LE MATEMATICHE
Vol. LXII (2007) - Fasc. I, pp. 41-46

ON REDUCIBILITY OF LINEAR DIFFERENCE SYSTEMS

ALI MAHMUD ATEIWI

We study problems related to the existence of a nondegenerate substitution (of the Lyapunov type) that reduces a system of linear difference equations to a system with constant coefficients.

Introduction.

In the present work, we study the problem of the possibility of the reduction of a linear difference system with variable coefficients in the Euclidean space $\mathbb{R}^d$ to a system with constant coefficients. In this case, it is important to find a change of variables that does not "spoil" the original system, i.e., the reduced system must be linear and must possess the same qualitative properties as the original problem. If, e.g., the original system is stable, then the reduced system must also be stable. In this connection, of special importance is the problem of the existence of periodic solutions of the original system because the same problem for the reduced system of difference equations (if this reduction is possible) can easily be solved.

Entrato in redazione il 28 Novembre 2006.
2000 *Mathematics Subject Classification*: 39A10, 34D20.
Key words and phrases. Difference equation, reducible system, Lyapunov substitution.

In the theory of differential equations, necessary and sufficient reducibility conditions are given by the Erugin theorem [1]. For systems of difference equations, this problem is inadequately studied. In this connection, one should mention the works [2]–[4], where the problem of the reducibility of linear difference equations with quasiperiodic coefficients was investigated. The method proposed in the works indicated is based on the idea of accelerated convergence, i.e., on constructing a sequence of quasiperiodic substitutions so that the appearing small denominators are "canceled out."

The present work has the following structure: In Sec. 1, we give necessary definitions and introduce the object of investigation.

In Sec. 2, we prove an analog of the Erugin theorem for a system of difference equations.

Section 3 is devoted to the investigation of the problem of reducibility to a system with zero matrix.

We now pass the the presentation of the main results of the work.

1. Statement of the problem.

In the Euclidean space $\mathbb{R}^d$, we consider a system of linear difference equations with variable coefficients

$$x_{n+1} = x_n + A_n x_n, \tag{1}$$

where $n = 0, 1, 2, ...,$ x_n is a vector from $\mathbb{R}^d$ and A_n is a $d \times d$ matrix. Note that the matrix A_n is variable.

We assume that $E + A_n$ is nondegenerate for all $n \geq 0$. This condition guarantees the uniqueness of a solution of system (1).

Definition 1. A $d \times d$ matrix L_n is called a discrete Lyapunov matrix if the following conditions are satisfied:

1) L_n is bounded for $n = 0, 1, 2, ...,$ i.e.,

$$\sup_{n \in \mathbb{N}} \|L_n\| < \infty;$$

2) $|det L_n| \geq m > 0$, where m is a certain positive number.

Note that it follows from the structure of the inverse matrix that the matrix

L_n^{-1} inverse to the Lyapunov matrix L_n is also a Lyapunov matrix.

Definition 2. The linear transformation

$$y = L_n x, \tag{2}$$

where L_n is a discrete Lyapunov matrix, is called the Lyapunov transformation.

Remark 1. By virtue of the definition of Lyapunov substitution, one can easily see that transformation (2) does not spoil the qualitative characteristics of the original system. For example, if solutions of the original system are bounded and stable, then the solutions of the transformed system possess the same properties. This explains the importance of substitutions of this type.

Definition 3. A linear difference system is called reducible if there exists a Lyapunov transformation that reduces it to a system with constant coefficients

$$y_{n+1} = y_n + By_n, \tag{3}$$

where B is a constant matrix.

The aim of this work is to establish reducibility conditions for systems of the type (1).

2. Reducibility of systems. Main result.

The following theorem is true:

Theorem 1. *The linear difference system (1) is reducible if and only if a certain fundamental matrix X_n of it is representable in the form*

$$X_n = L_n(E + B)^n, \tag{4}$$

where E is the $d \times d$ identity matrix and B is a certain constant $d \times d$ matrix.

Proof. First, we prove the necessity of the conditions of the theorem.

Assume that system (1) is reducible. This means that there exists a discrete Lyapunov substitution

$$x = L_n y \tag{5}$$

that transforms it into a linear system with constant coefficients

$$y_{n+1} = y_n + By_n, \tag{6}$$

where B is a certain constant matrix. It follows from (6) that its fundamental matrices have the form

$$Y_n = (E + B)^n C, \tag{7}$$

where C is an arbitrary nondegenerate $d \times d$ matrix. By virtue of (5), a fundamental matrix for (1) has the form

$$X_n = L_n(E + B)^n C.$$

Choosing $C = E$, where E is the identity matrix, we get (4).

The necessity is proved.

Let us prove the sufficiency. Assume that relation (4) is true. It follows from (4) that

$$L_n = X_n((E + B)^n)^{-1} = X_n((E + B)^{-1})^n.$$

In system (1), we perform the substitution

$$x = X_n\big((E + B)^{-1}\big)^n y.$$

We have

$$\begin{aligned} x_{n+1} &= X_{n+1}\big((E + B)^{-1}\big)^{n+1} y_{n+1} \\ &= X_n\big((E + B)^{-1}\big)^n y_n + A_n X_n\big((E + B)^{-1}\big)^n y_n. \end{aligned}$$

Then

$$X_{n+1}\big((E + B)^{-1}\big)^{n+1} y_{n+1} = (E + A_n)X_n\big((E + B)^{-1}\big)^n y_n.$$

Taking into account that X_n is a fundamental matrix of system (1), we conclude that it satisfies the equation

$$X_{n+1} = (E + A_n)X_n. \tag{9}$$

Substituting (9) in the last relation, we obtain

$$(E+A_n)X_n\big((E+B)^{-1}\big)^{n+1} y_{n+1} = (E+A_n)X_n\big((E+B)^{-1}\big)^{n+1}(E+B)y_n.$$

Canceling the nondegenerate matrices on both sides of this relation, we arrive at a system with constant coefficients:

$$y_{n+1} = y_n + By_n.$$

Thus, system (1) is reducible. The theorem is proved.

3. Reducibility to a system with zero matrix.

Assume that the linear system (1), where $E+A_n$ is a bounded matrix, can be reduced by a discrete Lyapunov transformation

$$x = L_n y$$

to the system

$$y_{n+1} = y_n. \tag{10}$$

Since a general solution of system (10) has the form $y_n = C$, where C is a constant, we conclude that if system (1) is transformed into (10) by a discrete Lyapunov substitution, then all solutions of (1) are bounded for $n = 0, 1, 2, \ldots$.

The following theorem is true:

Theorem 2. *Suppose that the following conditions are satisfied:*

1) all solutions of system (1) are bounded for $n = 0, 1, 2, \ldots$*;*

2) $\prod_{i=0}^{n-1} det(E + A_i) \geq a > 0$*, for any* $n = 0, 1, 2, \ldots$*, where* a *is a constant.*

Then system (1) is reducible to a system with zero matrix.

Proof. Let X_n be a fundamental matrix of system (1). Let us show that, in this case, X_n is a discrete Lyapunov matrix. Indeed, it follows from (1) that $\|X_n\| \leq C$, where C is a certain constant.

Using a discrete analog of the Ostrogradskii–Liouville formula [5], namely

$$det X_n = det X_0 \prod_{i=0}^{n-1} det(E + A_i),$$

and condition 2), we get

$$|det X_n| = |det X_0| \prod_{i=0}^{n-1} |det(E + A_i)| \geq |det X_0| a \geq C_2 > 0,$$

where C_2 is a certain constant. According to Definition 1, X_n is a Lyapunov matrix. In equation (1), we perform the substitution

$$x = X_n y.$$

We have

$$x_{n+1} = X_{n+1}y_{n+1} = X_n y_n + A_n X_n y_n.$$

However, $X_{n+1} = (E + A_n)X_n$, and, therefore,

$$(E + A_n)X_n y_{n+1} = (E + A_n)X_n y_n.$$

By virtue of the fact that the matrices $(E+A_n)$ and X_n are nondegenerate, the last relation yields

$$y_{n+1} = y_n.$$

The theorem is proved.

REFERENCES

[1] B. P. Demidovich, *Lectures on Mathematical Theory of Stability* [in Russian], Nauka, Moscow (1967)

[2] Yu. A. Mitropol'skii - A. M. Samoilenko - D. I. Martynyuk, *Systems of Evolution Equations with Periodic and Quasiperiodic Coefficients* [in Russian], Naukova Dumka, Kiev (1984)

[3] D. I. Martynyuk - N. A. Perestyuk, *"Reducibility of linear systems of difference equations with smooth right-hand side,"* Vychisl. Prikl. Mat., Issue 27, (1990), pp. 34–40

[4] D. I. Martynyuk - N. A. Perestyuk, *"On reducibility of linear systems of difference equations with quasiperiodic coefficients"* Vychisl. Prikl. Mat., Issue 29, (1992), pp. 116–127

[5] D. I. Martynyuk, *Lectures on Qualitative Theory of Difference Equations* [in Russian], Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1973)

Department of Mathematics and Statistics
Faculty of Science
Al-Hussein Bin Talal University
P.O. Box 20
Ma'an Jordan
e-mail: ateiwi@hotmail.com