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CERTAIN COMPLEX EQUATIONS CREATED BY INTEGRAL OPERATOR AND SOME OF THEIR APPLICATIONS TO ANALYTIC FUNCTIONS

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In this work, several novel results relating to complex equations constituted by an integral operator are first presented and some of their applications concerning multivalent functions which are analytic in the unit disk $\mathbb{U}$ are then emphasized.

1. Introduction, Notations and Definitions

Let us denote by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{U}$, $\mathcal{H}(\mathbb{U})$, $\mathcal{H}_p$ and $\mathcal{A}_p$ the set of natural numbers, the set of real numbers, the unit open disk which is the set $\{z \in \mathbb{C} : |z| < 1\}$, the set of all functions which are analytic in the disk $\mathbb{U}$, the set of analytic functions in the form:

$$\mathcal{H}_p = \left\{ f \in \mathcal{H}(\mathbb{U}) : \ f(z) = p + \sum_{n=p}^{\infty} b_n z^n \ \ (z \in \mathbb{U}) \right\}, \tag{1}$$

and the set of multivalently analytic functions in the form:

$$\mathcal{A}_p = \left\{ f \in \mathcal{H}(\mathbb{U}) : \ f(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \ \ (z \in \mathbb{U}) \right\}, \tag{2}$$

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respectively. Also let $\mathcal{H} := \mathcal{H}_1$ and $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. It is clear that $\mathcal{H}_p \subset \mathcal{H}(\mathbb{U})$ and $\mathcal{A}_p \subset \mathcal{H}(\mathbb{U})$.

For $\gamma > 0$ and $f \in \mathcal{A}_p$, we now recall the following integral operator:

$$\mathcal{P}^{\gamma}[f] := \mathcal{P}^{\gamma}[f(z)] = \frac{(p+1)^{\gamma}}{z\Gamma(z)} \int_0^z \Big(log\big(z/\kappa\big) \Big)^{\gamma-1} f(\kappa)\, d\kappa, \tag{3}$$

where the function Γ is well-known gamma function. When the operator given by (3) applies to a function $f \in \mathcal{A}_p$, we easily obtain that

$$\mathcal{P}^{\gamma}[f] = z^p + \sum_{n=p+1}^{\infty} \Big(\frac{p+1}{n+1} \Big)^{\gamma} a_n z^n, \tag{4}$$

for some $\gamma > 0$ and for all $f \in \mathcal{A}_p$. For the details and also some examples, one may look over the results or the works in the references in [4, 5]. In view of (4) and after some basic calculations, the following identity in relation with the integral operator:

$$z\frac{d}{dz}\Big(\mathcal{P}^{\gamma}[f]\Big) = z\big(\mathcal{P}^{\gamma}[f]\big)' = (p+1)\mathcal{P}^{\gamma-1}[f] - \mathcal{P}^{\gamma}[f] \tag{5}$$

can be easily obtained, where $\gamma > 0$ and $f \in \mathcal{A}_p$.

In the present investigation, several results related to both certain complex (differential) equations created by integral operator defined by (5) and (multivalently) analytic functions in the disk $\mathbb{U}$ are first determined and a number of those consequences are then pointed out.

For the main results, there is a need to recall the following result obtained by M. Nunokawa in [3].

Lemma 1.1. *Let $q(z)$ be in the class $\mathcal{H}$. If there exists a point z_0 in $\mathbb{U}$ such that*

$$\Re e\Big(q(z)\Big) > 0 \ \ \Big(|z| < |z_0|\Big) \ , \ \Re e\Big(q(z_0)\Big) = 0 \ \ \textit{and} \ \ q(z_0) \neq 0,$$

then

$$q(z_0) = ia \quad \textit{and} \quad \frac{zq'(z)}{q(z)}\bigg|_{z=z_0} = i\kappa\Big(a + \frac{1}{a}\Big),$$

where $\kappa \geq \frac{1}{2}$ and $a \in \mathbb{R}^*$.

2. The Main Results and Certain Applications

Now, we begin by setting and then by proving the following main result concerning certain complex (differential) equations and multivalently analytic functions.

Theorem 2.1. *Let a function $\Psi(z)$ belonging to the class $\mathcal{H}(\mathbb{U})$ satisfy any one of the cases of the following inequality:*

$$\Re e\Big(\Psi(z)\Big) > \begin{cases} \dfrac{\alpha}{2(\alpha-p)} & \textit{for } 0 \leq \alpha \leq \frac{p}{2} \\[2ex] \dfrac{\alpha-p}{2\alpha} & \textit{for } \frac{p}{2} \leq \lambda < p \end{cases} \tag{6}$$

If a function $f \in \mathcal{A}_p$ is a solution for the following complex equation:

$$\big(p+1\big) \cdot \Big(\mathcal{P}^{\gamma-1}[f]\Big) \; - \; \big(p+1+\Psi(z)\big) \cdot \Big(\mathcal{P}^{\gamma}[f]\Big) \; = \; 0 \tag{7}$$

then

$$\Re e\left(\frac{\mathcal{P}^{\gamma}[f]}{z^p}\right) > \frac{\alpha}{p} \quad \big(0 \leq \alpha < p; p \in \mathbb{N}; z \in \mathbb{U}\big). \tag{8}$$

Proof. With the help of (2) and (4), we define a function $q(z)$ by

$$\begin{aligned} \mathcal{P}^{\gamma}[f] &= z^p \left[1 + \sum_{n=1}^{\infty} \left(\frac{p+1}{n+p+1}\right)^{\gamma} a_{n+p} z^n\right] \\ &= z^p \left[\frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right) q(z)\right], \end{aligned} \tag{9}$$

where $z \in \mathbb{U}$, $\gamma > 0$, $0 \leq \alpha < p$ and $f(z) \in \mathcal{A}_p$. It is obvious that the function $q(z)$ is in the class $\mathcal{H}$. Shortly, $q(z)$ is analytic in $\mathbb{U}$ and $q(0) = 1$. By means of (9) together with (5), it is easily established that

$$\frac{z\Big(z^{-p}\mathcal{P}^{\gamma}[f]\Big)'}{z^{-p}\mathcal{P}^{\gamma}[f]} = \frac{z\Big(\mathcal{P}^{\gamma}[f]\Big)'}{\mathcal{P}^{\gamma}[f]} - p$$

or, equivalently,

$$\frac{z\Big(z^{-p}\mathcal{P}^{\gamma}[f]\Big)'}{z^{-p}\mathcal{P}^{\gamma}[f]} = \frac{(p+1)\mathcal{P}^{\gamma-1}[f]}{\mathcal{P}^{\gamma}[f]} - (p+1)\Big(=: \Psi(z)\Big). \tag{10}$$

In consideration of (10), it is clear that the complex function $\Psi(z)$ satisfies the complex equation given by (7). By the help of (9) and (10), we then obtain

$$\frac{(p+1)\mathcal{P}^{\gamma-1}[f]}{\mathcal{P}^{\gamma}[f]} - (p+1) = \frac{(1-\alpha/p)zq'(z)}{\alpha/p + (1-\alpha/p)q(z)}\Big(=: \Psi(z)\Big). \tag{11}$$

Now, suppose that there exists a point z_0 in $\mathbb{U}$ such that

$$\Re e\Big(q(z)\Big) > 0 \;\; \Big(|z| < |z_0|\Big) \;\; , \;\; \Re e\Big(q(z_0)\Big) = 0 \;\; \textit{and} \;\; q(z_0) \neq 0.$$

By applying the assertions of Lemma 1.1, which are

$$q(z_0) = ia \quad \text{and} \quad \left.\frac{zq'(z)}{q(z)}\right|_{z=z_0} = ic\left(a+\frac{1}{a}\right) \quad \left(c \geq \frac{1}{2}; a \in \mathbb{R}^*\right)$$

and using the following inequalities:

$$0 \leq \alpha \leq \frac{p}{2} \implies \frac{1+a^2}{a^2+\left(\frac{\alpha/p}{1-\alpha/p}\right)^2} \geq 1$$

and

$$\frac{p}{2} \leq \alpha < p \implies \frac{1+a^2}{1+a^2\left(\frac{1-\alpha/p}{\alpha/p}\right)^2} \geq 1,$$

respectively, (11) yields that

$$\begin{aligned}
\Re e\Big(\Psi(z_0)\Big) &= \Re e\left[\frac{zq'(z)}{q(z)} \cdot \left.\frac{(1-\alpha/p)q(z)}{\alpha/p+(1-\alpha/p)q(z)}\right|_{z=z_0}\right] \\
&= \Re e\left[ic\left(a+\frac{1}{a}\right) \cdot \frac{ia(1-\alpha/p)}{\alpha/p+ia(1-\alpha/p)}\right] \\
&= \frac{\frac{c\alpha}{p}\left(\frac{\alpha}{p}-1\right)\left(1+a^2\right)}{\left(\frac{\alpha}{p}\right)^2+a^2\left(1-\frac{\alpha}{p}\right)^2} \\
&\leq \frac{\frac{\alpha}{p}\left(\frac{\alpha}{p}-1\right)\left(1+a^2\right)}{2\left[\left(\frac{\alpha}{p}\right)^2+a^2\left(1-\frac{\alpha}{p}\right)^2\right]} \\
&\leq \begin{cases} \frac{\alpha}{2(\alpha-p)} & for \;\; 0 \leq \alpha \leq \frac{p}{2} \\ \frac{\alpha-p}{2\alpha} & for \;\; \frac{p}{2} \leq \lambda < p \end{cases}, \qquad (12)
\end{aligned}$$

since $c \geq 1/2$ and $\alpha/p - 1 < 0$. It is quite obvious that the inequalities given by (12) contradict with the assumptions given by (6), respectively. Hence, the statement given by (9) requires the following inequality:

$$\Re e\Big(q(z)\Big) = \Re e\left(\frac{\frac{\mathcal{P}^\gamma[f]}{z^p}-\frac{\alpha}{p}}{1-\frac{\alpha}{p}}\right) > 0 \quad \Big(0 \leq \alpha < p; p \in \mathbb{N}; z \in \mathbb{U}\Big),$$

which implies that the inequality given by (8). This completes the proof of Theorem 2.1. □

If we again define the related function $q(z)$, respectively, by the following form:

$$\frac{\big(\mathcal{P}^{\gamma}[f]\big)'}{z^{p-1}} = \alpha + (p-\alpha)q(z)$$

$$\big(0 \leq \alpha < p\ ;\ p \in \mathbb{N}\ ;\ \gamma > 0\ ;\ z \in \mathbb{U}\ ;\ f(z) \in \mathcal{A}_p\big),$$

$$\frac{z\big(\mathcal{P}^{\gamma}[f]\big)'}{\mathcal{P}^{\gamma}[f]} = \alpha + (p-\alpha)q(z)$$

$$\big(0 \leq \alpha < p\ ;\ p \in \mathbb{N}\ ;\ \gamma > 0\ ;\ z \in \mathbb{U}\ ;\ f(z) \in \mathcal{A}_p\big),$$

$$\frac{\mathcal{P}^{\gamma}[f]}{\mathcal{P}^{\beta}[f]} = \frac{\alpha}{p} + \Big(1 - \frac{\alpha}{p}\Big)q(z)$$

$$\big(0 \leq \alpha < p\ ;\ p \in \mathbb{N}\ ;\ \gamma > 0\ ;\ \beta > 0\ ;\ z \in \mathbb{U}\ ;\ f(z) \in \mathcal{A}_p\big),$$

and

$$\frac{\big(\mathcal{P}^{\gamma}[f]\big)'}{\big(\mathcal{P}^{\beta}[f]\big)'} = \frac{\alpha}{p} + \Big(1 - \frac{\alpha}{p}\Big)q(z)$$

$$\big(0 \leq \alpha < p\ ;\ p \in \mathbb{N}\ ;\ \gamma > 0\ ;\ \beta > 0\ ;\ z \in \mathbb{U}\ ;\ f(z) \in \mathcal{A}_p\big),$$

and then follow the ways and/or steps used in the proof of Theorem 2.1, we can easily arrive at the desired proofs. Their details are here omitted.

Theorem 2.2. *Let a function $\Psi(z)$ belonging to the class $\mathcal{H}\big(\mathbb{U}\big)$ satisfy any one of the cases of the following inequality*

$$\Re e\Big(\Psi(z)\Big) > \begin{cases} \dfrac{p\alpha}{2(\alpha-p)} & \text{for } 0 \leq \alpha \leq \frac{p}{2} \\[2ex] \dfrac{p(\alpha-p)}{2\alpha} & \text{for } \frac{p}{2} \leq \lambda < p \end{cases} \tag{13}$$

If a function $f \in \mathcal{A}_p$ is a solution for the following complex equation:

$$z\Big(\mathcal{P}^{\gamma}[f]\Big)'' - \big(p - 1 + \Psi(z)\big)\Big(\mathcal{P}^{\gamma}[f]\Big)' = 0,$$

then

$$\Re e\left(\frac{\big(\mathcal{P}^{\gamma}[f]\big)'}{z^{p-1}}\right) > \alpha \quad \big(0 \leq \alpha < p; p \in \mathbb{N}; z \in \mathbb{U}\big).$$

Theorem 2.3. *Let a function $\Psi(z)$ belonging to the class $\mathcal{H}(\mathbb{U})$ satisfy any one of the cases of the inequality in (13). If a function $f \in \mathcal{A}_p$ is a solution for the following complex equation:*

$$z\mathcal{P}^{\gamma}[f]\Big[z\big(\mathcal{P}^{\gamma}[f]\big)'\Big]' - z\Big[z\big(\mathcal{P}^{\gamma}[f]\big)'\Big]^2 - z\Psi(z)\mathcal{P}^{\gamma}[f]\Big(\mathcal{P}^{\gamma}[f]\Big)' = 0,$$

then

$$\Re e\left(\frac{z\big(\mathcal{P}^{\gamma}[f]\big)'}{\mathcal{P}^{\gamma}[f]}\right) > \alpha \quad \big(0 \leq \alpha < p; p \in \mathbb{N}; z \in \mathbb{U}\big).$$

Theorem 2.4. *Let a function $\Psi(z)$ belonging to the class $\mathcal{H}(\mathbb{U})$ satisfy any one of the cases of the inequality in (6). If a function $f \in \mathcal{A}_p$ is a solution for the following complex equation:*

$$z\mathcal{P}^{\beta}[f]\Big(\mathcal{P}^{\gamma}[f]\Big)' - z\mathcal{P}^{\gamma}[f]\Big(\mathcal{P}^{\beta}[f]\Big)' - \Psi(z)\mathcal{P}^{\beta}[f]\mathcal{P}^{\gamma}[f] = 0,$$

then

$$\Re e\left(\frac{\mathcal{P}^{\gamma}[f]}{\mathcal{P}^{\beta}[f]}\right) > \frac{\alpha}{p} \quad \big(0 \leq \alpha < p; p \in \mathbb{N}; z \in \mathbb{U}\big).$$

Theorem 2.5. *Let a function $\Psi(z)$ belonging to the class $\mathcal{H}(\mathbb{U})$ satisfy any one of the cases of the inequality in (6). If a function $f \in \mathcal{A}_p$ is a solution for the following complex equation:*

$$z\Big(\mathcal{P}^{\beta}[f]\Big)'\Big(\mathcal{P}^{\gamma}[f]\Big)'' - z\Big(\mathcal{P}^{\gamma}[f]\Big)'\Big(\mathcal{P}^{\beta}[f]\Big)'' - \Psi(z)\Big(\mathcal{P}^{\beta}[f]\Big)'\Big(\mathcal{P}^{\gamma}[f]\Big)' = 0,$$

then

$$\Re e\left(\frac{\big(\mathcal{P}^{\gamma}[f]\big)'}{\big(\mathcal{P}^{\beta}[f]\big)'}\right) > \frac{\alpha}{p} \quad \big(0 \leq \alpha < p; p \in \mathbb{N}; z \in \mathbb{U}\big).$$

As applications of all theorems above, when one focuses on Theorems 2.1-2.5, it is easily seen that there are several consequences of them which will be interesting in the theories of analytic functions and (certain) complex (differential) equations. For those, it is enough to choose suitable values of the parameter(s) in all the theorems. Of course, it is not possible to reveal all of them. But, particularly, for example, we want to present only one of those, which deals with an interesting result consisting of both certain complex differential equation and univalent function theory (see, for details, [1, 2, 6]), which is Corollary 2.6 below. The other possible consequences of all theorems (which are here omitted) are presented to the attention of the researchers who have been working on the theories of (complex) differential equations and/or analytic functions.

By taking $\gamma \to 0^-$ in Theorem 2.3, the following result related to multivalently starlikeness can be then obtained.

Corollary 2.6. *Let a function $\Psi(z)$ belonging to the class $\mathcal{H}(\mathbb{U})$ satisfy any one of the cases of the inequality in (13). If a function $w := w(z) := f(z) \in \mathcal{A}_p$ is a solution of the following nonlinear complex differential equation:*

$$z^2 ww'' - z(zw')^2 + (1 - \Psi(z))zww' = 0,$$

then w is multivalently starlike function of order α $(0 \leq \alpha < p; p \in \mathbb{N})$ in $\mathbb{U}$.

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REFERENCES

[1] P. L. Duren, *Grundlehren der Mathematischen Wissenchaften,* Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[2] A. W. Goodman, *Univalent functions*, Vols. I and II., Polygonal Publishing House, Washington-New Jersey, 1983.

[3] M. Nunokawa, *On properties of non-Carathéodory functions,* Proc. Japan Acad. Ser. A Math. Sci. 68 (1998), 152–153.

[4] I. B. Jung - Y. C. Kim - H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. 176 (1) (1993), 138–147.

[5] S. Shams - S. R. Kulkarni - J. M. Jahangiri, *Subordination properties of p-valent functions defined by integral operators*, Inter. J. Math. and Math. Sci. 2006 (2006), Article ID 94572, 3 pages.

[6] H. M. Srivastava - S. Owa (Editors), *Univalent Functions, Fractional Calculus and Their Applications,* Halsted Press, John Wiley and Sons. New York, Chichester, Brisbane, Toronto, 1989.

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