LE MATEMATICHE
Vol. LXX (2015) – Fasc. II, pp. 243–254
doi: 10.4418/2015.70.2.16

ON GRADED $n-$ABSORBING SUBMODULES

MOHAMMAD HAMODA - ARWA EID ASHOUR

Let G be a group with identity e. Let R be a $G-$graded commutative ring, M be a graded $R-$module and n be a positive integer. In this article, we introduce and study the concepts of graded $n-$absorbing submodules. Various properties of graded $n-$absorbing submodules are considered. For example, we show that if R is a Noetherian $G-$graded ring and M is a finitely generated graded $R-$module, then every nonzero proper graded submodule of M is a graded $n-$absorbing submodule of M for some positive integer n.

1. Introduction

Weakly prime ideals in a commutative ring with nonzero identity have been introduced and studied by Anderson and Smith in [4]. Various generalization of prime ideals were studied in [14–18, 20]. Prime submodules, weakly prime submodules, and primary submodules have been studied by various authors, see for example [11, 19, 24]. Graded prime and graded primary ideals of a commutative graded ring R with nonzero identity have been introduced and studied by Refai and Al-Zoubi in [23]. Graded prime and graded primary submodules of graded $R-$modules have been studied by Oral, Tekir and Agargun in [22]. Also, graded weakly prime submodules of graded $R-$modules have been studied by Atani in [8].

Entrato in redazione: 8 novembre 2014

AMS 2010 Subject Classification: 13A02, 16W50.

Keywords: graded prime submodules, graded weakly prime submodules, graded primary submodules, graded $n-$absorbing submodules.

The concept of $2-$absorbing ideals have been studied and investigated by Badawi in [12]. Weakly $2-$absorbing ideals and $n-$absorbing ideals of commutative rings have been studied by various authors. They prove many important results about these two concepts, see for example [5, 13]. Graded $2-$absorbing and weakly graded $2-$absorbing submodules have been studied by Al-Zoubi and Abou-Dawwas in [3].

In this paper, we characterize graded $n-$absorbing submodules in commutative rings, which are a generalization of graded prime ideals. The purpose of this paper is to explore some basic facts of these class of submodules. First, we show that if N is a graded $n-$absorbing submodule of M, then N_g is a $g-n-$absorbing R_e-submodule of M_g for every $g \in G$ (see Lemma 2.5). We show (Theorem 2.6) that if N is a graded submodule of a cyclic multiplication graded $R-$module, then N is graded $n-$absorbing submodule of M if and only if $(N :_R M)$ is graded $n-$absorbing ideal of R. Next, we give some characterizations of graded $n-$absorbing submodules (see section 2).

We start by recalling some background material. Let G be a group with identity e. By a $G-$graded commutative ring we mean a commutative ring R with nonzero identity together with a direct sum decomposition (as an additive group) $R = \bigoplus_{g\in G} R_g$ with the property that $R_gR_h \subseteq R_{gh}$ for all $g,\ h \in G$, here R_gR_h denotes the additive subgroup of R consisting of all finite sums of elements r_gs_h with $r_g \in R_g$ and $s_h \in R_h$. We denote this by (R,G). The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, we write $h(R) = \bigcup_{g\in G} R_g$. Moreover, if $R = \bigoplus_{g\in G} R_g$ is a G-graded ring, then R_e is a subring of R. $1_R \in R_e$ and R_g is an R_e-module for all $g \in G$. Let I be an ideal of R. For $g \in G$, let $I_g = I \bigcap R_g$. Then I is called a graded ideal of (R,G) if $I = \bigoplus_{g\in G} I_g$. In this case, I_g is called the $g-component$ of I for $g \in G$. Let $R = \bigoplus_{g\in G} R_g$ be a $G-$graded ring and let I be a graded ideal of R. Then the quotient ring R/I is also a $G-$graded ring. Indeed, $R/I = \bigoplus_{g\in G}(R/I)_g$, where $(R/I)_g = \{x+I : x \in R_g\}$. For simplicity, we will denote the graded ring (R,G) by R. Let R be a $G-$graded ring and M be an $R-$module. We say that M is a $G-$graded $R-$module (or graded $R-$module) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus_{g\in G} M_g$ (as abelian groups) and $R_gM_h \subseteq M_{gh}$ for all $g,h \in G$. Here, R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gs_h with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g\in G} M_g$ and the elements of $h(M)$ are called homogeneous. Let $M = \bigoplus_{g\in G} M_g$ be a graded $R-$module, then for all $g \in G$ the subgroup M_g of M is an R_e-module.

Let $M=\bigoplus_{g\in G}M_g$ be a graded $R-$module and N be a submodule of M. Then N is called a graded submodule of M if $N=\bigoplus_{g\in G}N_g$ where $N_g=N\bigcap M_g$ for $g\in G$. In this case N_g is called the $g-$component of N. Moreover, M/N becomes a $G-$graded $R-$module with $g-$component $(M/N)_g=(M_g+N)/N$ for $g\in G$. A graded $R-$module M is called cyclic if $M=Rm$, for some $m\in h(M)$. A graded $R-$module M is called a multiplication graded module if every submodule N of M has the form IM for some graded ideal I of R. A graded $R-$module M is defined to be a cancelation module if $IM=JM$ for graded ideals I and J of R implies $I=J$. Now, we have the following definition, see [7, 23].

Definition 1.1. A proper graded ideal I of a $G-$graded ring R is said to be graded prime (resp. graded weakly prime) ideal if whenever $a,b\in h(R)$ with $ab\in I$ (resp. $0\neq ab\in I$), then either $a\in I$ or $b\in I$.

Next, recall the following two definitions, see [8, 22].

Definition 1.2. A proper graded submodule N of a graded $R-$module M is said to be graded prime (resp. graded weakly prime) submodule if whenever $r\in h(R)$ and $m\in h(M)$ with $rm\in N$ (resp. $0\neq rm\in N$), then either $r\in(N:_R M)$ or $m\in N$.

Definition 1.3. A proper graded submodule N of a graded $R-$module M is said to be graded primary submodule if whenever $r\in h(R)$ and $m\in h(M)$ with $rm\in N$, then either $m\in N$ or $r^k\in(N:_R M)$ for some positive integer k.

If N is a graded prime (resp. graded primary) submodule of M, then $P:=(N:_R M)$ (resp. $P:=\sqrt{(N:_R M)}$) is a prime ideal of R. In this case, we say that N is a graded $P-$prime (resp. graded $P-$primary) submodule of M.
A submodule N of the $R-$module M is called a nilpotent submodule if $(N:_R M)^nN=0$ for some positive integer n, and $m\in M$ is said to be nilpotent if Rm is a nilpotent submodule of M, [1]. Assume that $Nil(M)$ is the set of all nilpotent elements of M, then $Nil(M)$ is a submodule of M provided that M is faithful module, and if in addition M is multiplication, then $Nil(M)=Nil(R)M=\bigcap P$, where the intersection runs of all prime submodules of M ([1],Th.6).
Al-Zoubi and Abou-Dawwas [3] define the graded $2-$absorbing and weakly graded $2-$absorbing ideals as follows:

Definition 1.4. A proper graded ideal I of a $G-$graded ring R is said to be graded $2-$absorbing (resp. weakly graded $2-$absorbing) ideal if whenever $r,s,t\in h(R)$ with $rst\in I$ (resp. $0\neq rst\in I$), then either $rs\in I$ or $rt\in I$ or $st\in I$.

In [5] Anderson and Badawi defined the $n-$absorbing ideals as follows:

Definition 1.5. Let R be a commutative ring with $1 \neq 0$ and n be a positive integer. A proper ideal I of R is called an $n-$absorbing ideal if whenever $a_1 a_2 \cdot \ldots \cdot a_{n+1} \in I$ for $a_1, a_2, \ldots, a_{n+1} \in R$, then there are n of the $a_i\,'s$ whose product is in I.

The motivation of this paper is to continue the studying of the graded $2-$absorbing and weakly graded $2-$absorbing submodules, also to extend the results of Anderson and Badawi [5], Oral, Tekir, and Agargun [22], and Al-Zoubi and Abu-Dawwas [3] to the graded $n-$absorbing submodules.

2. Main Results

Our starting point is the following definitions:

Definition 2.1. Let R be a $G-$graded ring and let n be a positive integer. A proper graded ideal I of R is said to be graded $n-$absorbing ideal if whenever $a_1, \ldots, a_{n+1} \in h(R)$ with $a_1 \cdot \ldots \cdot a_{n+1} \in I$, then there are n of the $a_i\,'s$ whose product is in I.

Definition 2.2. Let R be a $G-$graded ring, M be a graded $R-$module, N be a graded submodule of M, and let $g \in G$.
(i) We say that N_g is a $g-n-$absorbing submodule of R_e-module M_g,
if $N_g \neq M_g$; and whenever $a_1, \ldots, a_n \in R_e$ and $m \in M_g$ with $a_1 \cdot \ldots \cdot a_n m \in N_g$, then either $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$ or there are $n-1$ of the $a_i\,'s$ whose product with m is in N_g.
(ii) We say that N is a graded $n-$absorbing submodule of M, if $N \neq M$; and whenever $a_1, \ldots, a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \ldots \cdot a_n m \in N$, then either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or there are $n-1$ of the $a_i\,'s$ whose product with m is in N.

Remark 2.3. (1) It is clear that if N is a graded $n-$absorbing submodule of M, then it is a graded $m-$absorbing submodule of M for every integer $m \geq n$. Also if N_g is a $g-n-$absorbing submodule of R_e-module M_g, then it is a $g-m-$absorbing submodule of R_e-module M_g for every integer $m \geq n$.
(2) If N is a graded $n-$absorbing submodule of M for some positive integer n, then by following [5], define $w_M(N)$=min$\{n : N$ is a graded n-absorbing submodule of $M\}$; otherwise, the set $w_M(N) = \infty$ (we will just write $w(N)$ when the context is clear). Moreover, we define $w(M) = 0$. Therefore, for any graded submodule N of M, we have $w_M(N) \in \mathbb{N} \bigcup \{0, \infty\}$, with $w(N) = 1$ iff N is a graded prime submodule of M and $w(N) = 0$ iff $M = N$. Then $w(N)$ measures, in some sense, how far N is from being a graded prime submodule of M.

Lemma 2.4 ([6, 10]). *Let R be a $G-$graded ring, M be a graded $R-$module, and N be a graded submodule of M. Then the following hold:*
(i) $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.
(ii) rN and Rm are graded submodules of M, where $r \in h(R)$ and $m \in h(M)$.

Lemma 2.5. *Let R be a $G-$graded ring, M be a graded $R-$module, and N be a graded submodule of M. If N is a graded $n-$absorping submodule of M, then N_g is a $g-n-$absorbing R_e-submodule of M_g for every $g \in G$.*

Proof. Assume that N is a graded $n-$absorbing submodule of M. For $g \in G$, assume that $a_1 \cdot \ldots \cdot a_n m \in N_g \subseteq N$, where $a_1, \ldots, a_n \in R_e$ and $m \in M_g$. Since N is a graded $n-$absorping submodule of M, we have either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or there are $n-1$ of the $a_i\,'s$ whose product with m is in N. As $M_g \subseteq M$ and $N_g = N \bigcap M_g$, so either $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$ or there are $n-1$ of the $a_i\,'s$ whose product with m is in N_g. Hence N_g is a $g-n-$absorbing R_e-submodule of M_g for every $g \in G$. □

Theorem 2.6. *Let R be a $G-$graded ring, M be a cyclic multiplication graded $R-$module, and N a graded submodule of M. Then N is a graded $n-$absorbing submodule of M if and only if $(N :_R M)$ is a graded $n-$absorbing ideal of R.*

Proof. Suppose that $M = Rm$ for some $m \in h(M)$ is a cyclic multiplication graded $R-$module and N is a graded $n-$absorbing submodule of M. Assume that $a_1, \ldots, a_{n+1} \in h(R)$ with $a_1 \cdot \ldots \cdot a_{n+1} \in (N :_R M)$. For every $1 \leq i \leq n$, let $\widehat{a_i}$ be the element of R which is obtained by eliminating a_i from $a_1 \cdot \ldots \cdot a_n$. Assume that $\widehat{a_i} a_{n+1} \notin (N :_R M)$ for every $1 \leq i \leq n$. Then $\widehat{a_i} a_{n+1} m \notin N$. So it follows from $(a_1 \cdot \ldots \cdot a_n)(a_{n+1} m) \in N$ and the fact that N is a graded $n-$absorbing that $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$. Hence $(N :_R M)$ is a graded $n-$absorbing ideal of R.

Conversely, suppose that $(N :_R M)$ is a graded $n-$absorbing ideal of R. Let $a_1, \ldots, a_n \in h(R)$ and $x \in h(M)$ with $a_1 \cdot \ldots \cdot a_n x \in N$. Then there exists $a_{n+1} \in h(R)$ with $x = a_{n+1} m$. Thus $a_1 \cdot \ldots \cdot a_n a_{n+1} m \in N$. So $a_1 \cdot \ldots \cdot a_n a_{n+1} \in (N :_R m) = (N :_R M)$. Since $(N :_R M)$ is a graded $n-$absorbing ideal of R, so there are (n) of the $a_i\,'s$ whose product is in $(N :_R M)$. This implies that either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or there are $n-1$ of the $a_i\,'s$ whose product with x is in N. Therefore N is a graded $n-$absorbing submodule of M. □

Theorem 2.7. *Let R be a $G-$graded ring and M be a graded $R-$module. If N_j is a graded n_j-absorbing submodule of M for every $1 \leq j \leq k$, then $\bigcap_{j=1}^{k} N_j$ is a graded $n-$absorbing submodule of M for $n = n_1 + \ldots + n_k$.*

Proof. Let $a_1, \ldots, a_n \in h(R)$, $m \in h(M)$ and $N = \bigcap_{j=1}^{k} N_j$ with $a_1 \cdot \ldots \cdot a_n m \in N$ such that there are not $n-1$ of the $a_i\,'s$ whose product with m is in N. We want to show that $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$. As $a_1 \cdot \ldots \cdot a_n m \in N$, so $a_1 \cdot \ldots \cdot a_n m \in N_j$

for every $1 \leq j \leq k$. Therefore $a_1 \cdot \ldots \cdot a_n \in (N_j :_R M)$ for every $1 \leq j \leq k$ since N_j is a graded n_j-absorbing submodule of M and $n_j \leq n$. Therefore $a_1 \cdot \ldots \cdot a_n \in \bigcap_{j=1}^{k}(N_j :_R M) = (N :_R M)$. Hence $\bigcap_{j=1}^{k} N_j$ is a graded $n-$absorbing submodule of M. □

Note 2.8. The result of Theorem 2.7 may be recast using w function as

$$w(N_1 \bigcap \ldots \bigcap N_k) \leq w(N_1) + \ldots + w(N_k).$$

Theorem 2.9. *Let R be a $G-$graded ring, M be a graded $R-$module, and N, V be graded $R-$submodules of M with $V \subseteq N$. Then N is a graded $n-$absorbing submodule of M if and only if N/V is a graded $n-$absorbing $R-$submodule of M/V.*

Proof. Assume that N is a graded $n-$absorbing submodule of M.
Let $a_1, \ldots, a_n \in h(R)$, $m \in h(M)$, and $a_1 \cdot \ldots \cdot a_n(m+V) \in N/V$. Since N is a graded $n-$absorbing submodule of M and $a_1 \cdot \ldots \cdot a_n m \in N$, we have either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or there are $n-1$ of the $a_i\,'s$ whose product with m is in N. Hence either $a_1 \cdot \ldots \cdot a_n \in (N/V :_R M/V)$ or there are $n-1$ of the $a_i\,'s$ whose product with $(m+V)$ is in N/V. Therefore N/V is a graded $n-$absorbing $R-$submodule of M/V.
Conversely, suppose that N/V is a graded $n-$absorbing $R-$submodule of M/V. Let $a_1, \ldots, a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \ldots \cdot a_n m \in N$. Since N/V is a graded $n-$absorbing $R-$submodule of M/V and $a_1 \cdot \ldots \cdot a_n(m+V) \in N/V$, we conclude that either $a_1 \cdot \ldots \cdot a_n \in (N/V :_R M/V)$ or there are $n-1$ of the $a_i\,'s$ whose product with $(m+V)$ is in N/V and hence either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or there are $n-1$ of the $a_i\,'s$ whose product with m is in N. Therefore N is a graded $n-$absorbing submodule of M. □

Notation. Let R be a $G-$graded ring and $a_1, \ldots, a_n \in h(R)$. We denote by $\widehat{a_i}$ the element $a_1 \cdot \ldots \cdot a_{i-1}a_{i+1} \cdot \ldots \cdot a_n$. In this case the definition of a graded $n-$absorbing submodule can be reformulated as: the graded submodule N of the graded $R-$module M is called a graded $n-$absorbing if when $a_1, \ldots, a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \ldots \cdot a_n m \in N$, then either $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$ or $\widehat{a_i}m \in N$ for some $1 \leq i \leq n$. Similarly the definition of a $g-n-$absorbing submodule can be reformulated as: the $g-n-$absorbing submodule N_g of R_e-module M_g is called a $g-n-$absorbing if whenever $a_1, \ldots, a_n \in R_e$ and $m \in M_g$ with $a_1 \cdot \ldots \cdot a_n m \in N_g$, then either $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$ or $\widehat{a_i}m \in N_g$ for some $1 \leq i \leq n$.
The following theorem shows the relationship between graded $P-$primary submodules and graded $n-$absorbing submodules.

Theorem 2.10. *Let R be a $G-$graded ring, M be a graded $R-$module, and N be a graded submodule of M. If N is a graded $P-$primary submodule of M and $P^nM \subseteq N$ for some positive integer n, then N is a graded $n-$absorbing submodule of M.*

Proof. Assume that $a_1,\ldots,a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \ldots \cdot a_n m \in N$ such that $\widehat{a_i}m \notin N$ for every $1 \leq i \leq n$. We show that $a_1 \cdot \ldots \cdot a_n \in (N :_R M)$. For every $1 \leq i \leq n$, as $a_i\widehat{a_i}m \in N$ with $\widehat{a_i}m \notin N$ and N is a graded $P-$primary submodule of M, we have $a_i \in P$. Consequently, $a_1 \cdot \ldots \cdot a_n \in P^n \subseteq (N :_R M)$. Hence N is a graded $n-$absorbing submodule of M. □

Note 2.11. The result of Theorem 2.10 may be recast using w function as

$$w(N) \leq n.$$

Theorem 2.12. *Let R be a Notherian $G-$graded ring and M be a finitely generated graded $R-$module. Then every nonzero proper graded submodule of M is a graded $n-$absorbing submodule of M for some positive integer n.*

Proof. Let N be a graded $P-$primary submodule of M. Then $(N :_R M)$ is a graded $P-$primary ideal of R. Since R is a Notherian $G-$graded ring, then there exists a positive integer m such that $P^m \subseteq (N :_R M)$. Thus N is a graded $m-$absorbing submodule of M by Theorem 2.10. Now suppose that K is a proper graded submodule of M, we show that K is a graded $n-$absorbing submodule of M. Since M is a finitely generated graded $R-$module, then M is Notherian graded $R-$module. Assume that $K = N_1 \bigcap \ldots \bigcap N_k$ is a primary decomposition of K, where N_i is a graded P_i-primary submodule of M for any $1 \leq i \leq k$. By the first part of the proof, each N_i $(1 \leq i \leq k)$ is a graded m_i-absorbing submodule of M for some positive integer m_i. Thus, by Theorem 2.7, K is a graded $n-$absorbing submodule of M in which $n = m_1 + \cdots + m_k$. □

Recall that, the graded radical of a graded ideal I of a $G-$graded ring R denoted by $Gr(I)$ is the set of all $x \in R$ such that for each $g \in G$ there exists a positive integer $n_g > 0$ with ${x_g}^{n_g} \in I$. Note that if r is a homogeneous element of R, then $r \in Gr(I)$ iff $r^n \in I$ for some positive integer n (see[23], definition 1.1).

The graded radical of a graded submodule N of a graded $R-$module M denoted by $M - rad(N)$ is defined to be the intersection of all graded prime submodule of M containing N. According to [2], a proper submodule N of an $R-$module M is said to be divided if $N \subset Rm$ for all $m \in M \setminus N$. Also, a prime ideal P of a ring R is said to be a divided prime ideal if $P \subset xR$ for every $x \in R \setminus P$.

Theorem 2.13. *Let R be a $G-$graded ring, M be a finitely generated faithful multiplication graded $R-$module, and $K = PM$ be a divided graded prime submodule of M, where $P = (K :_R M)$ is a graded prime ideal of R. If $M - rad(N) = K$ and N is a graded $n-$absorbing submodule of M for some positive integer n, then N is graded $P-$primary submodule of M.*

Proof. First of all, by [25, Th.2.12], $M - rad(N) = \sqrt{(N :_R M)}M$. On the other hand, $M - rad(N) = K = PM$ ([25], Corollary 2.11). Moreover, every finitely generated faithful multiplication module is cancelation. Thus $M - rad(N) = \sqrt{(N :_R M)}M = K = PM = (K :_R M)M$ implies that $P = (K :_R M) = \sqrt{(N :_R M)}$. Assume that $am \in N$ but $a \notin P$. Then from $am \in K$, $a \notin (K :_R M)$ and K prime we get $m \in K$. By [2, Prop. 6], P is a divided prime ideal of R. So $P \subset Ra^{n-1}$ since $a \notin P$. Therefore, $K = PM \subset Ma^{n-1}$, and hence $m = a^{n-1}t$ for some $t \in M$. Now it follows from $a^n t = am \in N$ and $a^n \notin (N :_R M)$ that $m = a^{n-1}t \in N$ since N is a graded $n-$absorbing. Therefore N is a graded $P-$primary submodule of M. □

Theorem 2.14. *Let R be a $G-$graded ring and M be a finitely generated faithful multiplication graded $R-$module. Let $Nil(M) \subset P$ be divided graded prime submodule of M. Then P^n is a graded $n-$absorbing submodule of M for every positive integer n.*

Proof. Since M is a multiplication faithful module, we have $Nil(M) = Nil(R)M$. Also M is a cancelation module since every finitely generated faithful multiplication module is cancelation by [25]. Therefore $Nil(R) \subset (P :_R M)$ are divided prime ideals by [2, Prop.6]. It follows now from [5, Th.3.3] that $(P :_R M)^n$ is a graded $(P :_R M)-$primary ideal of R. Hence $P^n = (P :_R M)^n M$ is a graded $(P :_R M)-$primary submodule of M by [9, Cor. 2]. Therefore P^n is a graded n-absorbing submodule of M by Theorem 2.10. □

Corollary 2.15. *Let R be a $G-$graded integral domain and M be a faithful multiplication graded prime $R-$module. Let P be a nonzero divided graded prime submodule of M. Then P^n is a graded $n-$absorbing submodule of M for every positive integer n.*

Proof. By [1], $Nil(M) = 0$ is a divided prime submodule of M and therefore by the proof of Theorem 2.14, P^n is a graded $n-$absorbing submodule of M. □

In [21], we have the following $R(+)M$ construction:
Let R be a commutative ring with identity and M be an $R-$module. Then

$R(M) = R(+)M$ is a commutative ring with identity $(1_R, 0)$ under addition defined by $(r,m) + (s,n) = (r+s, m+n)$ and multiplication defined by

$$(r,m)(s,n) = (rs, rn+sm).$$

Note that $(0(+)M)^2 = 0$, so $0(+)M$ is nilpotent ideal with index 2. We view R as a subring of $R(+)M$ via $r \longmapsto (r,0)$. An ideal A is said to be homogeneous if $A = I(+)N$ for some ideal I of R and some submodule N of M.

Theorem 2.16. *Let R be a $G-$graded ring, I be a graded ideal of R, M be a graded $R-$module, and N be a graded submodule of M. If $I(+)N$ is a graded $n-$absorbing ideal of $R(M)$ such that $I(+)N$ is a homogeneous of $R(M)$, then I is a graded $n-$absorbing ideal of R and N is a graded $n-$absorbing submodule of M.*

Proof. Assume that $I(+)N$ is a graded $n-$absorbing ideal of $R(M)$.
Let $a_1, \ldots, a_{n+1} \in h(R)$ such that $a_1 \cdot \ldots \cdot a_{n+1} \in I$, then

$$(a_1, 0)(a_2, 0) \cdot \ldots \cdot (a_{n+1}, 0) \in I(+)N.$$

Since $I(+)N$ is a graded $n-$absorbing ideal of $R(M)$, then $\widehat{(a_i, 0)} \in I(+)N$ for some $1 \leq i \leq n$. So $\widehat{a_i} \in I$ for some $1 \leq i \leq n$ and hence I is a graded $n-$absorbing ideal of R. Now, let $a_1, \ldots, a_n \in h(R)$ and $m \in h(M)$ with $a_1 \cdot \ldots \cdot a_n m \in N$. Since $I(+)N$ is a homogeneous ideal of $R(M)$, we have $(a_1, 0)(a_2, 0) \cdot \ldots \cdot (a_n, 0)(0, m) \in I(+)N$. Since $I(+)N$ is a graded $n-$absorbing ideal of $R(M)$, so either $(a_1, 0)(a_2, 0) \cdot \ldots \cdot (a_n, 0) \in I(+)N$ or there exist $n-1$ of $(a_i, 0)'s$ whose product with $(0, m)$ is in $I(+)N$. Then $a_1 \cdot \ldots \cdot a_n \in I \subseteq (N :_R M)$ or there are $n-1$ of $a_i's$ whose product with m is in N and hence N is a graded $n-$absorbing submodule of M. □

Theorem 2.17. *Let R be a $G-$graded ring, M be a graded $R-$module, and N be a graded submodule of M. Let $g \in G$ such that N_g is a $g-n-$absorbing R_e-submodule of M_g. Then the following hold:*
For every R_e-submodule V of M_g and every $a_1, \ldots, a_n \in R_e$ such that $a_1 \cdot \ldots \cdot a_n V \subseteq N_g$, either $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$ or there are $n-1$ of the a_i 's whose product with V is contained in N_g.

Proof. Suppose that $a_1, \ldots, a_n \in R_e$, V is an R_e-submodule of M_g, and $a_1 \cdot \ldots \cdot a_n V \subseteq N_g$ such that $\widehat{a_i} v \notin N_g$ for every $1 \leq i \leq n$ and for some $v \in V$. We show that $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$. For every $1 \leq i \leq n$, as $a_i \widehat{a_i} v \in N_g$ with $\widehat{a_i} v \notin N_g$ and N_g is $g-n-$absorbing R_e-submodule of M_g, we conclude that $a_1 \cdot \ldots \cdot a_n \in (N_g :_{R_e} M_g)$. □

Theorem 2.18. *Let R be a $G-$graded ring, M be a graded $R-$module, and N be a graded submodule of M. Let $g \in G$ such that M_g is cyclic R_e-module. Then N_g is a $g-n-$absorbing R_e-submodule of M_g iff $(N_g :_{R_e} M_g)$ is a $g-n-$absorbing ideal of R_e.*

Proof. Suppose that N_g is a $g-n-$absorbing R_e-submodule of M_g such that $M_g = R_e x$ for some $x \in M_g$. Assume that $a_1, \dots, a_{n+1} \in R_e$ with $a_1 \cdot \dots \cdot a_{n+1} \in (N_g :_{R_e} M_g)$. For every $1 \leq i \leq n$, let $\widehat{a_i}$ be the element of R_e which is obtained by eliminating a_i from $a_1 \cdot \dots \cdot a_n$. Assume that $\widehat{a_i} a_{n+1} \notin (N_g :_{R_e} M_g)$ for every $1 \leq i \leq n$. Then $\widehat{a_i} a_{n+1} x \notin N_g$. So it follows from $(a_1 \cdot \dots \cdot a_n)(a_{n+1}x) \in N_g$ and the fact that N_g is a $g-n-$absorbing that $a_1 \cdot \dots \cdot a_n \in (N_g :_{R_e} M_g)$. Hence $(N_g :_{R_e} M_g)$ is a $g-n-$absorbing ideal of R_e.
Conversely; assume that $(N_g :_{R_e} M_g)$ is a $g-n-$absorbing ideal of R_e and let $a_1 \cdot \dots \cdot a_n m \in N_g$ for some $a_1, \dots, a_n \in R_e$ and for some $m \in M_g$. Since $M_g = R_e x$, then there exists $a_{n+1} \in R_e$ with $m = a_{n+1}x$. Then $a_1 \cdot \dots \cdot a_n a_{n+1} x \in N$. Hence $a_1 \cdot \dots \cdot a_n a_{n+1} \in (N_g :_{R_e} x) = (N_g :_{R_e} M_g)$. Since $(N_g :_{R_e} M_g)$ is a $g-n-$absorbing ideal of R_e, so there are n of the $a_i\,'s$ whose product is in $(N_g :_{R_e} M_g)$. This implies that either $a_1 \cdot \dots \cdot a_n \in (N_g :_{R_e} M_g)$ or there are $n-1$ of the $a_i\,'s$ whose product with m is in N_g. Therefore N_g is a $g-n-$absorbing R_e-submodule of M_g. □

Acknowledgements

The authors are grateful to Prof. M.H. Fahmy, and Prof. R.M. Salem, Math. Dept. Fac. of Sci. Al-Azhar Univ. Egypt, for their useful comments.

REFERENCES

[1] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra 36 (2008), 4620–4642.

[2] M. M. Ali, *Invertibility of multiplication modules III*, New Zealand J. Math. 39 (2009), 193–213.

[3] K. Al-Zoubi - R. Abu-Dawwas, *On graded $2-$absorbing and weakly graded $2-$absorbing submodules*, Journal of Mathematics Sciences and Applications 28 (2014), 45–60.

[4] D. D. Anderson - E. Smith, *Weakly prime ideals*, Houston J. Math. 29 (4) (2003), 831–840.

[5] D. F. Anderson - A. Badawi, *On $n-$absorbing ideals of commutative rings*, Comm. Algebra 39 (2011), 1646–1672.

[6] S. E. Atani, *On graded prime submodules*, Chiang Mai. J. Sci. 33 (1) (2006), 3–7.

[7] S. E. Atani, *On graded weakly prime ideals*, Turk. J. Math. 30 (2006), 351–358.

[8] S. E. Atani, *On graded weakly prime submodules*, Int. Math. Forum. 1 (2) (2006), 61–66.

[9] S. E. Atani - F. Callalp - U. Tekir, *A short note on the primary submodules of multiplication modules*, Inter. J. Algebra 1 (8) (2007), 381–384.

[10] S. E. Atani - F. Farzalipour, *On graded secondary modules*, Turk. J. Math. 31 (2007), 371–378.

[11] S. E. Atani - F. Farzalipour, *On weakly prime submodules*, Tamkang J. Math. 38 (3) (2007), 247–252.

[12] A. Badawi, *On* $2-$*absorbing ideals of commutative rings*, Bull Austral Math. Soc. 75 (3) (2007), 417–429.

[13] A. Badawi - Y. Darani, *On weakly* $2-$*absorbing ideals of commutative rings*, Houston J. Math. 39 (2013), 441–452.

[14] A. Badawi - U. Tekir - E. Yetkin, *On* $2-$*absorbing primary ideals in commutative rings*, Bull. Korean. Math. Soc. 51 (2014), 1163–1173.

[15] A. Badawi - U. Tekir - E. Yetkin , *On weakly* $2-$*absorbing primary ideals of commutative rings*, J. Korean Math. Soc. 52 (2015), 97–111.

[16] A. Y. Darani, *On* $2-$*absorbing and weakly* $2-$*absorbing ideals of commutative semirings*, Kyungpook Math. J. 52 (2012), 91–97.

[17] A. Y. Darani - E. R. Puczylowski, *On* $2-$*absorbing commutative semigroups and their applications to rings*, Semigroup Forum 86 (2013), 83–91.

[18] A. Y. Darani - F. Soheilnia, $2-$*absorbing and weakly* $2-$*absorbing submodules*, Thai J. Math. 9 (2011), 577–584.

[19] J.Dauns, *Prime modules*, J. reine. Angew. Math. 2 (1978), 156–181.

[20] M. Ebrahimpour - R. Nekooei, *On generalizations of prime ideals*, Comm. Algebra 40 (2012), 1268–1279.

[21] J. A. Huckaba, *Commutative rings with zero divisors*, New York, Marcel Dekker Inc, USA, 1988.

[22] K. H. Oral - U. Tekir - A. G. Agargun, *On graded prime and primary submodules*, Turk. J. Math. 35 (2011), 159-167.

[23] M. Refai - K. Al-Zoubi, *On graded primary ideals*, Turk. J. Math. 28 (2004), 217–229.

[24] P. F. Smith, *Primary modules over commutative rings*, Glasgow Math. J. 43 (2001), 103–111.

[25] P. F. Smith - Z. El-Bast, *Multiplication modules*, Comm. Algebra 16 (1988), 755–799.

MOHAMMAD HAMODA
Department of Mathematics
Al-Aqsa University, Gaza
Palestine, 4051.
e-mail: mamh_73@hotmail.com

ARWA EID ASHOUR
Department of Mathematics
The Islamic university of Gaza, Gaza
Palestine, 108.
e-mail: arashour@iugaza.edu.ps