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ON $\overline{H}$-FUNCTION

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Abstract

The present paper aim at the derivation of fractional calculus formulae for the $\overline{H}$-function due to Inayat-Hussain by the application of fractional calculus formulae due to Saigo and Meada involving a general class of polynomial $S_n^{a,b,\tau}(.)$.

1. Introduction

The $\overline{H}$-function introduced by Inayat-Hussain [1] in terms of Mellin Barnes type contour integral is defined by

$$\overline{H}(z)=\overline{H}^{m,n}_{p,q}=\overline{H}^{m,n}_{p,q}\left[z\,\middle|\,\begin{array}{l}(\alpha_j,A_j;a_j)_{1,n},(\alpha_j,A_j)_{n+1,p}\\(\beta_j,B_j)_{1,m},(\beta_j,B_j;b_j)_{m+1,q}\end{array}\right]$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\chi(s)z^s ds, \tag{1}$$

where

$$\chi(s)=\frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_js)\prod_{j=1}^{n}\{\Gamma(1-\alpha_j+A_js)\}^{a_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-\beta_j+B_js)\}^{b_j}\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_js)} \tag{2}$$

which contains fractional powers of some of the gamma functions $L=L_{i\infty}$ is a contour starting at the point $\tau-i\infty$, terminating at the point $\tau+i\infty$ with $\tau\in$

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$\Re = (-\infty, \infty)$. Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity; m, n, p, q are integers such that $1 \leq m \leq q, 0 \leq n \leq p$; $A_j > 0$ $(j = 1, \ldots, p)$, $B_j > 0$ $(j = 1, \ldots, q)$ and a_j $(j = 1, \ldots, p)$ and b_j $(j = 1, \ldots, q)$ are complex numbers. The exponents α_j $(j = 1, \ldots, n)$ and β_j $(j = m+1, \ldots, q)$ take on non integer values.

Also, from Inayat-Hussain [1], it follows that

$$\overline{H}^{m,n}_{p,q}[z] = o\left(|z|^{\zeta^*}\right) \text{ for small } z, \text{ where } \zeta^* = \min_{1 \leq j \leq m}\left[\Re\left(\frac{b_j}{B_j}\right)\right] \tag{3}$$

and

$$\overline{H}^{m,n}_{p,q}[z] = o\left(|z|^{\xi^*}\right) \text{ for large } z, \text{ where } \xi^* = \max_{1 \leq j \leq n} \Re\left[\alpha_j\left(\frac{a_j - 1}{A_j}\right)\right]. \tag{4}$$

When the exponents $\alpha_j = \beta_j = 1 \; \forall \; i$ and j, the $\overline{H}$-function reduces to the familiar Fox's H-function defined by Fox [4], and see also Mathai and Saxena [2] and [3].

Buschman and Srivastava [10, p. 4708] have shown that the sufficient condition for absolute convergence of the contour integral (1) is given by

$$\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{p} |A_j| > 0. \tag{5}$$

This condition evidently provides exponential decay of the integrand in (1), and the region of absolute convergence in (1) is

$$|\arg z| < \frac{1}{2}\pi\Omega. \tag{6}$$

Saxena [9, p. 127] has shown that $\overline{H}(z)$ makes sense and defines an analytic function of z in the following two cases:

I. $\Psi > 0$ and $0 < |z| < \infty$, where

$$\Psi = \sum_{j=1}^{m} |B_j| + \sum_{j=m+1}^{q} |B_j b_j| - \sum_{j=1}^{n} |A_j a_j| - \sum_{j=n+1}^{p} |A_j|. \tag{7}$$

II. $\Psi > 0$ and $0 < |z| < \theta^{-1}$ holds, where

$$\theta = \{\prod_{j=1}^{m} (B_j)^{-B_j}\}\{\prod_{j=1}^{n} (A_j)^{A_j a_j}\}\{\prod_{j=n+1}^{p} (A_j)^{A_j}\}\{\prod_{j=m+1}^{q} (B_j)^{-B_j b_j}\}. \tag{8}$$

A relation connecting $L^{\nu}(z)$, the polylogarithm of complex order ν, and the H-function is derived by Saxena [9, p.127, eq.(1.12)] as

$$L^{\nu}(z) = \overline{H}^{1,1}_{1,2}\left[z \mid \begin{matrix} (1,1;\nu) \\ (0,1),(0,1;\nu-1) \end{matrix}\right]. \tag{9}$$

An account of $L^{\nu}(z)$, the polylogarithm of complex order ν is available from the book by Marichev [8].
The generalized polynomial set due to Raizada [11] and is defined by the following Rodrigues type formula

$$S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l) = (At+B)^{-\alpha}(1-\tau t^r)^{-\frac{\beta}{\tau}} \times T_{k,l}^{m+n}\left[(At+B)^{\alpha+qn}(1-\tau t^r)^{\frac{\beta}{\tau+sn}}\right] \quad (10)$$

with the differential operator $T_{k,l}$ being defined as

$$T_{k,l} \equiv t^l(k+tD_t), \quad (11)$$

where $D_t = \frac{d}{dt}$. The explicit form of this generalized polynomial set ([11], p. 71, Eq.(2.34)) is

$$\begin{aligned} &S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l) \\ &= B^{qn}t^{l(m+n)}l^{m+n}(1-\tau t^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!} \\ &\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau t^r}{1-\tau t^r}\right)^p\left(\frac{At}{B}\right)^i. \end{aligned} \quad (12)$$

2. Generalized Fractional Calculus Operators

Let $\alpha,\beta,\eta \in C$ and $x \in \Re_+$; then the generalized fractional integration and fractional differentiation operators associated with Gauss hypergeometric function due to Saigo [5, 6] are defined as follows

$$\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt \quad (13)$$

$$(\Re(\alpha)>0);$$

$$= \frac{d^n}{dx^n}\left(I_{0+}^{\alpha+n,\beta-n,\eta-n}f\right)(x) \quad (14)$$

$$(\Re(\alpha)\le 0; n=[\Re(-\alpha)]+1);$$

$$\left(I_{-}^{\alpha,\beta,\eta}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)f(t)dt \quad (15)$$

$$(\Re(\alpha)>0);$$

$$= (-1)^n\frac{d^n}{dx^n}\left(I_{-}^{\alpha+n,\beta-n,\eta}f\right)(x) \quad (16)$$

$$(\Re(\alpha)\le 0; n=[\Re(-\alpha)]+1);$$

and

$$\left(D_{0+}^{\alpha,\beta,\eta}f\right)(x)=\left(I_{0+}^{-\alpha,-\beta,\alpha+\eta}f\right)(x)=\frac{d^n}{dx^n}\left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n}f\right)(x) \tag{17}$$

$$(\Re(\alpha)>0;n=[\Re(\alpha)]+1);$$

$$\left(D_{-}^{\alpha,\beta,\eta}f\right)(x)=\left(I_{-}^{-\alpha,-\beta,\alpha+\eta}f\right)(x)=(-1)^n\frac{d^n}{dx^n}\left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta}f\right)(x) \tag{18}$$

$$(\Re(\alpha)>0;n=[\Re(\alpha)]+1).$$

The Riemann-Liouville, Weyl and Erdelyi-Kober fractional calculus operators are recovered as special cases of the operators $I_{+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$ as shown below

$$\left(R_{0,x}^{\alpha}f\right)(x)=\left(I_{0+}^{\alpha,-\alpha,\eta}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt \tag{19}$$

$$(\Re(\alpha)>0);$$

$$=\frac{d^n}{dx^n}\left(R_{0,x}^{\alpha+n}f\right)(x) \tag{20}$$

$$(0<\Re(\alpha)+n\leq 1;n=1,2,3,\ldots);$$

$$\left(W_{x,\infty}^{\alpha}\right)=\left(I_{-}^{\alpha,-\alpha,\eta}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}f(t)dt \tag{21}$$

$$(\Re(\alpha)>0);$$

$$=(-1)^n\frac{d^n}{dx^n}\left(W_{x,\infty}^{\alpha+n}f\right)(x) \tag{22}$$

$$(0<\Re(\alpha)+n\leq 1;n=1,2,\ldots);$$

$$\left(E_{0,x}^{\alpha,\eta}f\right)(x)=\left(I_{0+}^{\alpha,0,\eta}f\right)(x)=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}t^{\eta}f(t)dt \tag{23}$$

$$(\Re(\alpha)>0);$$

$$\left(K_{x,\infty}^{\alpha,\eta}f\right)(x)=\left(I_{-}^{\alpha,0,\eta}f\right)(x)=\frac{x^{\eta}}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)dt \tag{24}$$

$$(\Re(\alpha)>0).$$

Now here the definition of the following generalized fractional integration and differentiation operators of any complex order involving Appell function F_3, due to Saigo and Meada [7, p.393, Eqs. (4.12) and (4. 13)] in the kernel in the following form.

Let $\alpha,\alpha',\beta,\beta',\gamma\in C$ and $x>0$, then the generalized fractional calculus operators involving the Appell function F_3 are defined by the following equations:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x t^{-\alpha'}(x-t)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)\times f(t)dt \tag{25}$$

$$(\Re(\gamma)>0);$$

$$=\frac{d^n}{dx^n}\left(I_{0+}^{\alpha,\alpha',\beta+n,\beta',\gamma+n}f\right)(x) \tag{26}$$

$$(\Re(\gamma)\leq 0;\, n=[-\Re(\gamma)]+1);$$

$$\left(I_{-}^{\alpha,\alpha'\beta,\beta',\gamma}f\right)(x)=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^{\infty}t^{-\alpha'}(t-x)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)$$

$$\times f(t)dt \tag{27}$$

$$(\Re(\gamma)>0);$$

$$=(-1)^n\frac{d^n}{dx^n}\left(I_{-}^{\alpha,\alpha',\beta,\beta'+n,\gamma+n}f\right)(x) \tag{28}$$

$$(\Re(\gamma)\leq 0;\, n=[-\Re(\gamma)]+1);$$

and

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x) \tag{29}$$

$$=\frac{d^n}{dx^n}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n}f\right)(x) \tag{30}$$

$$(\Re(\gamma)>0;\, n=[\Re(\gamma)]+1);$$

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(I_{-}^{-\alpha',\alpha,-\beta',-\beta,-\gamma}f\right)(x) \tag{31}$$

$$=(-1)^n\frac{d^n}{dx^n}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n}f\right)(x) \tag{32}$$

$$(\Re(\gamma)>0;\, n=[\Re(\gamma)]+1).$$

These operators reduce to that in (13)-(18) as the following.

$$\left(I_{0+}^{\alpha,0,\beta,\beta',\gamma}f\right)(x)=\left(I_{0+}^{\gamma,\alpha-\gamma,-\beta}f\right)(x)\,(\gamma\in C); \tag{33}$$

$$\left(I_{-}^{\alpha,0,\beta,\beta',\gamma}f\right)(x)=\left(I_{-}^{\gamma,\alpha-\gamma,-\beta}f\right)(x)\,(\gamma\in C); \tag{34}$$

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x)\,(\Re(\gamma)>0); \tag{35}$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x)\,(\Re(\gamma)>0). \tag{36}$$

Further from Saigo and Meada [7, p.394, eqs.(4.18)and (4.19)], we also have

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}f\right)(x)=\Gamma\left[\begin{array}{c}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha'\\ \rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta,\rho+\beta'\end{array}\right]$$

$$\times x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{37}$$

where $\Re(\gamma) > 0, \Re(\rho) > \max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')]$ and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} f\right)(x) =$$
$$\Gamma\left[\begin{array}{c} 1+\alpha+\alpha'-\gamma-\rho, 1+\alpha+\beta'-\gamma-\rho, 1-\beta-\rho \\ 1-\rho, 1+\alpha+\alpha'+\beta'-\gamma-\rho, 1+\alpha-\beta-\rho \end{array}\right] x^{\rho-\alpha-\alpha'+\gamma-1}, \quad (38)$$

where $\Re(\gamma) > 0, \Re(\rho) < 1+\min[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha-\beta'-\gamma)]$.
Here the symbol $\Gamma\left[\begin{array}{c} a,b,c \\ d,e,f \end{array}\right]$ will be employed to the represent the ratios of product of gamma function $\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}$.

3. Fractional integration of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ polynomial

Theorem 3.1. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C, \Re(\gamma) > 0$. Further let the constants m, n, $p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in \Re_+$ $(i = 1, \ldots, p;\ j = 1, \ldots, q)$, $\rho \in C, \sigma \in \Re_+$, $|\arg a| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents α_i $(i = 1, \ldots, n)$ and β_j $(j = m+1, \ldots, q) \notin N$ be given and satisfy the condition*

$$\sigma \max[\tau, \zeta^*] < \Re(\rho) + \min\left[0, \Re(\gamma-\alpha-\alpha'-\beta), \Re(\beta'-\alpha')\right]. \quad (39)$$

Then the fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}(at^{\sigma})\, S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1} B^{qn} l^{m+n} (1-\tau x^r)^{sn} \sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i} \frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p \left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p \left(\frac{Ax}{B}\right)^i$$

$$\times \overline{H}_{p+3,q+3}^{m,n+3}\left[ax^{\sigma}\Big|\begin{array}{c} (a_j,A_j;\alpha_j)_{1,n}, (a_j,A_j)_{n+1,p}, (\theta,\sigma), \\ (b_j,B_j)_{1,m}, (b_j,B_j;\beta_j)_{m+1,q}, (\theta+\alpha+\alpha'-\gamma,\sigma), \end{array}\right.$$
$$\left.\begin{array}{c} (\theta+\alpha+\alpha'+\beta-\gamma,\sigma), (\theta+\alpha'-\beta',\sigma) \\ (\theta+\alpha'+\beta-\gamma,\sigma), (\theta-\beta',\sigma) \end{array}\right]. \quad (40)$$

where $\theta = 1-\rho-i-l(m+n)-rp-rw$.

Proof. With (1) and (25), we have from (40)

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x t^{-\alpha'}(x-t)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)$$

$$\times\{t^{\rho-1}\frac{1}{2\pi i}\int_L\chi(s)(at^{\sigma})^s dsB^{qn}t^{l(m+n)}(1-\tau t^r)^{sn}l^{m+n}$$

$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau t^r}{1-\tau t^r}\right)^p\left(\frac{At}{B}\right)^i\}dt$$

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}$$

$$\times\frac{1}{2\pi i}\int_L\chi(s)a^s\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho+\sigma s+rw+rp+l(m+n)+i-1}\right)(x)ds. \quad (41)$$

Using (37) in the above expression, we obtain

$$=x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\frac{1}{2\pi i}\int_L\chi(s)$$

$$\times\Gamma\left[\begin{array}{c}1-\theta+\sigma s,1-\theta+\gamma-\alpha-\alpha'-\beta+\sigma s,1-\theta+\beta'-\alpha'+\sigma s\\ 1-\theta+\gamma-\alpha-\alpha'+\sigma s,1-\theta+\gamma-\alpha'-\beta+\sigma s,1-\theta+\beta'+\sigma s\end{array}\right]$$

$$\times\left(ax^{\sigma}\right)^s ds. \quad (42)$$

Which is the required result. □

On the other hand, if $\alpha'=0$ in Theorem 3.1, then by the relation (33), we arrive at

Corollary 3.2. *Let* $\alpha,\beta,\eta\in C$, $\Re(\alpha)>0$. *Further let the constants* $m,n,p,q\in N_0$, $a_i,b_j\in C$, $A_i,B_j\in\Re_+$ $(i=1,\ldots,p;\ j=1,\ldots,q)$, $\rho\in C,\sigma\in\Re_+,|\arg a|<\frac{\pi}{2}\Omega$, $\Omega>0$ *and the exponents* α_i $(i=1,\ldots,n)$ *and* β_j $(j=m+1,\ldots,q)\notin N$ *be given and satisfy the condition*

$$\sigma\max[\tau,\zeta^*]<\Re(\rho)+\min[0,\Re(\eta-\beta)]. \quad (43)$$

Then the fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\beta,\eta}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x) \tag{44}$$

$$= x^{\rho-\beta+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$
$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$
$$\times\overline{H}_{p+2,q+2}^{m,n+2}\left[ax^{\sigma}\,\middle|\,\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta,\sigma),(\theta+\beta-\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta+\beta,\sigma),(\theta-\alpha-\eta,\sigma)\end{matrix}\right].$$

Next, if we take $\beta=-\alpha$, in (44) implies that

$$\left(R_{0,x}^{\alpha}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$
$$= x^{\rho+\alpha+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$
$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$
$$\times\overline{H}_{p+1,q+1}^{m,n+1}\left[ax^{\sigma}\,\middle|\,\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta-\alpha,\sigma)\end{matrix}\right], \tag{45}$$

where

$$\Re\left[\rho+\sigma\min_{1\le j\le m}\left(\frac{b_j}{B_j}\right)\right]>0.$$

Also, if we take $\beta=0$ in (44), then we obtain

$$\left(E_{0,x}^{\alpha,\eta}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$
$$= x^{\rho+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$
$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$
$$\times\overline{H}_{p+1,q+1}^{m,n+1}\left[ax^{\sigma}\,\middle|\,\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta-\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta-\alpha-\eta,\sigma)\end{matrix}\right], \tag{46}$$

where

$$\Re\left[(\rho+\eta)+\sigma\min_{1\le j\le m}\left(\frac{b_j}{B_j}\right)\right]>0.$$

Theorem 3.3. *Let* $\alpha,\alpha',\beta,\beta',\gamma\in C, \Re(\gamma)>0$. *Further let the constants* $m,n,p,q\in N_0$, $a_i,b_j\in C$, $A_i,B_j\in\Re_+$ $(i=1,\ldots,p;\ j=1,\ldots,q)$, $\rho\in C,\sigma\in\Re_+$, $|\arg a|<\frac{\pi}{2}\Omega$, $\Omega>0$ *and the exponents* α_i $(i=1,\ldots,n)$ *and* β_j $(j=m+1,\ldots,q)\notin N$ *be given and satisfy the condition*

$$\sigma\min[\tau,\zeta^*]+1>\Re(\rho)+\max\left[\Re(\gamma-\alpha-\beta'),\Re(\beta),\Re(\gamma-\alpha-\alpha')\right]. \quad (47)$$

Then the fractional integral $I_-^{\alpha,\alpha',\beta,\beta',\gamma}$ *of the product* $\overline{H}$*-function and* $S_n^{a,b,\tau}(.)$ *exists and the following formula holds:*

$$\left(I_-^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}(at^\sigma)\,S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times\overline{H}_{p+3,q+3}^{m+3,n}\left[ax^\sigma\left|\begin{array}{l}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},\end{array}\right.\right.$$

$$\left.\begin{array}{l}(\theta,\sigma),(\theta+\alpha+\alpha'+\beta'-\gamma,\sigma),(\theta+\alpha-\beta,\sigma)\\(\theta+\alpha+\beta'-\gamma,\sigma),(\theta+\alpha'+\alpha-\gamma,\sigma),(\theta-\beta,\sigma)\end{array}\right]. \quad (48)$$

Proof. From equations (27) and (1), it follows that

$$\left(I_-^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}(at^\sigma)\,S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty t^{-\alpha}(t-x)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)$$

$$\times\{t^{\rho-1}\frac{1}{2\pi i}\int_L\chi(s)(at^\sigma)^s ds B^{qn}t^{l(m+n)}(1-\tau t^r)^{sn}l^{m+n}$$

$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau t^r}{1-\tau t^r}\right)^p\left(\frac{At}{B}\right)^i\}dt$$

Now interchanging the order of t and s-integrals, which is valid under the above stated conditions, we have

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}$$

$$\times \frac{1}{2\pi i}\int_L \chi(s)a^s\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho+\sigma s+rw+rp+l(m+n)+i-1}\right)(x)ds. \tag{49}$$

Applying the formula (38), the above expression becomes

$$= x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times \frac{1}{2\pi i}\int_L \chi(s)\Gamma\left[\begin{array}{c}\theta+\alpha+\alpha'-\gamma-\sigma s,\theta-\gamma+\alpha+\beta'-\sigma s,\theta-\beta-\sigma s\\ \theta-\sigma s,\theta-\gamma+\alpha+\alpha'+\beta'-\sigma s,\theta+\alpha-\beta-\sigma s\end{array}\right]$$

$$\times\left(ax^{\sigma}\right)^s ds. \tag{50}$$

Which is the required result. □

If we set $\alpha'=0$ in (48), then by the relation (34), we obtain the following result:

Corollary 3.4. *Let $\alpha,\beta,\eta\in C,\Re(\alpha)>0$. Further let the constants $m,n,p,q\in N_0,a_i,b_j\in C$, $A_i,B_j\in\Re_+$ $(i=1,\dots,p;\ j=1,\dots,q)$, $\rho\in C,\sigma\in\Re_+$, $|\arg a|<\frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents α_i $(i=1,\dots,n)$ and β_j $(j=m+1,\dots,q)\notin N$ be given and satisfy the condition*

$$\sigma\min[\tau,\zeta^*]+1>\Re(\rho)+\max\left[\Re(-\beta),\Re(-\eta)\right]. \tag{51}$$

Then the fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(I_{-}^{\alpha,\beta,\eta}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$= x^{\rho-\beta+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times\overline{H}_{p+2,q+2}^{m+2,n}\left[ax^{\sigma}\left|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta,\sigma),(\theta+\alpha+\beta+\eta,\sigma)\\ (b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta+\beta,\sigma),(\theta+\eta,\sigma)\end{array}\right.\right]. \tag{52}$$

If we take $\beta=-\alpha$, in (52), we get

$$\left(W_{x,\infty}^{\alpha}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_{n}^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=x^{\rho+\alpha+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^{r}\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^{i}(-p)_{e}(-i)_{j}(a)_{i}}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_{j}}{(1-a-i)_{j}}\left(\frac{-b}{\tau}-sn\right)_{p}\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^{r}}{1-\tau x^{r}}\right)^{p}\left(\frac{Ax}{B}\right)^{i}$$

$$\times\overline{H}_{p+1,q+1}^{m+1,n}\left[ax^{\sigma}\Big|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta,\sigma),(\theta+\alpha+\beta+\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta-\alpha,\sigma),\end{array}\right]. \tag{53}$$

Where

$$\Re\left[(\rho+\alpha)+\sigma\max_{1\le j\le n}\left(\alpha_j\frac{a_j-1}{A_j}\right)\right]<1.$$

Also for $\beta=0$ (52) gives the result

$$\left(K_{x,\infty}^{\alpha,\eta}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_{n}^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=x^{\rho+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^{r}\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^{i}(-p)_{e}(-i)_{j}(a)_{i}}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_{j}}{(1-a-i)_{j}}\left(\frac{-b}{\tau}-sn\right)_{p}\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^{r}}{1-\tau x^{r}}\right)^{p}\left(\frac{Ax}{B}\right)^{i}$$

$$\times\overline{H}_{p+1,q+1}^{m+1,n}\left[ax^{\sigma}\Big|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta+\alpha+\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta+\eta,\sigma),\end{array}\right]. \tag{54}$$

Where

$$\Re\left[(\rho-\eta)+\sigma\max_{1\le j\le n}\left(\alpha_j\frac{a_j-1}{A_j}\right)\right]<1.$$

4. Fractional differentiation of the product $\overline{H}-$ function and $S_n^{a,b,\tau}(.)$ polynomial

Theorem 4.1. *Let $\alpha,\alpha',\beta,\beta',\gamma\in C,\Re(\gamma)>0$. Further let the constants $m,n,p,q\in N_0$, $a_i,b_j\in C$, $A_i,B_j\in\Re_+$ $(i=1,\dots,p;\ j=1,\dots,q)$, $\rho\in C,\sigma\in\Re_+$, $|\arg a|<\frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents α_i $(i=1,\dots,n)$ and β_j $(j=m+1,\dots,q)\notin N$ be given and satisfy the condition*

$$\sigma\max[\tau,\zeta^*]<\Re(\rho)+\min\left[0,\Re(\alpha-\beta),\Re(\alpha+\alpha'+\beta'-\gamma)\right]. \tag{55}$$

Then the fractional derivative $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$= x^{\rho-\gamma+\alpha+\alpha'+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times\overline{H}_{p+3,q+3}^{m,n+3}\left[ax^{\sigma}\middle|\begin{array}{l}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},\end{array}\right.$$

$$\left.\begin{array}{c}(\theta,\sigma),(\theta-\alpha-\alpha'-\beta'+\gamma,\sigma),(\theta-\alpha+\beta,\sigma)\\(\theta-\alpha-\alpha'+\gamma,\sigma),(\theta-\alpha+\gamma-\beta',\sigma),(\theta+\beta,\sigma)\end{array}\right]. \tag{56}$$

Proof. Using (30), we have from (56)

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=\frac{d^k}{dx^k}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

where $k=[\Re(\gamma)+1]$.

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!} \tag{57}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}$$

$$\times\frac{d^k}{dx^k}\frac{1}{2\pi i}\int_L\chi(s)a^s\left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}t^{\rho+\sigma s+rw+rp+l(m+n)+i-1}\right)(x)ds.$$

Applying (37) to (57), we obtain

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}$$

$$\times(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}\frac{1}{2\pi i}\int_L\chi(s)a^s$$

$$\times\Gamma\left[\begin{array}{c}1-\theta+\sigma s,1-\theta-\gamma+\alpha+\alpha'+\beta'+\sigma s,1-\theta-\beta+\alpha+\sigma s\\1-\theta-\gamma+k+\alpha+\alpha'+\sigma s,1-\theta-\gamma+\alpha+\beta'+\sigma s,1-\theta-\beta+\sigma s\end{array}\right]$$

$$\times\frac{d^k}{dx^k}x^{\alpha+\alpha'-\gamma+k-\theta+\sigma s}ds.$$

Using $\frac{d^n}{dx^n}x^m=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$, where $m\geq n$ in the above expression, we obtain

$$=x^{\rho-\gamma+\alpha+\alpha'+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\frac{1}{2\pi i}\int_L\chi(s)$$

$$\times\Gamma\left[\begin{array}{c}1-\theta+\sigma s,1-\theta-\gamma+\alpha+\alpha'+\beta'+\sigma s,1-\theta-\beta+\alpha+\sigma s\\1-\theta-\gamma+\alpha+\alpha'+\sigma s,1-\theta-\gamma+\alpha+\beta'+\sigma s,1-\theta-\beta+\sigma s\end{array}\right]$$

$$\times\left(ax^\sigma\right)^s ds. \tag{58}$$

Which is the required result. □

The relation (35) indicates that Theorem 4.1 reduces to the following result:

Corollary 4.2. *Let $\alpha,\beta,\eta\in C,\Re(\alpha)>0$. Further let the constants $m,n,p,q\in N_0,a_i,b_j\in C$, $A_i,B_j\in\Re_+$ $(i=1,\dots,p;\ j=1,\dots,q)$, $\rho\in C,\sigma\in\Re_+$, $|\arg a|<\frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents α_i $(i=1,\dots,n)$ and β_j $(j=m+1,\dots,q)\notin N$ be given and satisfy the condition*

$$\sigma\max[\tau,\zeta^*]<\Re(\rho)+\min\left[0,\Re(\alpha+\eta+\beta)\right]. \tag{59}$$

Then the fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(D_{0+}^{\alpha,\beta,\eta}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^\sigma\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x) \tag{60}$$

$$=x^{\rho+\beta+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times\overline{H}_{p+2,q+2}^{m,n+2}\left[ax^\sigma\left|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta,\sigma),(\theta-\alpha-\beta-\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta-\beta,\sigma),(\theta-\eta,\sigma)\end{array}\right.\right].$$

Theorem 4.3. *Let $\alpha,\alpha',\beta,\beta',\gamma\in C,\Re(\gamma)>0$. Further let the constants $m,n,p,q\in N_0$, $a_i,b_j\in C$, $A_i,B_j\in\Re_+$ $(i=1,\dots,p;\ j=1,\dots,q)$, $\rho\in C,\sigma\in\Re_+$, $|\arg a|<\frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents α_i $(i=1,\dots,n)$ and β_j $(j=m+1,\dots,q)\notin N$ be given and satisfy the condition*

$$\sigma\min[\tau,\zeta^*]+1>\Re(\rho)+\max\left[\Re(\alpha+\alpha'+k-\gamma),\Re(-\beta'),\Re(\alpha'+\beta-\gamma)\right]. \quad (61)$$

Then the fractional derivative $D_-^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(D_-^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=x^{\rho-\gamma+\alpha+\alpha'+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times\overline{H}_{p+3,q+3}^{m+3,n}\left[ax^{\sigma}\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(\theta,\sigma),(\theta-\alpha-\alpha'-\beta+\gamma,\sigma),(\theta-\alpha'+\beta',\sigma)\\(\theta-\alpha-\alpha'+\gamma,\sigma),(\theta-\alpha'+\gamma-\beta,\sigma),(\theta+\beta',\sigma)\end{matrix}\right]. \quad (62)$$

Proof. From (32), we get

$$\left(D_-^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)=(-1)^k\frac{d^k}{dx^k}$$

$$\times\left(I_-^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x),$$

where $k=[\Re(\gamma)+1]$.

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}$$

$$\times(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}(-1)^k\frac{d^k}{dx^k}\frac{1}{2\pi i}\int_L\chi(s)a^s$$

$$\times\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}t^{\rho+\sigma s+rw+rp+l(m+n)+i-1}\right)(x)ds. \quad (63)$$

Applying the formula (38), we obtain

$$= B^{qn} l^{m+n} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \sum_{w=0}^{\infty} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn \right)_p \left(\frac{j+k+re}{l} \right)_{m+n}$$

$$\times (-\tau)^p \tau^w \left(\frac{A}{B} \right)^i \frac{(-sn+p)_w}{w!} \frac{1}{2\pi i} \int_L \chi(s) a^s$$

$$\times \Gamma \left[\begin{array}{c} \theta - \alpha' - \alpha + \gamma - k - \sigma s, \theta - \alpha' - \beta + \gamma - \sigma s, \theta + \beta' - \sigma s \\ \theta - \sigma s, \theta - \alpha' - \alpha - \beta + \gamma - \sigma s, \theta - \alpha' + \beta' - \sigma s \end{array} \right]$$

$$\times (-1)^k \frac{d^k}{dx^k} x^{\alpha + \alpha' - \gamma + k - \theta + \sigma s} ds.$$

$$= B^{qn} l^{m+n} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \sum_{w=0}^{\infty} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn \right)_p \left(\frac{j+k+re}{l} \right)_{m+n}$$

$$\times (-\tau)^p \tau^w \left(\frac{A}{B} \right)^i \frac{(-sn+p)_w}{w!} \frac{1}{2\pi i} \int_L \chi(s) a^s$$

$$\times \Gamma \left[\begin{array}{c} \theta - \alpha' - \alpha + \gamma - k - \sigma s, \theta - \alpha' - \beta + \gamma - \sigma s, \theta + \beta' - \sigma s \\ \theta - \sigma s, \theta - \alpha' - \alpha - \beta + \gamma - \sigma s, \theta - \alpha' + \beta' - \sigma s \end{array} \right]$$

$$\times \left(\theta - \alpha' - \alpha + \gamma - k - \sigma s \right)_k x^{\alpha + \alpha' - \gamma - \theta + \sigma s} ds.$$

$$= x^{\rho - \gamma + \alpha + \alpha' + l(m+n) - 1} B^{qn} l^{m+n} (1 - \tau x^r)^{sn} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn \right)_p \left(\frac{j+k+re}{l} \right)_{m+n} \left(\frac{-\tau x^r}{1 - \tau x^r} \right)^p \left(\frac{Ax}{B} \right)^i$$

$$\times \frac{1}{2\pi i} \int_L \chi(s) \Gamma \left[\begin{array}{c} \theta - \alpha' - \alpha + \gamma - \sigma s, \theta - \alpha' - \beta + \gamma - \sigma s, \theta + \beta' - \sigma s \\ \theta - \sigma s, \theta - \alpha' - \alpha - \beta + \gamma - \sigma s, \theta - \alpha' + \beta' - \sigma s \end{array} \right]$$

$$\times (a x^{\sigma})^s ds. \tag{64}$$

Which is the required result. □

Corollary 4.4. *Let $\alpha,\beta,\eta \in C, \Re(\alpha)>0$. Further let the constants $m,n,p,q \in N_0, a_i, b_j \in C$, $A_i, B_j \in \Re_+$ $(i=1,\ldots,p;\ j=1,\ldots,q)$, $\rho \in C, \sigma \in \Re_+$, $|\arg a| < \frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents α_i $(i=1,\ldots,n)$ and β_j $(j=m+1,\ldots,q) \notin N$ be given and satisfy the condition*

$$\sigma \min[\tau,\zeta^*]+1 > \Re(\rho)+\max\left[-\Re(\alpha+\eta),\Re(\beta+k)\right], \tag{65}$$

where $k=[\Re(\alpha)]+1$.
Then the fractional derivative $D_{-}^{\alpha,\beta,\eta}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(D_{-}^{\alpha,\beta,\eta}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x) \tag{66}$$

$$= x^{\rho+\beta+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times\overline{H}_{p+2,q+2}^{m+2,n}\left[ax^{\sigma}\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta,\sigma),(\theta-\beta+\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta-\beta,\sigma),(\theta+\alpha+\eta,\sigma)\end{matrix}\right.\right].$$

5. Fractional Integro-Differentiation of the product $\overline{H}$-function and $S_n^{a,b,\tau}$ polynomial

Theorem 5.1. *Let $\alpha,\alpha',\beta,\beta',\gamma \in C, \Re(\gamma)>0$. Further let the constants $m,n,$ $p,q\in N_0$, $a_i,b_j \in C$, $A_i,B_j \in \Re_+$ $(i=1,\ldots,p;\ j=1,\ldots,q)$, $\rho\in C,\sigma\in\Re_+$, $|\arg a| < \frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents α_i $(i=1,\ldots,n)$ and β_j $(j=m+1,\ldots,q)\notin N$ be given and satisfy the condition*

$$\sigma\max[\tau,\zeta^*] < \Re(\rho)+\min\left[0,\Re(\gamma-\alpha-\alpha'-\beta),\Re(\beta'-\alpha')\right]. \tag{67}$$

Then the fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}\left(at^{\sigma}\right)S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1}B^{qn}l^{m+n}\left(1-\tau x^r\right)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times\overline{H}^{m,n+3}_{p+3,q+3}\left[ax^{\sigma}\left|\begin{array}{l}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},\end{array}\right.\right.$$

$$\left.\begin{array}{c}(\theta,\sigma),(\theta+\alpha+\alpha'+\beta-\gamma,\sigma),(\theta+\alpha'-\beta',\sigma)\\(\theta+\alpha+\alpha'-\gamma,\sigma),(\theta+\alpha'+\beta-\gamma,\sigma),(\theta-\beta',\sigma)\end{array}\right]. \tag{68}$$

Proof. To prove (68) using equation (26), which represent integro-differentiation operator, we have

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\left[t^{\rho-1}\overline{H}^{m,n}_{p,q}(at^{\sigma})S^{a,b,\tau}_{n}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=\frac{d^k}{dx^k}\left(I^{\alpha,\alpha',\beta+k,\beta',\gamma+k}_{0+}\left[t^{\rho-1}\overline{H}^{m,n}_{p,q}(at^{\sigma})S^{a,b,\tau}_{n}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}$$

$$\times\frac{d^k}{dx^k}\frac{1}{2\pi i}\int_L\chi(s)a^s\left(I^{\alpha,\alpha',\beta+k,\beta',\gamma+k}_{0+}t^{\rho+\sigma s+rw+rp+l(m+n)+i-1}\right)(x)ds. \tag{69}$$

Using the formula (37), we obtain

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p(\tau)^w$$

$$\times\frac{1}{2\pi i}\int_L\chi(s)a^s\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}$$

$$\times\Gamma\left[\begin{array}{c}1-\theta+\sigma s,1-\theta+\gamma-\beta-\alpha-\alpha'+\sigma s,1-\theta+\beta'-\alpha'+\sigma s\\1-\theta+\gamma+k-\alpha-\alpha'+\sigma s,1-\theta+\gamma-\alpha'-\beta+\sigma s,1-\theta+\beta'+\sigma s\end{array}\right]$$

$$\times\frac{d^k}{dx^k}x^{\gamma+k-\alpha-\alpha'-\theta+\sigma s}ds.$$

Finally using $\frac{d^n}{dx^n}x^m=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$, where $m\geq n$, the above expression becomes

$$=x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\times\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\frac{1}{2\pi i}\int_L \chi(s)$$
$$\times\Gamma\left[\begin{array}{c}1-\theta+\sigma s, 1-\theta+\gamma-\beta-\alpha-\alpha'+\sigma s, 1-\theta+\beta'-\alpha'+\sigma s\\ 1-\theta+\gamma-\alpha-\alpha'+\sigma s, 1-\theta+\gamma-\alpha'-\beta+\sigma s, 1-\theta+\beta'+\sigma s\end{array}\right]$$
$$\times (ax^\sigma)^s\, ds. \tag{70}$$

Which is the required result. □

If we take $\alpha' = 0$ in (68), we arrive at

Corollary 5.2. *Let $\alpha,\beta,\eta \in C, \Re(\alpha) > 0$. Further let the constants $m,n,p,q \in N_0, a_i, b_j \in C$, $A_i, B_j \in \Re_+$ $(i = 1,\ldots,p;\ j = 1,\ldots,q)$, $\rho \in C, \sigma \in \Re_+$, $|\arg a| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents α_i $(i = 1,\ldots,n)$ and β_j $(j = m+1,\ldots,q) \notin N$ be given and satisfy the condition*

$$\sigma \max[\tau,\zeta^*] < \Re(\rho) + \min[0, \Re(\eta-\beta)]. \tag{71}$$

Then the fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\beta,\eta}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}(at^\sigma)\, S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x) \tag{72}$$
$$= x^{\rho-\beta+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$
$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$
$$\times\overline{H}_{p+2,q+2}^{m,n+2}\left[ax^\sigma\left|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\theta,\sigma),(\theta+\beta-\eta,\sigma)\\ (b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\theta+\beta,\sigma),(\theta-\alpha-\eta,\sigma)\end{array}\right.\right].$$

Theorem 5.3. *Let $\alpha,\alpha',\beta,\beta',\gamma \in C, \Re(\gamma) > 0$. Further let the constants m,n, $p,q \in N_0$, $a_i,b_j \in C$, $A_i,B_j \in \Re_+$ $(i = 1,\ldots,p;\ j = 1,\ldots,q)$, $\rho \in C, \sigma \in \Re_+$, $|\arg a| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents α_i $(i = 1,\ldots,n)$ and β_j $(j = m+1,\ldots,q) \notin N$ be given and satisfy the condition*

$$\sigma \min[\tau,\zeta^*] + 1 > \Re(\rho) + \max\left[\Re(\gamma-\alpha-\beta'), \Re(\beta), \Re(\gamma-\alpha-\alpha') + k\right]. \tag{73}$$

where $k = [-\Re(\gamma)] + 1$. Then the fractional integral $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following formula holds:

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\overline{H}_{p,q}^{m,n}(at^\sigma)\, S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right]\right)(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1} B^{qn} l^{m+n} (1-\tau x^r)^{sn} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn\right)_p \left(\frac{j+k+re}{l}\right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r}\right)^p \left(\frac{Ax}{B}\right)^i$$

$$\times \overline{H}^{m+3,n}_{p+3,q+3} \left[ax^{\sigma} \middle| \begin{array}{l} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, \end{array} \right.$$

$$\left. \begin{array}{c} (\theta, \sigma), (\theta+\alpha+\alpha'+\beta'-\gamma, \sigma), (\theta+\alpha-\beta, \sigma) \\ (\theta+\alpha+\beta'-\gamma, \sigma), (\theta+\alpha'+\alpha-\gamma, \sigma), (\theta-\beta, \sigma) \end{array} \right]. \tag{74}$$

Proof. In view of (28), it follows that

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left[t^{\rho-1} \overline{H}^{m,n}_{p,q} (at^{\sigma}) S_n^{a,b,\tau}(t; r, s, q, A, B, m, k, l) \right] \right)(x) \tag{75}$$

$$= (-1)^k \frac{d^k}{dx^k} \left(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} \left[t^{\rho-1} \overline{H}^{m,n}_{p,q} (at^{\sigma}) S_n^{a,b,\tau}(t; r, s, q, A, B, m, k, l) \right] \right)(x)$$

$$= B^{qn} l^{m+n} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \sum_{w=0}^{\infty} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn\right)_p \left(\frac{j+k+re}{l}\right)_{m+n} (-\tau)^p \tau^w \left(\frac{A}{B}\right)^i \frac{(-sn+p)_w}{w!}$$

$$\times (-1)^k \frac{d^k}{dx^k} \frac{1}{2\pi i} \int_L \chi(s) a^s \left(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} t^{\rho+\sigma s+rw+rp+l(m+n)+i-1} \right)(x) ds.$$

Applying the formula (38), we obtain

$$= B^{qn} l^{m+n} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \sum_{w=0}^{\infty} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$

$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn\right)_p \left(\frac{j+k+re}{l}\right)_{m+n}$$

$$\times (-\tau)^p (\tau)^w \left(\frac{A}{B}\right)^i \frac{(-sn+p)_w}{w!} \frac{1}{2\pi i} \int_L \chi(s) a^s$$

$$\times \Gamma \left[\begin{array}{c} \theta+\alpha+\alpha'-\gamma-k-\sigma s, \theta+\alpha+\beta'-\gamma-\sigma s, \theta-\beta-\sigma s \\ \theta-\sigma s, \theta-\gamma+\alpha+\alpha'+\beta'-\sigma s, \theta+\alpha-\beta-\sigma s \end{array} \right]$$

$$\times (-1)^k \frac{d^k}{dx^k} x^{\gamma+k-\alpha-\alpha'-\theta+\sigma s} ds.$$

$$= B^{qn} l^{m+n} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \sum_{w=0}^{\infty} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$
$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn \right)_p \left(\frac{j+k+re}{l} \right)_{m+n}$$
$$\times \frac{1}{2\pi i} \int_L \chi(s) a^s (-\tau)^p (\tau)^w \left(\frac{A}{B} \right)^i \frac{(-sn+p)_w}{w!}$$
$$\times \Gamma \left[\begin{array}{c} \theta + \alpha + \alpha' - \gamma - k - \sigma s, \theta + \alpha + \beta' - \gamma - \sigma s, \theta - \beta - \sigma s \\ \theta - \sigma s, \theta - \gamma + \alpha + \alpha' + \beta' - \sigma s, \theta + \alpha - \beta - \sigma s \end{array} \right]$$
$$\times \left(\theta + \alpha + \alpha' - \gamma - k - \sigma s \right)_k x^{\gamma - \alpha - \alpha' - \theta + \sigma s} ds.$$
$$= x^{\rho + \gamma - \alpha - \alpha' + l(m+n) - 1} B^{qn} l^{m+n} (1 - \tau x^r)^{sn} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$
$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn \right)_p \left(\frac{j+k+re}{l} \right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r} \right)^p \left(\frac{Ax}{B} \right)^i$$
$$\times \frac{1}{2\pi i} \int_L \chi(s) \Gamma \left[\begin{array}{c} \theta + \alpha + \alpha' - \gamma - k - \sigma s, \theta + \alpha + \beta' - \gamma - \sigma s, \theta - \beta - \sigma s \\ \theta - \sigma s, \theta - \gamma + \alpha + \alpha' + \beta' - \sigma s, \theta + \alpha - \beta - \sigma s \end{array} \right]$$
$$\times (ax)^\sigma ds. \tag{76}$$

Which is the required result. □

If we take $\alpha' = 0$ in (74), then the following result holds:

Corollary 5.4. *Let $\alpha, \beta, \eta \in C$, $\Re(\alpha) > 0$. Further let the constants $m, n, p, q \in N_0, a_i, b_j \in C$, $A_i, B_j \in \Re_+$ $(i = 1, \dots, p;\ j = 1, \dots, q)$, $\rho \in C, \sigma \in \Re_+$, $|\arg a| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents α_i $(i = 1, \dots, n)$ and β_j $(j = m+1, \dots, q) \notin N$ be given and satisfy the condition*

$$\sigma \min[\tau, \zeta^*] + 1 > \Re(\rho) + \max\left[-\Re(\beta) + [\Re(-\alpha)] + 1, -\Re(\eta)\right]. \tag{77}$$

Then the fractional integral $I_-^{\alpha,\beta,\eta}$ of the product $\overline{H}$-function and $S_n^{a,b,\tau}(.)$ exists and the following relation holds:

$$\left(I_-^{\alpha,\beta,\eta} \left[t^{\rho-1} \overline{H}_{p,q}^{m,n} (at^\sigma) S_n^{a,b,\tau}(t; r, s, q, A, B, m, k, l) \right] \right)(x)$$
$$= x^{\rho - \beta + l(m+n) - 1} B^{qn} l^{m+n} (1 - \tau x^r)^{sn} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \sum_{i=0}^{m+n} \sum_{j=0}^{i} \frac{(-1)^i (-p)_e (-i)_j (a)_i}{p! i! j! e!}$$
$$\times \frac{(-a-qn)_j}{(1-a-i)_j} \left(\frac{-b}{\tau} - sn \right)_p \left(\frac{j+k+re}{l} \right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r} \right)^p \left(\frac{Ax}{B} \right)^i$$
$$\times \overline{H}_{p+2,q+2}^{m+2,n} \left[ax^\sigma \middle| \begin{array}{c} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (\theta, \sigma), (\theta + \alpha + \beta + \eta, \sigma) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (\theta + \beta, \sigma), (\theta + \eta, \sigma) \end{array} \right]. \tag{78}$$

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