ON GRADED $n$–ABSORBING SUBMODULES

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Let $G$ be a group with identity $e$. Let $R$ be a $G$–graded commutative ring, $M$ be a graded $R$–module and $n$ be a positive integer. In this article, we introduce and study the concepts of graded $n$–absorbing submodules. Various properties of graded $n$–absorbing submodules are considered. For example, we show that if $R$ is a Noetherian $G$–graded ring and $M$ is a finitely generated graded $R$–module, then every nonzero proper graded submodule of $M$ is a graded $n$–absorbing submodule of $M$ for some positive integer $n$.

1. Introduction

Weakly prime ideals in a commutative ring with nonzero identity have been introduced and studied by Anderson and Smith in [4]. Various generalization of prime ideals were studied in [14–18, 20]. Prime submodules, weakly prime submodules, and primary submodules have been studied by various authors, see for example [11, 19, 24]. Graded prime and graded primary ideals of a commutative graded ring $R$ with nonzero identity have been introduced and studied by Refai and Al-Zoubi in [23]. Graded prime and graded primary submodules of graded $R$–modules have been studied by Oral, Tekir and Agargun in [22]. Also, graded weakly prime submodules of graded $R$–modules have been studied by Atani in [8].

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The concept of $2-$absorbing ideals have been studied and investigated by Badawi in [12]. Weakly $2-$absorbing ideals and $n-$absorbing ideals of commutative rings have been studied by various authors. They prove many important results about these two concepts, see for example [5, 13]. Graded $2-$absorbing and weakly graded $2-$absorbing submodules have been studied by Al-Zoubi and Abou-Dawwas in [3].

In this paper, we characterize graded $n-$absorbing submodules in commutative rings, which are a generalization of graded prime ideals. The purpose of this paper is to explore some basic facts of these class of submodules. First, we show that if $N$ is a graded $n-$absorbing submodule of $M$, then $N_g$ is a $g-n-$absorbing $R_e-$submodule of $M_g$ for every $g \in G$ (see Lemma 2.5). We show (Theorem 2.6) that if $N$ is a graded submodule of a cyclic multiplication graded $R-$module, then $N$ is graded $n-$absorbing submodule of $M$ if and only if $(N:_RM)$ is graded $n-$absorbing ideal of $R$. Next, we give some characterizations of graded $n-$absorbing submodules (see section 2).

We start by recalling some background material. Let $G$ be a group with identity $e$. By a $G-$graded commutative ring we mean a commutative ring $R$ with nonzero identity together with a direct sum decomposition (as an additive group) $R = \bigoplus_{g \in G} R_g$ with the property that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, here $R_g R_h$ denotes the additive subgroup of $R$ consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in R_h$. We denote this by $(R, G)$. The elements of $R_g$ are called homogeneous of degree $g$. If $x \in R$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $R_g$. Also, we write $h(R) = \bigcup_{g \in G} R_g$. Moreover, if $R = \bigoplus_{g \in G} R_g$ is a $G$-graded ring, then $R_e$ is a subring of $R$. $1_R \in R_e$ and $R_g$ is an $R_e-$module for all $g \in G$. Let $I$ be an ideal of $R$. For $g \in G$, let $I_g = I \cap R_g$. Then $I$ is called a graded ideal of $(R, G)$ if $I = \bigoplus_{g \in G} I_g$. In this case, $I_g$ is called the $g-$component of $I$ for $g \in G$. Let $R = \bigoplus_{g \in G} R_g$ be a $G-$graded ring and let $I$ be a graded ideal of $R$. Then the quotient ring $R/I$ is also a $G-$graded ring. Indeed, $R/I = \bigoplus_{g \in G} (R/I)_g$, where $(R/I)_g = \{x + I : x \in R_g\}$. For simplicity, we will denote the graded ring $(R, G)$ by $R$. Let $R$ be a $G-$graded ring and $M$ be an $R-$module. We say that $M$ is a $G-$graded $R-$module (or graded $R-$module) if there exists a family of subgroups $\{M_g\}_{g \in G}$ of $M$ such that $M = \bigoplus_{g \in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of $M$ consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g \in G} M_g$ and the elements of $h(M)$ are called homogeneous. Let $M = \bigoplus_{g \in G} M_g$ be a graded $R-$module, then for all $g \in G$ the subgroup $M_g$ of $M$ is an $R_e-$module.
Let \( M = \bigoplus_{g \in G} G_{M_g} \) be a graded \( R \)-module and \( N \) be a submodule of \( M \). Then \( N \) is called a graded submodule of \( M \) if \( N = \bigoplus_{g \in G} G_{N_g} \) where \( N_g = N \cap M_g \) for \( g \in G \). In this case \( N_g \) is called the \( g \)-component of \( N \). Moreover, \( M/N \) becomes a \( G \)-graded \( R \)-module with \( g \)-component \((M/N)_g = (M_g + N)/N \) for \( g \in G \). A graded \( R \)-module \( M \) is called cyclic if \( M = Rm \), for some \( m \in h(M) \). A graded \( R \)-module \( M \) is called a multiplication graded module if every submodule \( N \) of \( M \) has the form \( IM \) for some graded ideal \( I \) of \( R \). A graded \( R \)-module \( M \) is called a nilpotent submodule if whenever \( rm \in N \) for some \( m \in h(M) \). A graded \( R \)-module \( M \) is called a multiplication graded module if every submodule \( N \) of \( M \) has the form \( IM \) for some graded ideal \( I \) and \( J \) of \( R \) implies \( I = J \). Now, we have the following definition, see [7, 23].

**Definition 1.1.** A proper graded ideal \( I \) of a \( G \)-graded ring \( R \) is said to be graded prime (resp. graded weakly prime) ideal if whenever \( a, b \in h(R) \) with \( ab \in I \) (resp. \( 0 \neq ab \in I \)), then either \( a \in I \) or \( b \in I \).

Next, recall the following two definitions, see [8, 22].

**Definition 1.2.** A proper graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be graded prime (resp. graded weakly prime) submodule if whenever \( r \in h(R) \) and \( m \in h(M) \) with \( rm \in N \) (resp. \( 0 \neq rm \in N \)), then either \( r \in (N :_R M) \) or \( m \in N \).

**Definition 1.3.** A proper graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be graded primary submodule if whenever \( r \in h(R) \) and \( m \in h(M) \) with \( rm \in N \), then either \( m \in N \) or \( r^k \in (N :_R M) \) for some positive integer \( k \).

If \( N \) is a graded prime (resp. graded primary) submodule of \( M \), then \( P := (N :_R M) \) (resp. \( P := \sqrt{(N :_R M)} \)) is a prime ideal of \( R \). In this case, we say that \( N \) is a graded \( P \)-prime (resp. graded \( P \)-primary) submodule of \( M \).

A submodule \( N \) of the \( R \)-module \( M \) is called a nilpotent submodule if \((N :_R M)^n = 0 \) for some positive integer \( n \), and \( m \in M \) is said to be nilpotent if \( Rm \) is a nilpotent submodule of \( M \), [1]. Assume that \( Nil(M) \) is the set of all nilpotent elements of \( M \), then \( Nil(M) \) is a submodule of \( M \) provided that \( M \) is faithful module, and if in addition \( M \) is multiplication, then \( Nil(M) = Nil(R)M = \bigcap P \), where the intersection runs of all prime submodules of \( M \) ([1], Th.6).

Al-Zoubi and Abou-Dawwas [3] define the graded \( 2 \)-absorbing and weakly graded \( 2 \)-absorbing ideals as follows:

**Definition 1.4.** A proper graded ideal \( I \) of a \( G \)-graded ring \( R \) is said to be graded \( 2 \)-absorbing (resp. weakly graded \( 2 \)-absorbing) ideal if whenever \( rs, t \in h(R) \) with \( rst \in I \) (resp. \( 0 \neq rst \in I \)), then either \( rs \in I \) or \( rt \in I \) or \( st \in I \).

In [5] Anderson and Badawi defined the \( n \)-absorbing ideals as follows:
Definition 1.5. Let $R$ be a commutative ring with $1 \neq 0$ and $n$ be a positive integer. A proper ideal $I$ of $R$ is called an $n$–absorbing ideal if whenever $a_1a_2 \cdots a_{n+1} \in I$ for $a_1, a_2, \ldots, a_{n+1} \in R$, then there are $n$ of the $a_i$’s whose product is in $I$.

The motivation of this paper is to continue the studying of the graded $2$–absorbing and weakly graded $2$–absorbing submodules, also to extend the results of Anderson and Badawi [5], Oral, Tekir, and Agargun [22], and Al-Zoubi and Abu-Dawwas [3] to the graded $n$–absorbing submodules.

2. Main Results

Our starting point is the following definitions:

Definition 2.1. Let $R$ be a $G$–graded ring and let $n$ be a positive integer. A proper graded ideal $I$ of $R$ is said to be graded $n$–absorbing ideal if whenever $a_1, \ldots, a_{n+1} \in h(R)$ with $a_1 \cdot \cdots \cdot a_{n+1} \in I$, then there are $n$ of the $a_i$’s whose product is in $I$.

Definition 2.2. Let $R$ be a $G$–graded ring, $M$ be a graded $R$–module, $N$ be a graded submodule of $M$, and let $g \in G$.

(i) We say that $N_g$ is a $g – n$–absorbing submodule of $R_e$–module $M_g$, if $N_g \neq M_g$; and whenever $a_1, \ldots, a_n \in R_e$ and $m \in M_g$ with $a_1 \cdot \cdots \cdot a_n m \in N_g$, then either $a_1 \cdot \cdots \cdot a_n \in (N_g : R_e M_g)$ or there are $n – 1$ of the $a_i$’s whose product with $m$ is in $N_g$.

(ii) We say that $N$ is a graded $n$–absorbing submodule of $M$, if $N \neq M$; and whenever $a_1, \ldots, a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \cdots \cdot a_n m \in N$, then either $a_1 \cdot \cdots \cdot a_n \in (N : R M)$ or there are $n – 1$ of the $a_i$’s whose product with $m$ is in $N$.

Remark 2.3. (1) It is clear that if $N$ is a graded $n$–absorbing submodule of $M$, then it is a graded $m$–absorbing submodule of $M$ for every integer $m \geq n$. Also if $N_g$ is a $g – n$–absorbing submodule of $R_e$–module $M_g$, then it is a $g – m$–absorbing submodule of $R_e$–module $M_g$ for every integer $m \geq n$.

(2) If $N$ is a graded $n$–absorbing submodule of $M$ for some positive integer $n$, then by following [5], define $w_M(N) = \min\{n : N$ is a graded $n$-absorbing submodule of $M\}$; otherwise, the set $w_M(N) = \infty$ (we will just write $w(N)$ when the context is clear). Moreover, we define $w(M) = 0$. Therefore, for any graded submodule $N$ of $M$, we have $w_M(N) \in \mathbb{N} \cup \{0, \infty\}$, with $w(N) = 1$ iff $N$ is a graded prime submodule of $M$ and $w(N) = 0$ iff $M = N$. Then $w(N)$ measures, in some sense, how far $N$ is from being a graded prime submodule of $M$. 
Lemma 2.4 ([6, 10]). Let $R$ be a $G$–graded ring, $M$ be a graded $R$–module, and $N$ be a graded submodule of $M$. Then the following hold:
(i) $(N :_RM) = \{ r \in R : rM \subseteq N \}$ is a graded ideal of $R$.
(ii) $rN$ and $Rm$ are graded submodules of $M$, where $r \in h(R)$ and $m \in h(M)$.

Lemma 2.5. Let $R$ be a $G$–graded ring, $M$ be a graded $R$–module, and $N$ be a graded submodule of $M$. If $N$ is a graded $n$–absorbing submodule of $M$, then $N_g$ is a $g - n$–absorbing $R_e$–submodule of $M_g$ for every $g \in G$.

Proof. Assume that $N$ is a graded $n$–absorbing submodule of $M$. For $g \in G$, assume that $a_1 \cdots a_n m \in N_g \subseteq N$, where $a_1, \ldots, a_n \in R_e$ and $m \in M_g$. Since $N$ is a graded $n$–absorbing submodule of $M$, we have either $a_1 \cdots a_n \in (N :_RM)$ or there are $n - 1$ of the $a_i$'s whose product with $m$ is in $N$. As $M_g \subseteq M$ and $N_g = N \cap M_g$, so either $a_1 \cdots a_n \in (N :_RM)$ or there are $n - 1$ of the $a_i$'s whose product with $m$ is in $N_g$. Hence $N_g$ is a $g - n$–absorbing $R_e$–submodule of $M_g$ for every $g \in G$. \qed

Theorem 2.6. Let $R$ be a $G$–graded ring, $M$ be a cyclic multiplication graded $R$–module, and $N$ a graded submodule of $M$. Then $N$ is a graded $n$–absorbing submodule of $M$ if and only if $(N :_RM)$ is a graded $n$–absorbing ideal of $R$.

Proof. Suppose that $M = Rm$ for some $m \in h(M)$ is a cyclic multiplication graded $R$–module and $N$ is a graded $n$–absorbing submodule of $M$. Assume that $a_1, \ldots, a_{n+1} \in h(R)$ with $a_1 \cdots a_{n+1} \in (N :_RM)$. For every $1 \leq i \leq n$, let $\hat{a}_i$ be the element of $R$ which is obtained by eliminating $a_i$ from $a_1 \cdots a_n$. Assume that $\hat{a}_i a_{n+1} \notin (N :_RM)$ for every $1 \leq i \leq n$. Then $\hat{a}_i a_{n+1} m \notin N$. So it follows from $(a_1 \cdots a_n)(a_{n+1}m) \in N$ and the fact that $N$ is a graded $n$–absorbing that $a_1 \cdots a_n \in (N :_RM)$. Hence $(N :_RM)$ is a graded $n$–absorbing ideal of $R$.

Conversely, suppose that $(N :_RM)$ is a graded $n$–absorbing ideal of $R$. Let $a_1, \ldots, a_n \in h(R)$ and $x \in h(M)$ with $a_1 \cdots a_n x \in N$. Then there exists $a_{n+1} \in h(R)$ with $x = a_{n+1}m$. Thus $a_1 \cdots a_n a_{n+1}m \in N$. So $a_1 \cdots a_n a_{n+1} \in (N :_Rm) = (N :_RM)$. Since $(N :_RM)$ is a graded $n$–absorbing ideal of $R$, so there are $(n)$ of the $a_i$'s whose product is in $(N :_RM)$. This implies that either $a_1 \cdots a_n \in (N :_RM)$ or there are $n - 1$ of the $a_i$'s whose product with $x$ is in $N$. Therefore $N$ is a graded $n$–absorbing submodule of $M$. \qed

Theorem 2.7. Let $R$ be a $G$–graded ring and $M$ be a graded $R$–module. If $N_j$ is a graded $n_j$–absorbing submodule of $M$ for every $1 \leq j \leq k$, then $\bigcap_{j=1}^k N_j$ is a graded $n$–absorbing submodule of $M$ for $n = n_1 + \ldots + n_k$.

Proof. Let $a_1, \ldots, a_n \in h(R), m \in h(M)$ and $N = \bigcap_{j=1}^k N_j$ with $a_1 \cdots a_n m \in N$ such that there are not $n - 1$ of the $a_i$'s whose product with $m$ is in $N$. We want to show that $a_1 \cdots a_n \in (N :_RM)$. As $a_1 \cdots a_n m \in N$, so $a_1 \cdots a_n m \in N_j$
for every $1 \leq j \leq k$. Therefore $a_1 \cdot \ldots \cdot a_n \in (N_j :_R M)$ for every $1 \leq j \leq k$ since $N_j$ is a graded $n_j$–absorbing submodule of $M$ and $n_j \leq n$. Therefore $a_1 \cdot \ldots \cdot a_n \in \bigcap_{j=1}^k (N_j :_R M) = (N :_R M)$. Hence $\bigcap_{j=1}^k N_j$ is a graded $n$–absorbing submodule of $M$. \hfill \Box

Note 2.8. The result of Theorem 2.7 may be recast using $w$ function as

$$w(N_1 \bigcap \ldots \bigcap N_k) \leq w(N_1) + \ldots + w(N_k).$$

Theorem 2.9. Let $R$ be a $G$–graded ring, $M$ be a graded $R$–module, and $N$, $V$ be graded $R$–submodules of $M$ with $V \subseteq N$. Then $N$ is a graded $n$–absorbing submodule of $M$ if and only if $N/V$ is a graded $n$–absorbing $R$–submodule of $M/V$.

**Proof.** Assume that $N$ is a graded $n$–absorbing submodule of $M$.

Let $a_1, \ldots, a_n \in h(R)$, $m \in h(M)$, and $a_1 \cdot \ldots \cdot a_n(m + V) \in N/V$. Since $N$ is a graded $n$–absorbing submodule of $M$ and $a_1 \cdot \ldots \cdot a_n m \in N$, we have either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or there are $n – 1$ of the $a_i$'s whose product with $m$ is in $N$.

Hence either $a_1 \cdot \ldots \cdot a_n \in (N/V :_R M/V)$ or there are $n – 1$ of the $a_i$'s whose product with $(m + V)$ is in $N/V$. Therefore $N/V$ is a graded $n$–absorbing $R$–submodule of $M/V$.

Conversely, suppose that $N/V$ is a graded $n$–absorbing $R$–submodule of $M/V$. Let $a_1, \ldots, a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \ldots \cdot a_n m \in N$. Since $N/V$ is a graded $n$–absorbing $R$–submodule of $M/V$ and $a_1 \cdot \ldots \cdot a_n (m + V) \in N/V$, we conclude that either $a_1 \cdot \ldots \cdot a_n \in (N/V :_R M/V)$ or there are $n – 1$ of the $a_i$'s whose product with $(m + V)$ is in $N/V$ and hence either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or there are $n – 1$ of the $a_i$'s whose product with $m$ is in $N$. Therefore $N$ is a graded $n$–absorbing submodule of $M$. \hfill \Box

Notation. Let $R$ be a $G$–graded ring and $a_1, \ldots, a_n \in h(R)$. We denote by $\hat{a}_i$ the element $a_1 \cdot \ldots \cdot a_{i-1}a_{i+1} \cdot \ldots \cdot a_n$. In this case the definition of a graded $n$–absorbing submodule can be reformulated as: the graded submodule $N$ of the graded $R$–module $M$ is called a graded $n$–absorbing if when $a_1, \ldots, a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \ldots \cdot a_n m \in N$, then either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or $\hat{a}_i m \in N$ for some $1 \leq i \leq n$. Similarly the definition of a $g – n$–absorbing submodule can be reformulated as: the $g – n$–absorbing submodule $N_g$ of $R_e$–module $M_g$ is called a $g – n$–absorbing if whenever $a_1, \ldots, a_n \in R_e$ and $m \in M_g$ with $a_1 \cdot \ldots \cdot a_n m \in N_g$, then either $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$ or $\hat{a}_i m \in N_g$ for some $1 \leq i \leq n$.

The following theorem shows the relationship between graded $P$–primary submodules and graded $n$–absorbing submodules.
**Theorem 2.10.** Let \( R \) be a \( G \)-graded ring, \( M \) be a graded \( R \)-module, and \( N \) be a graded submodule of \( M \). If \( N \) is a graded \( P \)-primary submodule of \( M \) and \( P^nM \subseteq N \) for some positive integer \( n \), then \( N \) is a graded \( n \)-absorbing submodule of \( M \).

**Proof.** Assume that \( a_1, \ldots, a_n \in h(R) \) and \( m \in h(M) \) with \( a_1 \cdots a_nm \in N \) such that \( \hat{a}_im \nsubseteq N \) for every \( 1 \leq i \leq n \). We show that \( a_1 \cdots a_n \in (N :_RM) \). For every \( 1 \leq i \leq n \), as \( a_ia_i \in N \) with \( \hat{a}_im \nsubseteq N \) and \( N \) is a graded \( P \)-primary submodule of \( M \), we have \( a_i \in P \). Consequently, \( a_1 \cdots a_n \in P^n \subseteq (N :_RM) \). Hence \( N \) is a graded \( n \)-absorbing submodule of \( M \). \( \square \)

**Note 2.11.** The result of Theorem 2.10 may be recast using \( w \) function as

\[
w(N) \leq n.
\]

**Theorem 2.12.** Let \( R \) be a Noetherian \( G \)-graded ring and \( M \) be a finitely generated graded \( R \)-module. Then every nonzero proper graded submodule of \( M \) is a graded \( n \)-absorbing submodule of \( M \) for some positive integer \( n \).

**Proof.** Let \( N \) be a graded \( P \)-primary submodule of \( M \). Then \( (N :_RM) \) is a graded \( P \)-primary ideal of \( R \). Since \( R \) is a Noetherian \( G \)-graded ring, then there exists a positive integer \( m \) such that \( P^m \subseteq (N :_RM) \). Thus \( N \) is a graded \( m \)-absorbing submodule of \( M \) by Theorem 2.10. Now suppose that \( K \) is a proper graded submodule of \( M \), we show that \( K \) is a graded \( n \)-absorbing submodule of \( M \). Since \( M \) is a finitely generated graded \( R \)-module, then \( M \) is Noetherian graded \( R \)-module. Assume that \( K = N_1 \cap \cdots \cap N_k \) is a primary decomposition of \( K \), where \( N_i \) is a graded \( P_i \)-primary submodule of \( M \) for any \( 1 \leq i \leq k \). By the first part of the proof, each \( N_i \) \( (1 \leq i \leq k) \) is a graded \( m_i \)-absorbing submodule of \( M \) for some positive integer \( m_i \). Thus, by Theorem 2.7, \( K \) is a graded \( n \)-absorbing submodule of \( M \) in which \( n = m_1 + \cdots + m_k \). \( \square \)

Recall that, the graded radical of a graded ideal \( I \) of a \( G \)-graded ring \( R \) denoted by \( Gr(I) \) is the set of all \( x \in R \) such that for each \( g \in G \) there exists a positive integer \( n_g > 0 \) with \( x^g \in I \). Note that if \( r \) is a homogeneous element of \( R \), then \( r \in Gr(I) \) iff \( r^n \in I \) for some positive integer \( n \) (see[23], definition 1.1).

The graded radical of a graded submodule \( N \) of a graded \( R \)-module \( M \) denoted by \( M - rad(N) \) is defined to be the intersection of all graded prime submodule of \( M \) containing \( N \). According to [2], a proper submodule \( N \) of an \( R \)-module \( M \) is said to be divided if \( N \subseteq Rm \) for all \( m \in M \setminus N \). Also, a prime ideal \( P \) of a ring \( R \) is said to be a divided prime ideal if \( P \subseteq xR \) for every \( x \in R \setminus P \).
Theorem 2.13. Let $R$ be a $G$—graded ring, $M$ be a finitely generated faithful multiplication graded $R$—module, and $K = PM$ be a divided graded prime submodule of $M$, where $P = (K :_R M)$ is a graded prime ideal of $R$. If $M - \text{rad}(N) = K$ and $N$ is a graded $n$—absorbing submodule of $M$ for some positive integer $n$, then $N$ is graded $P$—primary submodule of $M$.

Proof. First of all, by [25, Th.2.12], $M - \text{rad}(N) = \sqrt{(N :_R M)}M$. On the other hand, $M - \text{rad}(N) = K = PM$ ([25], Corollary 2.11). Moreover, every finitely generated faithful multiplication module is cancelation. Thus $M - \text{rad}(N) = \sqrt{(N :_R M)}M = K = PM$ implies that $P = (K :_R M) = \sqrt{(N :_R M)}$. Assume that $am \in N$ but $a \notin P$. Then from $am \in K$, $a \notin (K :_R M)$ and $K$ prime we get $m \in K$. By [2, Prop. 6], $P$ is a divided prime ideal of $R$. So $P \subset Ra^{n-1}$ since $a \notin P$. Therefore, $K = PM \subset Ma^{n-1}$, and hence $m = a^n t$ for some $t \in M$. Now it follows from $a^n t = am \in N$ and $a^n \notin (N :_R M)$ that $m = a^n t \in N$ since $N$ is a graded $n$—absorbing. Therefore $N$ is a graded $P$—primary submodule of $M$.

Theorem 2.14. Let $R$ be a $G$—graded ring and $M$ be a finitely generated faithful multiplication graded $R$—module. Let $\text{Nil}(M) \subset P$ be divided graded prime submodule of $M$. Then $P^n$ is a graded $n$—absorbing submodule of $M$ for every positive integer $n$.

Proof. Since $M$ is a multiplication faithful module, we have $\text{Nil}(M) = \text{Nil}(R)M$. Also $M$ is a cancelation module since every finitely generated faithful multiplication module is cancelation by [25]. Therefore $\text{Nil}(R) \subset (P :_R M)$ are divided prime ideals by [2, Prop.6]. It follows now from [5, Th.3.3] that $(P :_R M)^n$ is a graded $(P :_R M)$—primary ideal of $R$. Hence $P^n = (P :_R M)^n M = (P :_R M)^n M$ is a graded $(P :_R M)$—primary submodule of $M$ by [9, Cor. 2]. Therefore $P^n$ is a graded $n$—absorbing submodule of $M$ by Theorem 2.10.

Corollary 2.15. Let $R$ be a $G$—graded integral domain and $M$ be a faithful multiplication graded prime $R$—module. Let $P$ be a nonzero divided graded prime submodule of $M$. Then $P^n$ is a graded $n$—absorbing submodule of $M$ for every positive integer $n$.

Proof. By [1], $\text{Nil}(M) = 0$ is a divided prime submodule of $M$ and therefore by the proof of Theorem 2.14, $P^n$ is a graded $n$—absorbing submodule of $M$. 

In [21], we have the following $R(+)^M$ construction:
Let $R$ be a commutative ring with identity and $M$ be an $R$—module. Then
$R(M) = R(+)M$ is a commutative ring with identity $(1_R, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$.

Note that $(0(+)M)^2 = 0$, so $0(+)M$ is nilpotent ideal with index 2. We view $R$ as a subring of $R(+)M$ via $r \mapsto (r, 0)$. An ideal $A$ is said to be homogeneous if $A = I(+)N$ for some ideal $I$ of $R$ and some submodule $N$ of $M$.

**Theorem 2.16.** Let $R$ be a $G$–graded ring, $I$ be a graded ideal of $R$, $M$ be a graded $R$–module, and $N$ be a graded submodule of $M$. If $I(+)N$ is a graded $n$–absorbing ideal of $R(M)$ such that $I(+)N$ is a homogeneous of $R(M)$, then $I$ is a graded $n$–absorbing ideal of $R$ and $N$ is a graded $n$–absorbing submodule of $M$.

**Proof.** Assume that $I(+)N$ is a graded $n$–absorbing ideal of $R(M)$. Let $a_1, \ldots, a_{n+1} \in h(R)$ such that $a_1 \cdot \ldots \cdot a_{n+1} \in I$, then

$$ (a_1, 0)(a_2, 0) \cdot \ldots \cdot (a_{n+1}, 0) \in I(+)N. $$

Since $I(+)N$ is a graded $n$–absorbing ideal of $R(M)$, then $(a_i, 0) \in I(+)N$ for some $1 \leq i \leq n$. So $a_i \in I$ for some $1 \leq i \leq n$ and hence $I$ is a graded $n$–absorbing ideal of $R$. Now, let $a_1, \ldots, a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \ldots \cdot a_nm \in N$. Since $I(+)N$ is a homogeneous ideal of $R(M)$, we have $(a_1, 0)(a_2, 0) \cdot \ldots \cdot (a_n, 0)(0, m) \in I(+)N$. Since $I(+)N$ is a graded $n$–absorbing ideal of $R(M)$, so either $(a_1, 0)(a_2, 0) \cdot \ldots \cdot (a_n, 0) \in I(+)N$ or there exist $n - 1$ of $(a_i, 0)'s$ whose product with $(0, m)$ is in $I(+)N$. Then $a_1 \cdot \ldots \cdot a_n \in I \subseteq (N :_R M)$ or there are $n - 1$ of $a_i'$s whose product with $m$ is in $N$ and hence $N$ is a graded $n$–absorbing submodule of $M$. \hfill \Box

**Theorem 2.17.** Let $R$ be a $G$–graded ring, $M$ be a graded $R$–module, and $N$ be a graded submodule of $M$. Let $g \in G$ such that $N_g$ is a $g$–$n$–absorbing $R_g$–submodule of $M_g$. Then the following hold:

For every $R_g$–submodule $V$ of $M_g$ and every $a_1, \ldots, a_n \in R_g$ such that $a_1 \cdot \ldots \cdot a_nV \subseteq N_g$, either $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_g} M_g)$ or there are $n - 1$ of the $a_i'$s whose product with $V$ is contained in $N_g$.

**Proof.** Suppose that $a_1, \ldots, a_n \in R_g$, $V$ is an $R_g$–submodule of $M_g$, and $a_1 \cdot \ldots \cdot a_nV \subseteq N_g$ such that $\hat{a}_iV \not\subseteq N_g$ for every $1 \leq i \leq n$ and for some $v \in V$. We show that $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_g} M_g)$. For every $1 \leq i \leq n$, as $a_i\hat{a}_iV \in N_g$ with $\hat{a}_iV \not\subseteq N_g$ and $N_g$ is $g$–$n$–absorbing $R_g$–submodule of $M_g$, we conclude that $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_g} M_g)$. \hfill \Box
Theorem 2.18. Let $R$ be a $G$–graded ring, $M$ be a graded $R$–module, and $N$ be a graded submodule of $M$. Let $g \in G$ such that $M_g$ is cyclic $R_e$–module. Then $N_g$ is a $g – n$–absorbing $R_e$–submodule of $M_g$ iff $(N_g :_{R_e} M_g)$ is a $g – n$–absorbing ideal of $R_e$.

Proof. Suppose that $N_g$ is a $g – n$–absorbing $R_e$–submodule of $M_g$ such that $M_g = R_e x$ for some $x \in M_g$. Assume that $a_1, \ldots, a_{n+1} \in R_e$ with $a_1 \cdot \ldots \cdot a_{n+1} \in (N_g :_{R_e} M_g)$. For every $1 \leq i \leq n$, let $\widehat{a}_i$ be the element of $R_e$ which is obtained by eliminating $a_i$ from $a_1 \cdot \ldots \cdot a_n$. Assume that $\widehat{a}_i a_{n+1} \notin (N_g :_{R_e} M_g)$ for every $1 \leq i \leq n$. Then $\widehat{a}_i a_{n+1} x \notin N_g$. So it follows from $(a_1 \cdot \ldots \cdot a_n)(a_{n+1} x) \in N_g$ and the fact that $N_g$ is a $g – n$–absorbing that $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$. Hence $(N_g :_{R_e} M_g)$ is a $g – n$–absorbing ideal of $R_e$.

Conversely; assume that $(N_g :_{R_e} M_g)$ is a $g – n$–absorbing ideal of $R_e$ and let $a_1 \cdot \ldots \cdot a_m \in N_g$ for some $a_1, \ldots, a_n \in R_e$ and for some $m \in M_g$. Since $M_g = R_e x$, then there exists $a_{n+1} \in R_e$ with $m = a_{n+1} x$. Then $a_1 \cdot \ldots \cdot a_n a_{n+1} x \in N$. Hence $a_1 \cdot \ldots \cdot a_n a_{n+1} \in (N_g :_{R_e} x) = (N_g :_{R_e} M_g)$. Since $(N_g :_{R_e} M_g)$ is a $g – n$–absorbing ideal of $R_e$, so there are $n$ of the $a_i$’s whose product is in $(N_g :_{R_e} M_g)$. This implies that either $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$ or there are $n – 1$ of the $a_i$’s whose product with $m$ is in $N_g$. Therefore $N_g$ is a $g – n$–absorbing $R_e$–submodule of $M_g$.

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REFERENCES


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