\frac{1}{\alpha},b)\in\bigcap_{i=1}^n(\mu_i\times\mu_i)^{-1}(\tilde{1}),$$

but $(b+\frac{1}{\alpha},b)\notin U$ and this is a contradiction. Therefore $\mathfrak{U}$ is strictly finer than $\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+)$. Now, we prove that $(\overline{\mathbb{R}}_+)^2$ has the same *u*-bounded subsets under the convex quasiuniform structures $\mathfrak{U}$ and $\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+)$. Indeed, let $B\subset(\overline{\mathbb{R}}_+)^2$ be *u*-bounded under $\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+)$ and $(a,b)\in B$, then $\alpha a\leq\alpha b+1$ for all $\alpha>0$. This shows that $a\leq b$ and then $(a,b)\in U$. Therefore $B$ is *u*-bounded under $\mathfrak{U}$. Now, we realize that $[\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+)]_\tau=\mathfrak{U}$, since $(\overline{\mathbb{R}}_+,\mathfrak{U})$ is a *uc*-cone and then it is bornological. The locally convex cone $(\overline{\mathbb{R}}_+,\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+))$ is not bornological, since $\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+)$ is not equivalent to $[\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+)]_\tau=\mathfrak{U}$. Also, $(\overline{\mathbb{R}}_+,\mathfrak{U}_\sigma(\overline{\mathbb{R}}_+,\overline{\mathbb{R}}^*_+))$ is not *b*-bornological, since every *b*-bornological locally convex cone is bornological.

**Remark 2.4.** The locally convex cone $(\overline{\mathbb{R}}_+,\mathfrak{U})$ from above is *b*-bornological. Indeed, if $T:(\overline{\mathbb{R}}_+,\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$, where $(\mathcal{Q},\mathcal{W})$ is a *uc*-cone, is bounded below, and $W\in\mathcal{W}$, then $(T\times T)(\{0\}\times\overline{\mathbb{R}}_+)\subseteq W$. This shows that $(T\times T)(U)\subseteq W$, since $U=uch(\{0\}\times\overline{\mathbb{R}}_+)$.

The linear operator $T$ from a locally convex cone $(\mathcal{P},\mathfrak{U})$ into a locally convex cone $(\mathcal{Q},\mathcal{W})$ is called weakly continuous if

$$T:(\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*))\to(\mathcal{Q},\mathcal{W}_\sigma(\mathcal{Q},\mathcal{Q}^*)),$$

is continuous. For a locally convex cone $(\mathcal{P},\mathfrak{U})$, we set $\mathcal{P}^*_\sigma=(\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*))^*$, $\mathcal{P}^*_\tau=(\mathcal{P},\mathfrak{U}_\tau)^*$ and $\mathcal{P}^*_{b\tau}=(\mathcal{P},\mathfrak{U}_{b\tau})^*$ and call them the weak dual, bornological dual and *b*-bornological dual of $(\mathcal{P},\mathfrak{U})$, respectively. We note that $\mathcal{P}^*_\tau$ consists of all *u*-bounded linear functionals on $(\mathcal{P},\mathfrak{U})$. Similarly, $\mathcal{P}^*_{b\tau}$ consists of all bounded below linear functionals on $(\mathcal{P},\mathfrak{U})$.

**Proposition 2.5.** *Every weakly continuous linear functional is u-bounded (and then it is bounded below).*

*Proof.* Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and $\mu:\mathcal{P}\to\overline{\mathbb{R}}$ be a weakly continuous linear functional. Then $\mu$ is weakly $u$-bounded i.e. $\mu$ maps $u$-bounded subsets of $(\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*))$ into $u$-bounded subsets of $(\overline{\mathbb{R}},\tilde{\mathcal{V}}_\sigma(\overline{\mathbb{R}},\overline{\mathbb{R}}^*))$. Since every $u$-bonded subset is weakly $u$-bounded and $\tilde{\mathcal{V}}_\sigma(\overline{\mathbb{R}},\overline{\mathbb{R}}^*)$ is equivalent to $\tilde{\mathcal{V}}$, we conclude that $\mu$ is $u$-bounded. □

**Corollary 2.6.** *For locally convex cone* $(\mathcal{P},\mathfrak{U})$*, we have* $\mathcal{P}^*\subseteq\mathcal{P}^*_\sigma\subseteq\mathcal{P}^*_\tau\subseteq\mathcal{P}^*_{b\tau}$.

Every continuous linear operator is weakly continuous by Proposition 3.15 from [1], but for the converse we have:

**Proposition 2.7** ([1], Proposition 3.16)**.** *Let* $(\mathcal{Q},\mathcal{W})$ *be a locally convex cone such that* $Q^2$ *has the same u-bounded subsets under* $\mathcal{W}$ *and* $W_\sigma(\mathcal{Q},\mathcal{Q}^*)$*. If* $(\mathcal{P},\mathfrak{U})$ *is a bornological cone and the linear operator* $T:\mathcal{P}\to\mathcal{Q}$ *is weakly continuous, then it is continuous.*

**Theorem 2.8.** *Let* $(\mathcal{P},\mathfrak{U})$ *be a bornological locally convex cone. Then every linear functional on* $(\mathcal{P},\mathfrak{U})$ *is continuous if and only if it is weakly continuous.*

*Proof.* Let $\mu:\mathcal{P}\to\overline{\mathbb{R}}$ be a linear functional. If $\mu$ is continuous, then it is weakly continuous by Proposition 3.15 from [1]. Now, let $\mu$ be weakly continuous. Since $\tilde{\mathcal{V}}$ is equivalent to $\tilde{\mathcal{V}}_\sigma(\overline{\mathbb{R}},\overline{\mathbb{R}}^*)$ on $\overline{\mathbb{R}}$, we conclude that $(\overline{\mathbb{R}})^2$ has the same $u$-bounded subsets under the convex quasiuniform structures $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}_\sigma(\overline{\mathbb{R}},\overline{\mathbb{R}}^*)$. Now, Proposition 2.7, implies that $\mu$ is continuous. □

**Corollary 2.9.** *If* $(\mathcal{P},\mathfrak{U})$ *is a bornological locally convex cone, then* $\mathcal{P}^*=\mathcal{P}^*_\sigma=\mathcal{P}^*_\tau$*. If* $(\mathcal{P},\mathfrak{U})$ *is b-bornological, then* $\mathcal{P}^*=\mathcal{P}^*_\sigma=\mathcal{P}^*_\tau=\mathcal{P}^*_{b\tau}$.

Suppose that $(\mathcal{P},\mathfrak{U})$ is a locally convex cone. Generally, $\mathfrak{U}_{b\tau}$ is finer than $\mathfrak{U}_\tau$ on $\mathcal{P}$. If $(\mathcal{P},\mathfrak{U})$ is bornological, then $\mathfrak{U}$ is equivalent to $\mathfrak{U}_\tau$. If $(\mathcal{P},\mathfrak{U})$ is $b$-bornological, then the convex quasiuniform structures $\mathfrak{U}$, $\mathfrak{U}_\tau$ and $\mathfrak{U}_{b\tau}$ are equivalent on $\mathcal{P}$. We know that every $b$-bornological locally convex cone is bornological but the converse is not true in general. Here, we present an example of bornological locally convex cone which is not $b$-bornological.

**Example 2.10.** We consider the locally convex subcone $(\overline{\mathbb{R}}_+,\tilde{\mathcal{V}})$ of the locally convex cone $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$. Then $(\overline{\mathbb{R}}_+,\tilde{\mathcal{V}})$ is a $uc$-cone and then it is bornological. But is not $b$-bornological, since the functional $\overline{\infty}$ on $\overline{\mathbb{R}}_+$ acting as:

$$\overline{\infty}(a)=\begin{cases}0 & a=0,\\ +\infty & \text{else}\end{cases}$$

is bounded below on $(\overline{\mathbb{R}}_+,\tilde{\mathcal{V}})$, but it is not continuous. We note that $\overline{\infty}$ is not $u$-bounded. For $(\overline{\mathbb{R}}_+,\tilde{\mathcal{V}})$, we have $\tilde{\mathcal{V}}=\tilde{\mathcal{V}}_\tau$, but $\tilde{\mathcal{V}}_{b\tau}$ is strictly finer than $\tilde{\mathcal{V}}$.

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