In this paper, we define the $q$-analogue of Mellin transform symmetric under interchange of $q$ and $q^{-1}$, and present some of its main properties and explore the possibility of using the integral transform to solve a class of differential $q$-differences equations.

1. Introduction

The study of $q$-analysis is an old subject, which dates back to the end of the 19th century. The subject of $q$-analysis concerns mainly the properties of the so-called $q$-special functions, which are the extensions of the classical special functions based on a parameter, or the base, $q$.

It is well known that one of the purposes of integral transforms like Fourier, and Mellin is to solve differential equations using these $q$-special functions.

In this present paper, we are concerned with the study of the $q$-analogue of the Mellin transform using the symmetric $q$-Jackson integral. We also discuss its properties and we give its inversion formula. Furthermore, a $q$-analogue of the Titchmarsh theorem is proved and we solve respectively the $q$-diffusion and the $q$-wave equations using the symmetric $q$-derivative operator and the symmetric $q$-Mellin transform.
This paper is organized as follows. In Section 2, we present some preliminary results and notations useful in the following sections. In Section 3 we introduce the symmetric $q$-Mellin transform, we discuss its definition domain and its properties. Special attention is devoted to the inversion formula. In Section 4 we study the $q$-analogue of the convolution product. Finally, in Section 5, a $q$-analogue of the Titchmarsh theorem and a solutions of the $q$-diffusion and the $q$-wave equations are given.

2. Notations and preliminaries

Throughout this paper, we will fix $q > 0$, $q \neq 1$. We recall some usual notions and notations used in $q$-theory (see [1–3, 5, 7–11]).

Let $a \in \mathbb{C}$, the symmetric $q$-numbers $\widetilde{[a]}_q$ and symmetric $q$-factorials $\widetilde{[n]}_q!$ are defined by

$$\widetilde{[a]}_q = \frac{q^a - q^{-a}}{q - q^{-1}},$$

and

$$\widetilde{[n]}_q! = [1]_q[2]_q \ldots [n]_q.$$  \/(1)

Clearly, these two symmetric $q$-analogues satisfy

$$\widetilde{[a]}_q = q^{-a+1} [a]_q,$$

and

$$\widetilde{[n]}_q! = q^{-n(n-1)/2} [n]_q!$$  \/(2)

where

$$[a]_q = \frac{1 - q^a}{1 - q}.$$  \/(3)

It is easy to prove that

$$\widetilde{n}_q = \widetilde{n}_q^{-1} = -\widetilde{n}_q,$$

$$\widetilde{n+m}_q = q^n \widetilde{m}_q + q^{-m} \widetilde{n}_q = q^m \widetilde{n}_q + q^{-n} \widetilde{m}_q,$$

$$\widetilde{0}_q = 0, \widetilde{1}_q = 1.$$  \/(5)

The symmetric $q$-shifted factorial is defined by (see [11]):

$$\left( \widetilde{[a]}_q \right)_m = \left\{ \begin{array}{ll}
\frac{\widetilde{[a]}_q \widetilde{[a+1]}_q \ldots \widetilde{[a+m-1]}_q}{1} & \text{if } m = 1, 2, \\
& \ldots ,
\frac{q^{-m(m-1)/2} (-1)^m (q^{2a},q^{2})_m}{(q^{-1}-q)_m} & \text{if } m = 0.
\end{array} \right.$$  \/(6)

$$= \left\{ \begin{array}{ll}
\frac{q^{-m(m-1)/2} (-1)^m (q^{2a},q^{2})_m}{(q^{-1}-q)_m} & \text{if } m = 1, 2, \\
& \ldots ,
1 & \text{if } m = 0.
\end{array} \right.$$  \/(7)
where
\[(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad n \in \mathbb{N}^*.\] (8)

The symmetric \(q\)-hypergeometric series \(n \tilde{\Psi}_{n-1}\) is defined by (see [11]):
\[n \tilde{\Psi}_{n-1} \left(\frac{a_1, \ldots, a_n}{b_1, \ldots, b_{n-1}}; q; z\right) = \sum_{m=0}^{\infty} \left[ \frac{[\hat{a}_1]_q}{[b_1]_q} \cdots \frac{[\hat{a}_n]_q}{[b_{n-1}]_q} \right] m^m \frac{z^m}{[m]_q!}.\] (9)

For arbitrary number of numerator and denominator parameters, we introduce the generalized symmetric \(q\)-hypergeometric series as:
for \(q \in ]0, 1[,\)
\[r \tilde{\Psi}_s \left(\frac{a_1, \ldots, a_r}{b_1, \ldots, b_s}; q; z\right) = \sum_{m=0}^{\infty} \left[ \frac{[\hat{a}_1]_q}{[b_1]_q} \cdots \frac{[\hat{a}_r]_q}{[b_s]_q} \right] m^m \frac{z^m}{[m]_q!},\] (10)
and for \(q > 1,\)
\[r \tilde{\Psi}_s \left(\frac{a_1, \ldots, a_r}{b_1, \ldots, b_s}; q; z\right) = \sum_{m=0}^{\infty} \left[ \frac{[\hat{a}_1]_q}{[b_1]_q} \cdots \frac{[\hat{a}_r]_q}{[b_s]_q} \right] m^m \frac{z^m}{[m]_q!}.\] (11)

We introduce the symmetric \(q\)-binomial theorem, expressed with the symmetric \(q\)-hypergeometric series by
\[1 \tilde{\Psi}_0 \left(\frac{a}{-}; q; z\right) = \sum_{m=0}^{\infty} \frac{[\hat{a}^2]_q}{[m]_q!} z^m = \frac{(q^a z; q^2)_\infty}{(q^{-a} z; q^2)_\infty}.\] (12)

The symmetric \(q\)-derivative \(\tilde{D}_q f\), of a function \(f\) is given by
\[\tilde{D}_q f (x) = \begin{cases} \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}, & \text{if } x \neq 0, \\ f'(0), & \text{when } f'(0) \text{ exists} \end{cases};\] (13)

\[= D_q f \left(q^{-1}x\right),\]

where
\[D_q f (x) = \begin{cases} \frac{f(qx) - f(x)}{(q - 1)x}, & \text{if } x \neq 0, \\ f'(0), & \text{provided } f'(0) \text{ exists} \end{cases}.\] (14)
For $n \in \mathbb{N}$, we note
\[ \tilde{D}_q^1 = \tilde{D}_q, \quad \tilde{D}_q^n = \tilde{D}_q (\tilde{D}_q^{n-1}) . \] (15)

The symmetric $q$-derivative has the following property
\[ \tilde{D}_q (f(x)g(x)) = f(qx)\tilde{D}_q g(x) + g(q^{-1}x) \tilde{D}_q f(x) . \] (16)

The symmetric $q$-Jackson integrals are defined by (see [2])
\[ \int_{a}^{0} f(x) d_q x = (q^{-1} - q) a \sum_{n=1,3,\ldots}^{+\infty} f(aq^n) q^n , \quad q \in ]0, 1[ \] (17)
\[ \int_{0}^{\infty} f(x) d_q x = |q^{-1} - q| \sum_{n=\pm 1, \pm 3, \ldots}^{\infty} f(q^n) q^n , \]
and the $q$-Jackson integrals are given by:
for $q \in ]0, 1[$:
\[ \int_{a}^{0} f(x) dx = (1 - q) a \sum_{n=0}^{+\infty} f(aq^n) q^n , \] (18)
\[ \int_{0}^{\infty} f(x) dx = (1 - q) \sum_{n=0}^{+\infty} f(q^n) q^n ; \]
\[ \int_{-\infty}^{+\infty} f(x) dx = (1 - q) \sum_{n=0}^{+\infty} f(q^n) q^n + (1 - q) \sum_{n=0}^{+\infty} f(-q^n) q^n , \] (19)
provided the sums converge absolutely. Using these symmetric $q$-integrals, we set
\[ L_q^1(\tilde{R}_{q,+}) = \left\{ f : \int_{0}^{\infty} |f(x)| d_q x < \infty. \right\} , \] (20)
where $\tilde{R}_{q,+}$ is the set
\[ \tilde{R}_{q,+} = \left\{ q^{2n+1}, \quad n \in \mathbb{Z} \right\} , \] (21)
and we write for $p > 0$,
\[ L_q^1(\mathbb{R}_{q,+}) = \left\{ f : \int_{0}^{+\infty} |f(x)| d_q x < \infty. \right\} , \] (22)
\[ L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left( \int_{-\infty}^{+\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty. \right\} , \]
where
\[ \mathbb{R}_{q,+} = \left\{ q^n, \quad n \in \mathbb{Z} \right\}, \mathbb{R}_q = \left\{ \pm q^n, \quad n \in \mathbb{Z} \right\} . \] (23)
The symmetric improper $q$-integral is defined by

$$\int_0^{\infty/A} f(x) \tilde{d}_q x = |q^{-1} - q| \sum_{k \in \mathbb{Z}} f\left(\frac{q^{2k+1}}{A}\right) \frac{q^{2k+1}}{A}. \quad (24)$$

In the case $A = q^{2n}$, we can write

$$\int_0^{\infty/q^{2n}} f(x) \tilde{d}_q x = \int_0^{\infty} f(x) \tilde{d}_q x. \quad (25)$$

The symmetric $q$-Jackson integral in a generic interval $[a, b]$ is given by

$$\int_a^b f(x) \tilde{d}_q x = \int_a^b f(x) \tilde{d}_q x - \int_0^a f(x) \tilde{d}_q x, \quad (26)$$

we have, in particular

$$\int_{q^{m+1}}^{q^{m-1}} f(x) \tilde{d}_q x = (q^{-1} - q) q^m f(q^m). \quad (27)$$

**Theorem 2.1.**

1. If $F$ is any anti $q$-derivative of the function $f$, namely $\tilde{D}_q F = f$, continuous at $x = 0$, then

$$\int_0^a f(x) \tilde{d}_q x = F(a) - F(0). \quad (28)$$

2. For any function $f$ we have:

$$\tilde{D}_q \left( \int_0^x f(t) \tilde{d}_q t \right) = f(x). \quad (29)$$

3. A symmetric $q$-analogue of the integration by parts formula is given by

$$\int_0^a f(qx) \tilde{D}_q g(x) \tilde{d}_q x = f(b) g(b) - f(a) g(a) + \int_0^a g(q^{-1}x) \tilde{D}_q f(x) \tilde{d}_q x. \quad (30)$$

The symmetric $q$-analogues of the exponential function are given by

$$\tilde{E}^z_q = \begin{cases} 
1 \tilde{\Psi}_1 \left( \begin{array}{c} 1 \\
1 \\
(1 \quad q^{-1} - q) z \end{array} \right) = \sum_{n \geq 0} q^{-\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} & \text{for } 0 < q < 1 \\
\tilde{\Psi}_1 \left( \begin{array}{c} 1 \\
1 \\
q \quad (q - q^{-1}) z \end{array} \right) = \sum_{n \geq 0} q^{-\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} & \text{for } q > 1
\end{cases} \quad (31)$$
and

\[
\bar{\xi}_q = \begin{cases} 
2\bar{\Psi}_0 \left( 1, 1 ; q; \frac{z}{q-1-q} \right) = \sum_{n \geq 0} q^{n(n-1)} \frac{z^n}{[n]_q!} & \text{for } 0 < q < 1 \\
2\bar{\Psi}_0 \left( 1, 1 ; q; \frac{z}{q-1-q} \right) = \sum_{n \geq 0} q^{-n(n-1)} \frac{z^n}{[n]_q!} & \text{for } q > 1 
\end{cases}
\]

The symmetric $q$-Gamma function $\tilde{\Gamma}_q$ is defined by (see [3])

\[
\tilde{\Gamma}_q (x) = q^{-x(x-1)(x-2)/2} \Gamma_{q^2} (x), \quad x \neq 0, -1, -2, \ldots,
\]

where

\[
\Gamma_q (x) = \begin{cases} 
(1-q)^{1-x} \frac{(q; q)_\infty}{(q^2; q)_\infty} & \text{for } q \in [0, 1[ \\
q^{-x(x-1)} \frac{(q^{-1}; q^{-1})_\infty}{(q^2(1-x); q^2)_\infty} & \text{for } q > 1 
\end{cases}
\]

It is well known that it satisfies

\[
\tilde{\Gamma}_q (x+1) = [x]_q \tilde{\Gamma}_q (x), \quad \tilde{\Gamma}_q (1) = 1 \quad \text{and} \quad \tilde{\Gamma}_{q^{-1}} (x) = \tilde{\Gamma}_q (x).
\]

**Theorem 2.2.** For any $x > 0$ we have:

For $q \in [0, 1[,$

\[
\tilde{\Gamma}_q (x) = q^{-x(x-1)/2} \int_0^\infty \frac{(q^{-1}-q)}{t^x t^{-1}} \tilde{\xi}_q^{-1} d\tilde{q}_t,
\]

where

\[
K_{q^2} (x) = \frac{(-q^2, -1; q^2)_\infty}{(-q^2 x, -q^2(1-x); q^2)_\infty},
\]

and

\[
\tilde{\Gamma}_q (x) = q^{-x(x+1)/2} \int_0^\infty \frac{(q^{-1}-q)}{t^{x-1}} \tilde{\xi}_q^{-1} d\tilde{q}_t.
\]

Moreover, if $\text{Log} (q^{-1} - q) / \text{Log} (q) \in 2\mathbb{Z},$ we obtain

\[
\tilde{\Gamma}_q (x) = q^{-x(x+1)}/2 \int_0^\infty t^{x-1} \tilde{\xi}_q^{-1} d\tilde{q}_t, \quad q \in [0, 1[.
\]

Recently, R. L. Rubin [9] introduced a $q$-derivative operator $\partial_q$ as follows

\[
\partial_q (f) (z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z}.
\]
The \( q^2 \)-analogue of exponential function is given by

\[
e (z; q^2) = \cos (-iz; q^2) + i \sin (-iz; q^2),
\]

where cosine and sine are the \( q \)-trigonometric functions defined by:

\[
\cos (x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!}, \tag{42}
\]

\[
\sin (x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}.
\]

In [9], R.L. Rubin defined the \( q^2 \)-analogue Fourier transform by

\[
\hat{f} (x; q^2) = F_q (f) (x) = K \int_{-\infty}^{+\infty} f(t) e (-itx; q^2) \, dq_t, \quad x \in \mathbb{R}_q, \tag{43}
\]

where

\[
K = \frac{(q; q^2)_\infty}{2 (q^2 q^2)_\infty (1-q)^{1/2}}. \tag{44}
\]

We remind the following properties:

1. If \( f(u), uf(u) \in L_q^1 (\mathbb{R}_q) \), then

\[
\partial_q (F_q (f))(x) = F_q (-iu f(u))(x). \tag{45}
\]

2. If \( f \) and \( \partial_q f \in L_q^1 (\mathbb{R}_q) \), then

\[
F_q (\partial_q f) (x) = ix F_q (f)(x). \tag{46}
\]

3. For \( f \in L_q^2 (\mathbb{R}_q) \), we have

\[
f(t) = K \int_{-\infty}^{+\infty} F_q (f)(x) e \left( itx; q^2 \right) \, dq_x, \quad t \in \mathbb{R}_q. \tag{47}
\]

3. The symmetric \( q \)-Mellin transform

**Definition 3.1.** Let \( f \) be a function defined on \( \mathbb{R}_{q,+} \) we define the symmetric \( q \)-Mellin transform of \( f \) as

\[
\tilde{M}_q (f) (s) = \tilde{M}_q [f(t)] (s) = \int_{0}^{\infty} t^{s-1} f(t) \, dq_t \quad q \in [0, 1[. \tag{48}
\]
Theorem 3.2. Let $f$ be a function defined on $\mathbb{R}_q^+$ and let $u,v \in \mathbb{R}$ with $u > v$. We suppose
\[ f(x) = O_{o+} (x^u) \quad \text{and} \quad f(x) = O_{\pm\infty} (x^v). \] (49)
Then $\tilde{M}_q(f)(s)$ exists in the strip $(-u, -v)$.

Theorem 3.3. If $f$ is a function defined on $\mathbb{R}_q^+$, then $\tilde{M}_q(f)(s)$ is analytic on the strip $\langle \alpha_{q,f}, \beta_{q,f} \rangle$ and we have
\[ \forall s \in \langle \alpha_{q,f}, \beta_{q,f} \rangle, \frac{d}{ds} \tilde{M}_q(f)(s) = \tilde{M}_q [\log(t) f(t)](s). \] (50)

3.1. Properties

In the following subsection, we give some interesting properties of the symmetric $q$-Mellin transform.

1. For $a = q^{2n}$, $n \in \mathbb{Z}$ and $s \in \langle \alpha_{q,f}, \beta_{q,f} \rangle$, we have
\[ \tilde{M}_q [f(at)](s) = a^{-s} \tilde{M}_q(f)(s). \] (51)

2. For $s \in \langle -\beta_{q,f}, -\alpha_{q,f} \rangle$, we have
\[ \tilde{M}_q \left[ f \left( \frac{1}{t} \right) \right](s) = \tilde{M}_q(f)(-s). \] (52)

3. For $s \in \langle 1 - \beta_{q,f}, 1 - \alpha_{q,f} \rangle$, we have
\[ \tilde{M}_q \left[ \frac{1}{t} f \left( \frac{1}{t} \right) \right](s) = \tilde{M}_q(f)(1 - s). \] (53)

4. For $s \in \langle \alpha_{q,f}, \beta_{q,f} \rangle$, we have
\[ \tilde{M}_q \left[ t \tilde{D}_q^2 f(t) \right](s) = \left[ -s \right]_{q^2} \tilde{M}_q(f)(s). \] (54)

5. For $s \in \langle \alpha_{q,f} + 1, \beta_{q,f} + 1 \rangle$, we have
\[ \tilde{M}_q \left[ \tilde{D}_q^2 f(t) \right](s) = \left[ 1 - s \right]_{q^2} \tilde{M}_q(f)(s - 1). \] (55)

By induction, we obtain that, for $n \in \mathbb{N}^*$ and $s \in \langle \alpha_{q,f} + n, \beta_{q,f} + n \rangle$,
\[ \tilde{M}_q \left[ \tilde{D}_q^{(n)} f(t) \right](s) = \left[ 1 - s \right]_{q^2} \ldots \left[ n - s \right]_{q^2} \tilde{M}_q(f)(s - n). \]
6. Given $\rho$ a positive odd integer and $s \in \langle \rho \alpha_{q^\rho, f}, \rho \beta_{q^\rho, f} \rangle$, we have

$$\tilde{M}_q [f (t^\rho)] (s) = \left[ \frac{1}{\rho} \right]_{q^\rho} \tilde{M}_{q^\rho} (f) \left( \frac{s}{\rho} \right).$$

7. Let $(\mu_k)_k$ be a sequence of $\mathbb{R}_{q^+, \mathbb{R}_{q^+, \mathbb{R}}}$, let $(\lambda_k)_k$ be a sequence of $\mathbb{C}$, and let $f$ be a suitable function, then we have

$$\tilde{M}_q \left( \sum_{k \geq 0} \lambda_k f (\mu_k t) \right) (s) = \left( \sum_{k \geq 0} \lambda_k \mu_k^s \right) \tilde{M}_q (f) (s),$$

provided the sums converge.

### 3.2. The symmetric $q$-Mellin inversion formula

**Theorem 3.4.** Let $f$ be a function defined over $\mathbb{R}_{q^+, \mathbb{R}}$ and $c \in ]\alpha_{q, f}, \beta_{q, f} [$, then

$$\forall x \in \mathbb{R}_{q^+, \mathbb{R}} \quad \frac{1}{2i \pi} \frac{\log (q)}{(q^{-1} - q)} \int_{c-\pi/\log(q)}^{c+\pi/\log(q)} \tilde{M}_q (f) (s) x^{-s} ds = f (x). \quad (56)$$

**Proof.** Let $c \in ]\alpha_{q, f}, \beta_{q, f} [$ and $x = q^{2k+1} \in \mathbb{R}_{q^+, \mathbb{R}}$, we have:

$$\frac{1}{2i \pi} \frac{\log (q)}{(q^{-1} - q)} \int_{c-\pi/\log(q)}^{c+\pi/\log(q)} \tilde{M}_q (f) (s) x^{-s} ds = \frac{\log (q)}{2i \pi} \int_{c-\pi/\log(q)}^{c+\pi/\log(q)} \left( \sum_{n \in \mathbb{Z}} f (q^{2n+1}) (q^{2n+1})^s \right) (q^{2k+1})^{-s} ds,$$

since the series $\sum_{n \in \mathbb{Z}} f (q^{2n+1}) (q^{2n-2k})^s$ converge uniformly with respect to $s$, one gets:

$$\frac{1}{2i \pi} \frac{\log (q)}{(q^{-1} - q)} \int_{c-\pi/\log(q)}^{c+\pi/\log(q)} \tilde{M}_q (f) (s) x^{-s} ds = \frac{i \log (q)}{2i \pi} \sum_{n \in \mathbb{Z}} q^{2c(n-k)} f (q^{2n+1}) \int_{-\pi/\log(q)}^{\pi/\log(q)} q^{2i(n-k)t} dt = \sum_{n \in \mathbb{Z}} q^{2c(n-k)} f (q^{2n+1}) \delta_{n,k} = f \left( q^{2k+1} \right). \quad \square$$
3.3. **Symmetric $q$-Mellin’s convolution product**

**Definition 3.5.** The symmetric $q$-Mellin convolution product of the functions $f$ and $g$ is the function $f \ast_{\tilde{M}_q} g$ defined by

$$f \ast_{\tilde{M}_q} g (x) = \int_0^\infty f(t) g\left(\frac{q^{-x}}{t}\right) \tilde{d}_q t, \quad x \in \mathbb{R}_{q,+}, \quad (57)$$

provided the symmetric $q$-integral exists.

**Theorem 3.6.** If the symmetric $q$-Mellin convolution product of $f$ and $g$ exists, then

$$f \ast_{\tilde{M}_q} g = g \ast_{\tilde{M}_q} f, \quad (58)$$

$$\tilde{M}_q \left[ f \ast_{\tilde{M}_q} g \right] (s) = q^{-s} \tilde{M}_q (f) (s) \tilde{M}_q (g) (s). \quad (59)$$

**Proof.**

$$\tilde{M}_q \left[ f \ast_{\tilde{M}_q} g \right] (s) = \int_0^\infty x^{s-1} f \ast_{\tilde{M}_q} g (x) \tilde{d}_q x$$

$$= (q^{-1} - q) \int_0^\infty x^{s-1} \left( \sum_{n \in \mathbb{Z}} f (q^{2n+1}) g (x q^{-2n}) \right) \tilde{d}_q x$$

$$= (q^{-1} - q) \sum_{n \in \mathbb{Z}} f (q^{2n+1}) \int_0^\infty x^{s-1} g (x q^{-2n}) \tilde{d}_q x$$

$$= (q^{-1} - q) \sum_{n \in \mathbb{Z}} f (q^{2n+1}) (q^{-2n})^{-s} \tilde{M}_q (g) (s)$$

$$= q^{-s} \tilde{M}_q (g) (s) \tilde{M}_q (f) (s). \quad \square$$

**Theorem 3.7.** For the suitable functions $f$ and $g$, we have the following relations:

$$\frac{1}{2i\pi} \frac{\log (q)}{(q^{-1} - q)} \int_{c-i\pi/\log (q)}^{c+i\pi/\log (q)} \tilde{M}_q (f) (s) \tilde{M}_q (g) (1 - s) \, ds = \int_0^\infty g (x) f (x) \tilde{d}_q x, \quad (60)$$

and

$$\frac{1}{2i\pi} \frac{\log (q)}{(q^{-1} - q)} \int_{c-i\pi/\log (q)}^{c+i\pi/\log (q)} q^{-s} \tilde{M}_q (f) (s) \tilde{M}_q (g) (s) \, ds = \int_0^\infty f (t) g\left(\frac{q}{t}\right) \tilde{d}_q t. \quad (61)$$

**Proof.** In order to prove the first relation, let $c \in \mathbb{R}$ such that $c \in [\alpha_{q,f}, \beta_{q,f} \cap (1 - \beta_{q,g}, 1 - \alpha_{q,g}]$. We put $I(c) = [c - i\pi/\text{Log} (q), c + i\pi/\text{Log} (q)]$. From the symmetric $q$-Mellin inversion formula, and the relation

$$\sup_{s \in I(c)} \left| (q^{2n+1})^{1-s} g (q^{2n+1}) \tilde{M}_q (f) (s) \right| = q^{(1-c)(2n+1)} \sup_{s \in I(c)} \left| \tilde{M}_q (f) (s) \right|,$$
we obtain

\[
\frac{1}{2i\pi} \log(q) \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \mathcal{M}_q(f)(s) \hat{M}_q(g)(1-s) \, ds
\]

\[
= \frac{1}{2i\pi} \log(q) \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \sum_{n \in \mathbb{Z}} (q^{2n+1})^{1-s} g(q^{2n+1}) \hat{M}_q(f)(s) \, ds
\]

\[
= \frac{\log(q)}{2i\pi} \int_{0}^{\infty} g(x) \left( \int_{q^{-1}}^{q} x^{-s} \hat{M}_q(f)(s) \, ds \right) \tilde{d}_q x
\]

\[
= \int_{0}^{\infty} g(x) f(x) \tilde{d}_q x. \quad \square
\]

4. Applications

4.1. Symmetric q-integral equations

**Theorem 4.1.** Let \( K \) and \( g \) be functions defined on \( \tilde{R}_{q,+} \). We suppose that \( \langle \alpha_{q,L}, \beta_{q,L} \rangle \cap \langle 1 - \beta_{q,g}, 1 - \alpha_{q,g} \rangle \) is not empty, for a suitable real \( c \), we put

\[
\forall x \in \tilde{R}_{q,+}, \quad L(x) = \frac{q^{-1} \log(q)}{2i\pi (q^{-1} - q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \frac{x^{-s}}{\tilde{M}_q(K)(1-s)} \, ds. \quad (62)
\]

Then the q-symmetric integral equation

\[
\int_{0}^{\infty} f(x) K(q^{-1}xt) \tilde{d}_q x = g(t), \quad t \in \tilde{R}_{q,+}, \quad (63)
\]

has as solution

\[
f(x) = \int_{0}^{\infty} g(t) L(q^{-1}xt) \tilde{d}_q t, \quad x \in \tilde{R}_{q,+}. \quad (64)
\]

In addition, if

\[
\tilde{M}_q(K)(s) \tilde{M}_q(K)(1-s) = q^{-1}, \quad (65)
\]

then equation (63) has the solution:

\[
f(x) = \int_{0}^{\infty} g(t) K(q^{-1}xt) \tilde{d}_q t, \quad x \in \tilde{R}_{q,+}. \quad (66)
\]

**Proof.** By taking the symmetric q-Mellin transform in the equation (63), we obtain

\[
\tilde{M}_q(f)(s) \tilde{M}_q(K)(1-s) = q^{-s} \tilde{M}_q(g)(1-s). \]
From the relation (62), we deduce that
\[ \tilde{M}_q(f)(s) = q^s \tilde{M}_q(L)(s) \tilde{M}_q(g)(1 - s). \]

Then, for \( c' \in \langle \alpha_{q,L}, \beta_{q,L} \rangle \cap \langle 1 - \beta_{q,g}, 1 - \alpha_{q,g} \rangle \), we get:
\[ \forall x \in \tilde{R}_{q,+}, \quad f(x) = \frac{\log(q)}{2i\pi(q^{-1} - q)} \int_{c' + i\pi/(\log(q))}^{c' - i\pi/(\log(q))} \tilde{M}_q(L)(s) \tilde{M}_q(g)(1 - s) (q^{-1}x)^{-s} ds. \]

Finally, by the relation (60), we have:
\[ \forall x \in \tilde{R}_{q,+}, \quad f(x) = \int_0^\infty g(t) L(q^{-1}xt) \tilde{d}_q t. \]

### 4.2. Symmetric q-analogue of the Titchmarsh theorem

**Theorem 4.2.** Let \( K \) be a function defined on \( \tilde{R}_{q,+} \). Suppose that \( \langle \alpha_{q,K}, \beta_{q,K} \rangle \) is not empty. If the integral equation
\[ f(x) = \int_0^\infty K(q^{-1}xu) \tilde{d}_q u \int_0^\infty K(q^{-1}yu) f(y) \tilde{d}_q y, \] (67)
has a suitable solution \( f \) then, for every \( s \in \mathbb{C} \) such that \( s \) and \( 1 - s \) are in \( \langle \alpha_{q,K}, \beta_{q,K} \rangle \), we have
\[ \tilde{M}_q(K)(s) \tilde{M}_q(K)(1 - s) = q^{-1}. \]

**Proof.** The integral equation (67) is written as a pair of reciprocal formulas as follows
\[ 1. \quad g(x) = \int_0^\infty K(q^{-1}yx) f(y) \tilde{d}_q y, \]
\[ 2. \quad f(x) = \int_0^\infty K(q^{-1}yx) g(y) \tilde{d}_q y. \]

By taking the symmetric \( q \)-Mellin transform in (1) and (2) at \( s \), we get
\[ \tilde{M}_q(g)(s) = q^s \tilde{M}_q(f)(1 - s) \tilde{M}_q(K)(s), \]
and
\[ \tilde{M}_q(f)(s) = q^s \tilde{M}_q(g)(1 - s) \tilde{M}_q(K)(s), \]
changing \( s \) into \( 1 - s \) in one of these equations and multiplying, we deduce that
\[ \tilde{M}_q(K)(s) \tilde{M}_q(K)(1 - s) = q^{-1}. \]
4.3. Symmetric $q$-diffusion equation

We assume that $\frac{\log(q^{-1}-q)}{\log(q)} \in 2\mathbb{Z}$. We consider the following symmetric $q$-diffusion equation:

$$\tilde{D}_{q^2} u(x,t) = (\partial_{q^x})^2 u(x,q^2t), \quad x \in \mathbb{R}_q \text{ and } t \in \tilde{\mathbb{R}}_{q^+}$$

subject to the initial condition

$$u(x,0) = f(x), \quad f \in L^2_q(\mathbb{R}_q).$$

By taking a the $q^2$-Fourier transform in $x$ and the symmetric $q$-Mellin transform in $t$, we obtain

$$\left[ s - 1 \right]_{q^2} U(\lambda, s - 1) = \lambda^2 q^{-2s} U(\xi, s).$$

The general solution is [6]

$$U(\lambda, s) = C(\lambda) \lambda^{-2s} q^{s(s+1)} \tilde{\Gamma}_{q^2}(s),$$

where $C(\lambda)$ is a function of $\lambda$ only. For the relation (39), the inversion symmetric $q$-Mellin transform of $\lambda^{-2s} q^{s(s+1)} \tilde{\Gamma}_{q^2}(s)$ is

$$\frac{\log(q)}{2i\pi(q^{-1}-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \lambda^{-2s} q^{s(s+1)} \tilde{\Gamma}_{q^2}(s) t^{-s} ds = \xi^{-\lambda^2 t} q^{-2s t}.$$  \(72\)

Then

$$u(x,t) = K \int_{-\infty}^{+\infty} C(\lambda) \xi^{-\lambda^2 t} q^{-2s} e(i\lambda x, q^2) d_q\lambda.$$  \(73\)

For $t = 0$, we get

$$u(x,0) = K \int_{-\infty}^{+\infty} C(\lambda) e(i\lambda x, q^2) d_q\lambda = f(x),$$  \(74\)

so

$$C(\lambda) = K \int_{-\infty}^{+\infty} f(x) e(-i\lambda x, q^2) d_qx = \hat{f}(\lambda, q^2).$$  \(75\)

Therefore, a solution of (68) is

$$u(x,t) = K \int_{-\infty}^{+\infty} \hat{f}(\lambda, q^2) \xi^{-\lambda^2 t} q^{-2s} e(i\lambda x, q^2) d_q\lambda.$$  \(76\)
4.4. Symmetric $q$-wave equation

We assume that \( \frac{\log(q^{-1} - q)}{\log(q)} \in 2\mathbb{Z} \). Let’s consider the following symmetric $q$-wave equation:

\[
(\tilde{D}_{q^2,t})^2 u(x, q^{-2}t) = (\partial_{q,x})^2 u(x, q^2t), \quad x \in \mathbb{R}_q and t \in \tilde{\mathbb{R}}_{q,+},
\]

(77)

with the initial conditions

\[
u(x, 0) = f(x), \quad \tilde{D}_{q^2,t}u(x, 0) = g(x), \quad f, g \in L^2_q(\mathbb{R}_q).
\]

(78)

By applying the $q^2$-Fourier and the symmetric $q$-Mellin transform, we obtain

\[
[s - 1]_{q^2}[s - 2]_{q^2}U(\lambda, s - 2) = -\lambda^2 q^{-2(s-1)}U(\lambda, s).
\]

(79)

A solution of the equation (79) is given by:

\[
U(\lambda, s) = \left[ C(\lambda) (-i\lambda)^{-s} + C'(\lambda) (i\lambda)^{-s} \right] q^{s(s+1)} \tilde{\Gamma}_{q^2}(s),
\]

(80)

where $C(\lambda)$ and $C'(\lambda)$ are functions of $\lambda$ only.

From the symmetric $q$-Mellin inversion formula, we get

\[
\hat{u}(\lambda, t) = \left[ C(\lambda) \tilde{\xi}_q^{i\lambda t} + C'(\lambda) \tilde{\xi}_q^{-i\lambda t} \right],
\]

(81)

where $\hat{u}(\lambda, t)$ is the $q^2$-Fourier transform of $u(x, t)$ with respect to $x$.

Now we rewrite (81) in terms of the symmetric $q$-Sine and the $q$-Cosine function which are defined by

\[
\tilde{\text{Sin}}_q(x) = \frac{\tilde{\xi}_q - \tilde{\xi}_q^{-ix}}{2i}, \quad \tilde{\text{Cos}}_q(x) = \frac{\tilde{\xi}_q + \tilde{\xi}_q^{-ix}}{2}.
\]

(82)

The result is

\[
\hat{u}(\lambda, t) = D(\lambda) \tilde{\text{Cos}}_{q^2}(\lambda t) + D'(\lambda) \tilde{\text{Sin}}_{q^2}(\lambda t),
\]

(83)

where $D(\lambda)$ and $D'(\lambda)$ are functions of $\lambda$.

Now, the inverse $q^2$-Fourier transform of (83) gives:

\[
u(x, t) = K \int_{-\infty}^{+\infty} \left( D(\lambda) \tilde{\text{Cos}}_{q^2}(\lambda t) + D'(\lambda) \tilde{\text{Sin}}_{q^2}(\lambda t) \right) e(i\lambda x, q^2) d_q\lambda.
\]

(84)

By taking $t = 0$ in (84), we get $D(\lambda) = \hat{f}(\lambda, q^2)$.

On the other hand, by using the relations

\[
\tilde{D}_{q^2,x} \tilde{\text{Sin}}_{q^2}(\lambda t) = -\lambda \tilde{\text{Sin}}_{q^2}(q^2\lambda t),
\]

\[
\tilde{D}_{q^2,x} \tilde{\text{Cos}}_{q^2}(\lambda t) = -\lambda \tilde{\text{Cos}}_{q^2}(q^2\lambda t),
\]

we have

\[
u(x, 0) = f(x), \quad \tilde{D}_{q^2,x}u(x, 0) = g(x), \quad f, g \in L^2_q(\mathbb{R}_q).
\]
and

\[ \tilde{D}_{q^2} \tilde{\cos}_{q^2} (\lambda t) = \lambda \tilde{\cos}_{q^2} (q^2 \lambda t), \]

we get

\[ \hat{g} (\lambda, q^2) = D' (\lambda) \lambda. \]  

(85)

Therefore the final solution of (77) is

\[ u(x, t) = K \int_{-\infty}^{+\infty} \left( \hat{f} (\lambda, q^2) \tilde{\cos}_{q^2} (\lambda t) + \frac{\hat{g} (\lambda, q^2)}{\lambda} \tilde{\sin}_{q^2} (\lambda t) \right) e(i\lambda x, q^2) \, d\lambda. \]  

(86)

REFERENCES

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