LE MATEMATICHE
Vol. LXX (2015) – Fasc. II, pp. 255–270
doi: 10.4418/2015.70.2.17

THE SYMMETRIC MELLIN TRANSFORM IN QUANTUM CALCULUS

KAMEL BRAHIM - BOCHRA NEFZI - ANIS BSAISSA

In this paper, we define the q-analogue of Mellin transform symmetric under interchange of q and q^{-1}, and present some of its main properties and explore the possibility of using the integral transform to solve a class of differential q-differences equations.

1. Introduction

The study of q-analysis is an old subject, which dates back to the end of the 19^{th} century. The subject of q-analysis concerns mainly the properties of the so-called q-special functions, which are the extensions of the classical special functions based on a parameter, or the base, q.

It is well known that one of the purposes of integral transforms like Fourier, and Mellin is to solve differential equations using these q-special functions.

In this present paper, we are concerned with the study of the q-analogue of the Mellin transform using the symmetric q-Jackson integral. We also discuss its properties and we give its inversion formula. Furthermore, a q-analogue of the Titchmarsh theorem is proved and we solve respectively the q-diffusion and the q-wave equations using the symmetric q-derivative operator and the symmetric q-Mellin transform.

Entrato in redazione: 13 novembre 2014

AMS 2010 Subject Classification: 33D60, 33D05, 33D15.
Keywords: symmetric q-Mellin transform, symmetric q-gamma function.

This paper is organized as follows. In Section 2, we present some preliminary results and notations useful in the following sections. In Section 3 we introduce the symmetric q-Mellin transform, we discuss its definition domain and its properties. Special attention is devoted to the inversion formula. In Section 4 we study the q-analogue of the convolution product. Finally, in Section 5, a q-analogue of the Titchmarsh theorem and a solutions of the q-diffusion and the q-wave equations are given.

2. Notations and preliminaries

Throughout this paper, we will fix $q > 0$, $q \neq 1$. We recall some usual notions and notations used in q-theory (see [1–3, 5, 7–11]).
Let $a \in \mathbb{C}$, the symmetric q-numbers $\widetilde{[a]}_q$ and symmetric q-factorials $\widetilde{[n]}_q!$ are defined by

$$\widetilde{[a]}_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \tag{1}$$

and

$$\widetilde{[n]}_q! = \widetilde{[1]}_q \widetilde{[2]}_q \dots \widetilde{[n]}_q. \tag{2}$$

Clearly, these two symmetric q-analogues satisfy

$$\widetilde{[a]}_q = q^{-a+1}\,[a]_{q^2} \qquad \text{and} \qquad \widetilde{[n]}_q! = q^{\frac{-n(n-1)}{2}}\,[n]_{q^2}!, \tag{3}$$

where

$$[a]_q = \frac{1-q^a}{1-q}. \tag{4}$$

It is easy to prove that

$$\begin{aligned} \widetilde{[n]}_q &= \widetilde{[n]}_{q^{-1}} = -\widetilde{[-n]}_q, \\ \widetilde{[n+m]}_q &= q^n \widetilde{[m]}_q + q^{-m}\widetilde{[n]}_q = q^m \widetilde{[n]}_q + q^{-n}\widetilde{[m]}_q \\ \widetilde{[0]}_q &= 0, \widetilde{[1]}_q = 1. \end{aligned} \tag{5}$$

The symmetric q-shifted factorial is defined by (see [11]):

$$\left(\widetilde{[a]}_q\right)_m = \begin{cases} \widetilde{[a]}_q \widetilde{[a+1]}_q \dots \widetilde{[a+m-1]}_q & \text{if } m = 1,2,\dots \\ 1 & \text{if } m = 0 \end{cases}, \tag{6}$$

$$= \begin{cases} q^{\frac{-m(m-1)}{2}} \frac{q^{-am}}{(q^{-1}-q)^m}\left(q^{2a};q^2\right)_m & \text{if } m = 1,2,\dots \\ 1 & \text{if } m = 0 \end{cases}, \tag{7}$$

where

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \qquad n \in \mathbb{N}^*. \tag{8}$$

The symmetric q-hypergeometric series ${}_n\tilde{\Psi}_{n-1}$ is defined by (see [11]):

$${}_n\tilde{\Psi}_{n-1}\left(\begin{array}{c} a_1,\ldots,a_n \\ b_1,\ldots,b_{n-1}\end{array};q;z\right) = \sum_{m=0}^{\infty} \frac{\left(\widetilde{[a_1]_q}\right)_m \cdots \left(\widetilde{[a_n]_q}\right)_m}{\left(\widetilde{[b_1]_q}\right)_m \cdots \left(\widetilde{[b_{n-1}]_q}\right)_m} \frac{z^m}{\widetilde{[m]_q}!}. \tag{9}$$

For arbitrary number of numerator and denominator parameters, we introduce the generalized symmetric q-hypergeometric series as:

for $q \in]0,1[$,

$${}_r\tilde{\Psi}_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{array};q;z\right) = \sum_{m=0}^{\infty} \frac{\left(\widetilde{[a_1]_q}\right)_m \cdots \left(\widetilde{[a_r]_q}\right)_m}{\left(\widetilde{[b_1]_q}\right)_m \cdots \left(\widetilde{[b_s]_q}\right)_m} \left[\frac{q^{\frac{-m(m-1)}{2}}}{(q^{-1}-q)^m}\right]^{1+s-r} \frac{z^m}{\widetilde{[m]_q}!}, \tag{10}$$

and for $q > 1$,

$${}_r\tilde{\Psi}_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{array};q;z\right) = \sum_{m=0}^{\infty} \frac{\left(\widetilde{[a_1]_q}\right)_m \cdots \left(\widetilde{[a_r]_q}\right)_m}{\left(\widetilde{[b_1]_q}\right)_m \cdots \left(\widetilde{[b_s]_q}\right)_m} \left[\frac{q^{\frac{m(m-1)}{2}}}{(q-q^{-1})^m}\right]^{1+s-r} \frac{z^m}{\widetilde{[m]_q}!}. \tag{11}$$

We introduce the symmetric q-binomial theorem, expressed with the symmetric q-hypergeometric series by

$${}_1\tilde{\Psi}_0\left(\begin{array}{c} a \\ - \end{array};q;z\right) = \sum_{m=0}^{\infty} \frac{\left(\widetilde{[a]_q}\right)_m}{\widetilde{[m]_q}!} z^m = \frac{(q^a z;q^2)_\infty}{(q^{-a}z;q^2)_\infty}. \tag{12}$$

The symmetric q-derivative $\tilde{D}_q f$, of a function f is given by

$$\begin{aligned}\tilde{D}_q f(x) &= \begin{cases} \frac{f(qx)-f\left(q^{-1}x\right)}{(q-q^{-1})x}, if\ x \neq 0, \\ f'(0), \text{ when } f'(0) \text{ exists}\end{cases}; \\ &= D_{q^2} f\left(q^{-1}x\right),\end{aligned} \tag{13}$$

where

$$D_q f(x) = \begin{cases} \frac{f(qx)-f(x)}{(q-1)x}, \text{if } x \neq 0, \\ f'(0), \text{ provided } f'(0) \text{ exists}\end{cases}. \tag{14}$$

For $n \in \mathbb{N}$, we note

$$\tilde{D}_q^1 = \tilde{D}_q, \qquad \tilde{D}_q^n = \tilde{D}_q\left(\tilde{D}_q^{n-1}\right). \tag{15}$$

The symmetric q-derivative has the following property

$$\tilde{D}_q\left(f(x)g(x)\right) = f(qx)\tilde{D}_q g(x) + g\left(q^{-1}x\right)\tilde{D}_q f(x). \tag{16}$$

The symmetric q-Jackson integrals are defined by (see [2])

$$\int_0^a f(x)\widetilde{d}_q x = \left(q^{-1}-q\right)a \sum_{n=1,3,\ldots}^{+\infty} f(aq^n)q^n, \quad q \in]0,1[\tag{17}$$

$$\int_0^\infty f(x)\widetilde{d}_q x = \left|q^{-1}-q\right| \sum_{n=\pm1,\pm3,\ldots} f(q^n)q^n,$$

and the q-Jackson integrals are given by:
for $q \in]0,1[$:

$$\int_0^a f(x)d_q x = (1-q)a\sum_{n=0}^{+\infty} f(aq^n)q^n, \tag{18}$$

$$\int_0^\infty f(x)d_q x = (1-q)\sum_{n=0}^{+\infty} f(q^n)q^n,$$

$$\int_{-\infty}^{+\infty} f(x)d_q x = (1-q)\sum_{n=0}^{+\infty} f(q^n)q^n + (1-q)\sum_{n=0}^{+\infty} f(-q^n)q^n, \tag{19}$$

provided the sums converge absolutely. Using these symmetric q-integrals, we set

$$L_q^1\left(\widetilde{\mathbb{R}}_{q,+}\right) = \left\{ f : \int_0^\infty |f(x)|\widetilde{d}_q x < \infty. \right\}, \tag{20}$$

where $\widetilde{\mathbb{R}}_{q,+}$ is the set

$$\widetilde{\mathbb{R}}_{q,+} = \left\{q^{2n+1}, \quad n \in \mathbb{Z}\right\}, \tag{21}$$

and we write for $p > 0$,

$$L_q^1(\mathbb{R}_{q,+}) = \left\{ f : \int_0^{+\infty} |f(x)|d_q x < \infty. \right\}, \tag{22}$$

$$L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{+\infty} |f(x)|^p d_q x\right)^{\frac{1}{p}} < \infty. \right\},$$

where

$$\mathbb{R}_{q,+} = \{q^n, \quad n \in \mathbb{Z}\}, \mathbb{R}_q = \{\pm q^n, \quad n \in \mathbb{Z}\}. \tag{23}$$

The symmetric improper q-integral is defined by

$$\int_0^{\infty/A} f(x)\,\widetilde{d}_q x = \left|q^{-1}-q\right| \sum_{k\in\mathbb{Z}} f\left(\frac{q^{2k+1}}{A}\right)\frac{q^{2k+1}}{A}. \tag{24}$$

In the case $A = q^{2n}$, we can write

$$\int_0^{\infty/q^{2n}} f(x)\,\widetilde{d}_q x = \int_0^{\infty} f(x)\,\widetilde{d}_q x. \tag{25}$$

The symmetric q-Jackson integral in a generic interval $[a,b]$ is given by

$$\int_a^b f(x)\,\tilde{d}_q x = \int_0^b f(x)\,\tilde{d}_q x - \int_0^a f(x)\,\tilde{d}_q x, \tag{26}$$

we have, in particular

$$\int_{q^{m+1}}^{q^{m-1}} f(x)\,\tilde{d}_q x = \left(q^{-1}-q\right) q^m f(q^m). \tag{27}$$

Theorem 2.1. *1. If F is any anti q-derivative of the function f, namely $\tilde{D}_q F = f$, continuous at $x = 0$, then*

$$\int_0^a f(x)\,\widetilde{d}_q x = F(a) - F(0). \tag{28}$$

2. For any function f we have:

$$\tilde{D}_q\left(\int_0^x f(t)\,\widetilde{d}_q t\right) = f(x). \tag{29}$$

3. A symmetric q-analogue of the integration by parts formula is given by

$$\int_0^a f(qx)\,\tilde{D}_q g(x)\,\widetilde{d}_q x = f(b)g(b) - f(a)g(a) + \int_0^a g\left(q^{-1}x\right)\tilde{D}_q f(x)\,\widetilde{d}_q x. \tag{30}$$

The symmetric q-analogues of the exponential function are given by

$$\tilde{E}_q^z = \begin{cases} {}_1\tilde{\Psi}_1\left(\begin{matrix}1\\1\end{matrix};q;\left(q^{-1}-q\right)z\right) = \sum_{n\geqslant 0} q^{-\frac{n(n-1)}{2}} \frac{z^n}{\widetilde{[n]}_q!} & \text{for } 0<q<1 \\ {}_1\tilde{\Psi}_1\left(\begin{matrix}1\\1\end{matrix};q;\left(q-q^{-1}\right)z\right) = \sum_{n\geqslant 0} q^{\frac{n(n-1)}{2}} \frac{z^n}{\widetilde{[n]}_q!} & \text{for } q>1 \end{cases} \tag{31}$$

$$\tilde{e}_q^z = {}_1\tilde{\Psi}_0\left(1;-;q;z\right) = \sum_{n\geqslant 0} \frac{z^n}{\widetilde{[n]}_q!}, \tag{32}$$

and

$$\tilde{\xi}_q^z = \begin{cases} {}_2\tilde{\Psi}_0\left(\begin{array}{c} 1,1 \\ - \end{array};q;\frac{z}{q^{-1}-q}\right) = \sum_{n\geqslant 0} q^{\frac{n(n-1)}{2}} \frac{z^n}{\widetilde{[n]}_q!} & \text{for } 0<q<1 \\ {}_2\tilde{\Psi}_0\left(\begin{array}{c} 1,1 \\ - \end{array};q;\frac{z}{q-q^{-1}}\right) = \sum_{n\geqslant 0} q^{-\frac{n(n-1)}{2}} \frac{z^n}{\widetilde{[n]}_q!} & \text{for } q>1 \end{cases}.$$

The symmetric q-Gamma function $\tilde{\Gamma}_q$ is defined by (see [3])

$$\tilde{\Gamma}_q(x) = q^{-(x-1)(x-2)/2}\Gamma_{q^2}(x),\ x \neq 0,-1,-2,\ldots, \tag{33}$$

where

$$\Gamma_q(x) = \begin{cases} (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} & \text{for} \quad q\in\,]0,1[\\ q^{\frac{x(x-1)}{2}}(q-1)^{1-x}\frac{\left(q^{-1};q^{-1}\right)_\infty}{\left(q^{-x};q^{-1}\right)_\infty} & \text{for} \quad q>1 \end{cases}. \tag{34}$$

It is well known that it satisfies

$$\tilde{\Gamma}_q(x+1) = \widetilde{[x]}_q\tilde{\Gamma}_q(x), \quad \tilde{\Gamma}_q(1) = 1 \quad \text{and} \quad \tilde{\Gamma}_{q^{-1}}(x) = \tilde{\Gamma}_q(x). \tag{35}$$

Theorem 2.2. *For any $x>0$ we have:*
For $q\in\,]0,1[$,

$$\tilde{\Gamma}_q(x) = q^{-x(x-3)/2}K_{q^2}(x)\int_0^{\infty/\left(q^{-1}-q\right)} t^{x-1}\tilde{E}_q^{-t}\tilde{d}_qt, \tag{36}$$

where

$$K_{q^2}(x) = \frac{\left(-q^2,-1;q^2\right)_\infty}{\left(-q^{2x},-q^{2(1-x)};q^2\right)_\infty}, \tag{37}$$

and

$$\tilde{\Gamma}_q(x) = q^{-x(x+1)/2}\int_0^{\infty/\left(q^{-1}-q\right)} t^{x-1}\tilde{\xi}_q^{-t}\tilde{d}_qt. \tag{38}$$

Moreover, if $Log\left(q^{-1}-q\right)/Log(q)\in 2\mathbb{Z}$, we obtain

$$\tilde{\Gamma}_q(x) = q^{\frac{-x(x+1)}{2}}\int_0^{\infty} t^{x-1}\tilde{\xi}_q^{-t}\tilde{d}_qt, \quad q\in\,]0,1[. \tag{39}$$

Recently, R. L. Rubin [9] introduced a q-derivative operator ∂_q as follows

$$\partial_q(f)(z) = \frac{f\left(q^{-1}z\right)+f\left(-q^{-1}z\right)-f(qz)+f(-qz)-2f(-z)}{2(1-q)z}. \tag{40}$$

The q^2-analogue of exponential function is given by

$$e\left(z;q^2\right)=\cos\left(-iz;q^2\right)+i\sin\left(-iz;q^2\right), \tag{41}$$

where cosine and sine are the q-trigonometric functions defined by:

$$\cos\left(x;q^2\right)=\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n}}{[2n]_q!}, \tag{42}$$
$$\sin\left(x;q^2\right)=\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n+1}}{[2n+1]_q!}.$$

In [9], R.L. Rubin defned the q^2-analogue Fourier transform by

$$\hat{f}\left(x;q^2\right)=F_q(f)(x)=K\int_{-\infty}^{+\infty}f(t)\,e\left(-itx;q^2\right)d_qt,\quad x\in\mathbb{R}_q, \tag{43}$$

where

$$K=\frac{\left(q;q^2\right)_\infty}{2\left(q^2;q^2\right)_\infty(1-q)^{1/2}}. \tag{44}$$

We remind the following properties:

1. If $f(u), uf(u)\in L_q^1(\mathbb{R}_q)$, then

$$\partial_q(F_q(f))(x)=F_q(-iuf(u))(x). \tag{45}$$

2. If f *and* $\partial_q f\in L_q^1(\mathbb{R}_q)$, then

$$F_q(\partial_q f)(x)=ixF_q(f)(x). \tag{46}$$

3. For $f\in L_q^2(\mathbb{R}_q)$, we have

$$f(t)=K\int_{-\infty}^{+\infty}F_q(f)(x)\,e\left(itx;q^2\right)d_qx,\quad t\in\mathbb{R}_q. \tag{47}$$

3. The symmetric q-Mellin transform

Definition 3.1. Let f be a function defined on $\tilde{\mathbb{R}}_{q,+}$ we define the symmetric q-Mellin transform of f as

$$\tilde{M}_q(f)(s)=\tilde{M}_q[f(t)](s)=\int_0^{\infty}t^{s-1}f(t)\,\tilde{d}_qt\quad q\in\,]0,1[. \tag{48}$$

Theorem 3.2. *Let f be a function defined on $\tilde{\mathbb{R}}_{q,+}$ and let $u, v \in \mathbb{R}$ with $u > v$. We suppose*

$$f(x) = O_{o^+}(x^u) \qquad \text{and} \qquad f(x) = O_{+\infty}(x^v). \tag{49}$$

Then $\tilde{M}_q(f)(s)$ exists in the strip $\langle -u, -v \rangle$.

Theorem 3.3. *If f is a function defined on $\tilde{\mathbb{R}}_{q,+}$, then $\tilde{M}_q(f)(s)$ is analytic on the strip $\left\langle \alpha_{q,f}, \beta_{q,f} \right\rangle$ and we have*

$$\forall s \in \left\langle \alpha_{q,f}, \beta_{q,f} \right\rangle, \frac{d}{ds}\tilde{M}_q(f)(s) = \tilde{M}_q[Log(t)f(t)](s). \tag{50}$$

3.1. Properties

In the following subsection, we give some interesting properties of the symmetric q-Mellin transform.

1. For $a = q^{2n}$, $\quad n \in \mathbb{Z}$ and $s \in \left\langle \alpha_{q,f}, \beta_{q,f} \right\rangle$, we have

$$\tilde{M}_q[f(at)](s) = a^{-s}\tilde{M}_q(f)(s). \tag{51}$$

2. For $s \in \left\langle -\beta_{q,f}, -\alpha_{q,f} \right\rangle$, we have

$$\tilde{M}_q\left[f\left(\frac{1}{t}\right)\right](s) = \tilde{M}_q(f)(-s). \tag{52}$$

3. For $s \in \left\langle 1-\beta_{q,f}, 1-\alpha_{q,f} \right\rangle$, we have

$$\tilde{M}_q\left[\frac{1}{t}f\left(\frac{1}{t}\right)\right](s) = \tilde{M}_q(f)(1-s). \tag{53}$$

4. For $s \in \left\langle \alpha_{q,f}, \beta_{q,f} \right\rangle$, we have

$$\tilde{M}_q\left[t\tilde{D}_{q^2}f(t)\right](s) = \widetilde{[-s]}_{q^2}\tilde{M}_q(f)(s). \tag{54}$$

5. For $s \in \left\langle \alpha_{q,f}+1, \beta_{q,f}+1 \right\rangle$, we have

$$\tilde{M}_q\left[\tilde{D}_{q^2}f(t)\right](s) = \widetilde{[1-s]}_{q^2}\tilde{M}_q(f)(s-1). \tag{55}$$

By induction, we obtain that, for $n \in \mathbb{N}^*$ and $s \in \left\langle \alpha_{q,f}+n, \beta_{q,f}+n \right\rangle$,

$$\tilde{M}_q\left[\tilde{D}_{q^2}^{(n)}f(t)\right](s) = \widetilde{[1-s]}_{q^2}\dots\widetilde{[n-s]}_{q^2}\tilde{M}_q(f)(s-n).$$

6. Given ρ a positive odd integer and $s \in \langle \rho\alpha_{q^\rho,f}, \rho\beta_{q^\rho,f} \rangle$, we have

$$\tilde{M}_q\left[f\left(t^\rho\right)\right](s) = \widetilde{\left[\frac{1}{\rho}\right]}_{q^\rho} \tilde{M}_{q^\rho}(f)\left(\frac{s}{\rho}\right).$$

7. Let $(\mu_k)_k$ be a sequence of $\mathbb{R}_{q,+} \setminus \tilde{\mathbb{R}}_{q,+}$, let $(\lambda_k)_k$ be a sequence of $\mathbb{C}$, and let f be a suitable function, then we have

$$\tilde{M}_q\left[\sum_{k\geqslant 0} \lambda_k f\left(\mu_k t\right)\right](s) = \left(\sum_{k\geqslant 0} \frac{\lambda_k}{\mu_k^s}\right) \tilde{M}_q(f)(s),$$

provided the sums converge.

3.2. The symmetric q-Mellin inversion formula

Theorem 3.4. *Let f be a function defined over $\tilde{\mathbb{R}}_{q,+}$, and let $c \in \left]\alpha_{q,f}, \beta_{q,f}\right[$, then*

$$\forall x \in \tilde{\mathbb{R}}_{q,+}, \quad \frac{1}{2i\pi} \frac{\log(q)}{(q^{-1}-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \tilde{M}_q(f)(s) x^{-s} ds = f(x). \tag{56}$$

Proof. Let $c \in \left]\alpha_{q,f}, \beta_{q,f}\right[$ and $x = q^{2k+1} \in \tilde{\mathbb{R}}_{q,+}$, we have:

$$\begin{aligned}
&\frac{1}{2i\pi} \frac{\log(q)}{(q^{-1}-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \tilde{M}_q(f)(s) x^{-s} ds \\
&= \frac{\log(q)}{2i\pi} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \left(\sum_{n\in\mathbb{Z}} f\left(q^{2n+1}\right)\left(q^{2n+1}\right)^s\right)\left(q^{2k+1}\right)^{-s} ds \\
&= \frac{\log(q)}{2i\pi} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \left(\sum_{n\in\mathbb{Z}} f\left(q^{2n+1}\right)\left(q^{2n-2k}\right)^s\right) ds,
\end{aligned}$$

since the series $\sum_{n\in\mathbb{Z}} f\left(q^{2n+1}\right)\left(q^{2n-2k}\right)^s$ converge uniformly with respect to s, one gets:

$$\begin{aligned}
&\frac{1}{2i\pi} \frac{\log(q)}{(q^{-1}-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \tilde{M}_q(f)(s) x^{-s} ds \\
&= \frac{i\log(q)}{2i\pi} \sum_{n\in\mathbb{Z}} q^{2c(n-k)} f\left(q^{2n+1}\right) \int_{-\pi/\log(q)}^{\pi/\log(q)} q^{2i(n-k)t} dt \\
&= \sum_{n\in\mathbb{Z}} q^{2c(n-k)} f\left(q^{2n+1}\right) \delta_{n,k} \\
&= f\left(q^{2k+1}\right).
\end{aligned}$$

□

3.3. Symmetric q-Mellin's convolution product

Definition 3.5. The symmetric q-Mellin convolution product of the functions f and g is the function $f\star_{\tilde{M}_q} g$ defined by

$$f\star_{\tilde{M}_q} g(x)=\int_0^{\infty} f(t)\, g\left(q\frac{x}{t}\right)\frac{\tilde{d}_q t}{t},\qquad x\in\tilde{\mathbb{R}}_{q,+},\tag{57}$$

provided the symmetric q-integral exists.

Theorem 3.6. *If the symmetric q-Mellin convolution product of f and g exists, then*

$$f\star_{\tilde{M}_q} g=g\star_{\tilde{M}_q} f,\tag{58}$$

$$\tilde{M}_q\left[f\star_{\tilde{M}_q} g\right](s)=q^{-s}\tilde{M}_q(f)(s)\tilde{M}_q(g)(s).\tag{59}$$

Proof.

$$\begin{aligned}\tilde{M}_q\left[f\star_{\tilde{M}_q} g\right](s)&=\int_0^{\infty} x^{s-1} f\star_{\tilde{M}_q} g(x)\,\tilde{d}_q x\\&=\left(q^{-1}-q\right)\int_0^{\infty} x^{s-1}\left(\sum_{n\in\mathbb{Z}} f\left(q^{2n+1}\right) g\left(xq^{-2n}\right)\right)\tilde{d}_q x\\&=\left(q^{-1}-q\right)\sum_{n\in\mathbb{Z}} f\left(q^{2n+1}\right)\int_0^{\infty} x^{s-1} g\left(xq^{-2n}\right)\tilde{d}_q x\\&=\left(q^{-1}-q\right)\sum_{n\in\mathbb{Z}} f\left(q^{2n+1}\right)\left(q^{-2n}\right)^{-s}\tilde{M}_q(g)(s)\\&=q^{-s}\tilde{M}_q(g)(s)\tilde{M}_q(f)(s).\end{aligned}$$

□

Theorem 3.7. *For the suitable functions f and g, we have the following relations:*

$$\frac{1}{2i\pi}\frac{\log(q)}{(q^{-1}-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)}\tilde{M}_q(f)(s)\tilde{M}_q(g)(1-s)\,ds=\int_0^{\infty} g(x)f(x)\,\tilde{d}_q x,\tag{60}$$

and

$$\frac{1}{2i\pi}\frac{\log(q)}{(q^{-1}-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} q^{-s}\tilde{M}_q(f)(s)\tilde{M}_q(g)(s)\,ds=\int_0^{\infty} f(t)\,g\left(\frac{q}{t}\right)\frac{\tilde{d}_q t}{t}.\tag{61}$$

Proof. In order to prove the first relation, let $c\in\mathbb{R}$ such that $c\in\,]\alpha_{q,f},\beta_{q,f}[\,\cap\,]1-\beta_{q,g},1-\alpha_{q,g}[$. We put $I(c)=[c-i\pi/Log(q),c+i\pi/Log(q)]$. From the symmetric q-Mellin inversion formula, and the relation

$$\sup_{s\in I(c)}\left|\left(q^{2n+1}\right)^{1-s} g(q^{2n+1})\tilde{M}_q(f)(s)\right|=q^{(1-c)(2n+1)}\left|g(q^{2n+1})\right|\sup_{s\in I(c)}\left|\tilde{M}_q(f)(s)\right|,$$

we obtain

$$\begin{aligned}
&\frac{1}{2i\pi}\frac{\log(q)}{(q^{-1}-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)}\tilde{M}_q(f)(s)\tilde{M}_q(g)(1-s)\,ds\\
&=\frac{1}{2i\pi}\log(q)\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)}\sum_{n\in\mathbb{Z}}\left(q^{2n+1}\right)^{1-s}g\left(q^{2n+1}\right)\tilde{M}_q(f)(s)\,ds\\
&=\frac{\log(q)}{2i\pi}\sum_{n\in\mathbb{Z}}q^{2n+1}g\left(q^{2n+1}\right)\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)}\left(q^{2n+1}\right)^{-s}\tilde{M}_q(f)(s)\,ds\\
&=\frac{1}{2i\pi}\frac{\log(q)}{(q^{-1}-q)}\int_0^\infty g(x)\left(\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)}x^{-s}\tilde{M}_q(f)(s)\,ds\right)\tilde{d}_qx\\
&=\int_0^\infty g(x)f(x)\,\tilde{d}_qx.
\end{aligned}$$

□

4. Applications

4.1. Symmetric q-integral equations

Theorem 4.1. *Let K and g be functions defined on $\tilde{\mathbb{R}}_{q,+}$. We suppose that $\langle\alpha_{q,L},\beta_{q,L}\rangle\cap\langle 1-\beta_{q,g},1-\alpha_{q,g}\rangle$ is not empty, for a suitable real c, we put*

$$\forall x\in\tilde{\mathbb{R}}_{q,+},\ L(x)=\frac{q^{-1}}{2i\pi}\frac{\log(q)}{(q^{-1}-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)}\frac{x^{-s}}{\tilde{M}_q(K)(1-s)}ds. \tag{62}$$

Then the q-symmetric integral equation

$$\int_0^\infty f(x)K\left(q^{-1}xt\right)\tilde{d}_qx=g(t),\qquad t\in\tilde{\mathbb{R}}_{q,+}, \tag{63}$$

has as solution

$$f(x)=\int_0^\infty g(t)L\left(q^{-1}xt\right)\tilde{d}_qt,\qquad x\in\tilde{\mathbb{R}}_{q,+}. \tag{64}$$

In addition, if

$$\tilde{M}_q(K)(s)\tilde{M}_q(K)(1-s)=q^{-1}, \tag{65}$$

then equation (63) has the solution:

$$f(x)=\int_0^\infty g(t)K\left(q^{-1}xt\right)\tilde{d}_qt,\qquad x\in\tilde{\mathbb{R}}_{q,+}. \tag{66}$$

Proof. By taking the symmetric q-Mellin transform in the equation (63), we obtain

$$\tilde{M}_q(f)(s)\tilde{M}_q(K)(1-s)=q^{s-1}\tilde{M}_q(g)(1-s).$$

From the relation (62), we deduce that

$$\tilde{M}_q(f)(s) = q^s \tilde{M}_q(L)(s)\tilde{M}_q(g)(1-s).$$

Then, for $c' \in \langle \alpha_{q,L}, \beta_{q,L} \rangle \cap \langle 1-\beta_{q,g}, 1-\alpha_{q,g} \rangle$, we get:
$\forall x \in \tilde{\mathbb{R}}_{q,+}$,

$$\begin{aligned} f(x) &= \frac{\log(q)}{2i\pi(q^{-1}-q)} \int_{c'-i\pi/\log(q)}^{c'+i\pi/\log(q)} \tilde{M}_q(L)(s)\tilde{M}_q(g)(1-s)\left(q^{-1}x\right)^{-s} ds \\ &= \frac{\log(q)}{2i\pi(q^{-1}-q)} \int_{c'-i\pi/\log(q)}^{c'+i\pi/\log(q)} \tilde{M}_q\left[L\left(q^{-1}xt\right)\right](s)\tilde{M}_q(g)(1-s)\, ds. \end{aligned}$$

Finally, by the relation (60), we have:

$$\forall x \in \tilde{\mathbb{R}}_{q,+}, \qquad f(x) = \int_0^\infty g(t) L\left(q^{-1}xt\right) \tilde{d}_q t. \qquad \square$$

4.2. Symmetric q-analogue of the Titchmarsh theorem

Theorem 4.2. *Let K be a function defined on $\tilde{\mathbb{R}}_{q,+}$. Suppose that $\langle \alpha_{q,K}, \beta_{q,K} \rangle$ is not empty. If the integral equation*

$$f(x) = \int_0^\infty K\left(q^{-1}xu\right) \tilde{d}_q u \int_0^\infty K\left(q^{-1}yu\right) f(y)\, \tilde{d}_q y, \tag{67}$$

has a suitable solution f then, for every $s \in \mathbb{C}$ such that s and $1-s$ are in $\langle \alpha_{q,K}, \beta_{q,K} \rangle$, we have

$$\tilde{M}_q(K)(s)\tilde{M}_q(K)(1-s) = q^{-1}.$$

Proof. The integral equation (67) is written as a pair of reciprocal formulas as follows

1. $g(x) = \int_0^\infty K\left(q^{-1}yx\right) f(y)\, \tilde{d}_q y,$

2. $f(x) = \int_0^\infty K\left(q^{-1}yx\right) g(y)\, \tilde{d}_q y.$

By taking the symmetric q-Mellin transform in (1) and (2) at s, we get

$$\tilde{M}_q(g)(s) = q^s \tilde{M}_q(f)(1-s)\tilde{M}_q(K)(s),$$

and

$$\tilde{M}_q(f)(s) = q^s \tilde{M}_q(g)(1-s)\tilde{M}_q(K)(s),$$

changing s into $1-s$ in one of these equations and multiplying, we deduce that

$$\tilde{M}_q(K)(s)\tilde{M}_q(K)(1-s) = q^{-1}. \qquad \square$$

4.3. Symmetric q-diffusion equation

We assume that $\frac{Log\left(q^{-1}-q\right)}{Log(q)} \in 2\mathbb{Z}$. We consider the following symmetric q-diffusion equation:

$$\tilde{D}_{q^2,t}u(x,t) = (\partial_{q,x})^2 u\left(x,q^2t\right), \quad x \in \mathbb{R}_q \text{ and } t \in \tilde{\mathbb{R}}_{q,+}, \tag{68}$$

subject to the initial condition

$$u(x,0) = f(x), \qquad f \in L_q^2(\mathbb{R}_q). \tag{69}$$

By taking a the q^2-Fourier transform in x and the symmetric q-Mellin transform in t, we obtain

$$\widetilde{[s-1]}_{q^2}U(\lambda,s-1) = \lambda^2 q^{-2s}U(\xi,s). \tag{70}$$

The general solution is [6]

$$U(\lambda,s) = C(\lambda)\lambda^{-2s}q^{s(s+1)}\tilde{\Gamma}_{q^2}(s), \tag{71}$$

where $C(\lambda)$ is a function of λ only. For the relation (39), the inversion symmetric q-Mellin transform of $\lambda^{-2s}q^{s(s+1)}\tilde{\Gamma}_{q^2}(s)$ is

$$\frac{log(q)}{2i\pi(q^{-1}-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \lambda^{-2s}q^{s(s+1)}\tilde{\Gamma}_{q^2}(s)t^{-s}ds = \tilde{\xi}_{q^2}^{-\lambda^2 t}. \tag{72}$$

Then

$$u(x,t) = K\int_{-\infty}^{+\infty} C(\lambda)\,\tilde{\xi}_{q^2}^{-\lambda^2 t}e\left(i\lambda x,q^2\right)d_q\lambda. \tag{73}$$

For $t=0$, we get

$$u(x,0) = K\int_{-\infty}^{+\infty} C(\lambda)\,e\left(i\lambda x,q^2\right)d_q\lambda = f(x), \tag{74}$$

so

$$C(\lambda) = K\int_{-\infty}^{+\infty} f(x)\,e\left(-i\lambda x,q^2\right)d_qx = \hat{f}\left(\lambda,q^2\right). \tag{75}$$

Therefore, a solution of (68) is

$$u(x,t) = K\int_{-\infty}^{+\infty} \hat{f}\left(\lambda,q^2\right)\tilde{\xi}_{q^2}^{-\lambda^2 t}e\left(i\lambda x,q^2\right)d_q\lambda. \tag{76}$$

4.4. Symmetric q-wave equation

We assume that $\frac{Log(q^{-1}-q)}{Log(q)} \in 2\mathbb{Z}$. Let's consider the following symmetric q-wave equation:

$$\left(\tilde{D}_{q^2,t}\right)^2 u\left(x,q^{-2}t\right) = \left(\partial_{q,x}\right)^2 u\left(x,q^2t\right), \quad x \in \mathbb{R}_q \text{ and } t \in \tilde{\mathbb{R}}_{q,+}, \tag{77}$$

with the initial conditions

$$u(x,0) = f(x),\ \tilde{D}_{q^2,t}u(x,0) = g(x),\ f,g \in L_q^2(\mathbb{R}_q). \tag{78}$$

By applying the q^2-Fourier and the symmetric q-Mellin transform, we obtain

$$\widetilde{[s-1]}_{q^2}\widetilde{[s-2]}_{q^2}U(\lambda,s-2) = -\lambda^2 q^{-2(2s-1)}U(\lambda,s). \tag{79}$$

A solution of the equation (79) is given by:

$$U(\lambda,s) = \left[C(\lambda)(-i\lambda)^{-s} + C'(\lambda)(i\lambda)^{-s}\right] q^{s(s+1)}\tilde{\Gamma}_{q^2}(s), \tag{80}$$

where $C(\lambda)$ and $C'(\lambda)$ are functions of λ only.
From the symmetric q-Mellin inversion formula, we get

$$\hat{u}(\lambda,t) = \left[C(\lambda)\,\widetilde{\xi}_{q^2}^{\,i\lambda t} + C'(\lambda)\,\widetilde{\xi}_{q^2}^{\,-i\lambda t}\right], \tag{81}$$

where $\hat{u}(\lambda,t)$ is the q^2-Fourier transform of $u(x,t)$ with respect to x.
Now we rewrite (81) in terms of the symmetric q-Sine and the q-Cosine function which are defined by

$$\widetilde{Sin}_q(x) = \frac{\widetilde{\xi}_q^{\,ix} - \widetilde{\xi}_q^{\,-ix}}{2i}, \quad \widetilde{Cos}_q(x) = \frac{\widetilde{\xi}_q^{\,ix} + \widetilde{\xi}_q^{\,-ix}}{2}. \tag{82}$$

The result is

$$\hat{u}(\lambda,t) = D(\lambda)\widetilde{Cos}_{q^2}(\lambda t) + D'(\lambda)\widetilde{Sin}_{q^2}(\lambda t), \tag{83}$$

where $D(\lambda)$ and $D'(\lambda)$ are functions of λ.

Now, the inverse q^2-Fourier transform of (83) gives:

$$u(x,t) = K\int_{-\infty}^{+\infty}\left(D(\lambda)\widetilde{Cos}_{q^2}(\lambda t) + D'(\lambda)\widetilde{Sin}_{q^2}(\lambda t)\right) e\left(i\lambda x, q^2\right) d_q\lambda. \tag{84}$$

By taking $t=0$ in (84), we get $D(\lambda) = \hat{f}\left(\lambda,q^2\right)$.

On the other hand, by using the relations

$$\tilde{D}_{q^2,t}\widetilde{Sin}_{q^2}(\lambda t) = -\lambda\widetilde{Sin}_{q^2}\left(q^2\lambda t\right),$$

and

$$\tilde{D}_{q^2,t}\widetilde{Cos}_{q^2}(\lambda t) = \lambda \widetilde{Cos}_{q^2}\left(q^2\lambda t\right),$$

we get

$$\hat{g}\left(\lambda, q^2\right) = D'(\lambda)\lambda. \tag{85}$$

Therefore the final solution of (77) is

$$u(x,t) = K\int_{-\infty}^{+\infty}\left(\hat{f}\left(\lambda, q^2\right)\widetilde{Cos}_{q^2}(\lambda t) + \frac{\hat{g}\left(\lambda, q^2\right)}{\lambda}\widetilde{Sin}_{q^2}(\lambda t)\right) e\left(i\lambda x, q^2\right) d_q\lambda. \tag{86}$$

REFERENCES

[1] R. Askey, *The q-gamma and q-beta functions*, App. Anal. 8 (1978), 125–141.

[2] I. M. Burban - A. U. Klimyk, *p,q-differentiation, p,q-integration, and p,q-hypergeometric functions related to quantum groups*, Integral Transform Special Funct. 2 (1994), 15–36.

[3] K. Brahim - Y. Sidomou, *On some symmetric q-special functions*, Le Matematiche 68 (2) (2013), 107–122.

[4] K. Brahim - R. Ouanes, *Some applications of the q-Mellin transform*, Tamsui Oxford journal of mathematical sciences 26 (3) (2010), 335–343.

[5] G. Gasper - M. Rahmen, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its application, vol. 35 Cambridge Univ. Press, Cambridge, UK, 1990.

[6] C. L. Ho, *On the use of Mellin transform to a class of q-difference-differential equations*, Physics Letters A, 268 (2000), 217–223.

[7] V. G. Kac - P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

[8] D. S. Moak, *The q-gamma function for $q > 1$*, Aequat. Math. 20 (1980), 278–285.

[9] R. L. Rubin, *A q^2-Analogue Operator for q^2-analogue Fourier Analysis*, J. Math. Analys. App. 212 (1997), 571–582.

[10] R. L. Rubin, *Duhamel Solutions of non-Homogenous q^2-Analogue Wave Equations*, Proc. of Amer. Math. Soc. 135 (3) (2007), 777–785.

[11] V. Sahai - S. Yadav, *Representations of two parameter quantum algebras and p,q-special functions*, J. Math. Anal. Appl. 335 (2007), 268–279.

[12] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed. Oxford University Press, 1948.

KAMEL BRAHIM
Department of Mathematics
Faculty of Science of Tunis, Tunisia.
e-mail: kamel.brahim@ipeit.rnu.tn

BOCHRA NEFZI
Department of Mathematics
Faculty of Science of Tunis, Tunisia.
e-mail: asimellinbochra@gmail.com

ANIS BSAISSA
Institut Préparatoire aux Études d'Ingénieur de Nabeul
e-mail: ipeibsaissa@yahoo.fr