SOME PROPERTIES OF A NEW SUBCLASS OF ANALYTIC UNIVALENT FUNCTIONS

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The purpose of the present paper is to study the integral operator of the form
\[ \int_0^z \left\{ \frac{D^n f(t)}{t} \right\}^\delta dt \]
where \( f \) belongs to the subclass \( C(n, \alpha, \beta) \) and \( \delta \) is a real number. We obtain integral characterization for the subclass \( C(n, \alpha, \beta) \) and also prove distortion, rotation and radii theorem for this class. Relevant connections of the results presented here with various known results are briefly indicated.

1. Introduction

Let \( A \) denote the class of functions \( f \) of the form
\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1) \]
which are analytic in the open unit disk \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \) and satisfy the normalization condition \( f(0) = f'(0) - 1 = 0 \). Let \( S \) be the subclass of \( A \) consisting of functions of the form (1) which are also univalent in \( U \).
A function $f$ of $S$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$, denoted by $f \in S^*(\alpha)$, if and only if
\[ \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha , \quad z \in U, \]
and is said to be convex of order $\alpha (0 \leq \alpha < 1)$, denoted by $f \in K(\alpha)$, if and only if
\[ \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha , \quad z \in U. \]

The classes $S^*$ and $K$ of starlike and convex functions, respectively, are identified by $S^*(0) \equiv S^*$ and $K(0) \equiv K$.

In 1983, Salagean [17], introduced a derivative operator known as Salagean operator which is defined as follows:

Let $f(z) \in A$ and be of the form (1). Then we define:
\[
D^0 f(z) = f(z) \\
D^1 f(z) = zf'(z) \\
space \ldots \ldots \ldots \\
D^n f(z) = D(D^{n-1} f(z)), \quad n \in \mathbb{N}.
\]

Thus
\[ D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k. \tag{2} \]

A function $f$ of $A$ belongs to the class $S(n, \alpha)$ of functions of the form (1) satisfying the condition
\[ \Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \alpha , \quad z \in U, \tag{3} \]
where $D^n$ stands for the Salagean operator.

The class $S(n, \alpha)$ was first introduced by Salagean [17] and further studied by Kadioğlu [2].

It should be worthy to note that $S(0, \alpha) = S^*(\alpha)$ and $S(1, \alpha) = K(\alpha)$.

A function $f$ of $A$ belongs to the class $C(n, \alpha, \beta)$ if there exists a function $F \in S^*(\alpha)$ such that
\[ \left| \arg \frac{D^n f(z)}{F(z)} \right| < \frac{\beta \pi}{2}, \quad z \in U, \]
where $n \in \mathbb{N}_0$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$.

By specializing the parameters in $C(n, \alpha, \beta)$ we obtain the following known subclasses of $A$ studied earlier by various researchers.
(1) $C(0, \alpha, \beta) \equiv CS^*(\alpha, \beta)$ studied by Mishra [7].

(2) $C(1, \alpha, \beta) \equiv C(\alpha, \beta)$ studied by Mishra [7].

(3) $C(0, 0, \beta) \equiv CS^*(\beta)$ studied by Reade [14].

(4) $C(1, 0, \beta) \equiv C(\beta)$ studied by Kaplan [3].

(5) $C(0, 0, 1) \equiv S^*$ studied by Roberston [15], (see also [1], [19]).

(6) $C(1, 0, 1) \equiv K$ studied by Roberston [15], (see also [1], [19]).

In the present paper, we study the integral operator

$$h(z) = \int_0^z \left\{ \frac{D^n f(t)}{t} \right\}^\delta dt$$

where $n \in N_0$ and $\delta$ is a real number. For $n = 0$ and $n = 1$ this integral operator was studied by Kim [4], Merkes and Wright [6], Mishra [7], Nunokawa ([8], [9]), Pfaltzgraff [11], Royster [16], Patil and Thakare [10] and Shukla and Kumar [18], (see also [13]).

To prove our main results, we shall require the following definition and lemmas.

**Definition 1.1.** Let $P(\alpha)$ denote the class of functions of the form $P(z) = 1 + \sum_{k=1}^{\infty} p_kz^k$ which are regular in $U$ and satisfy $\Re\{P(z)\} > \alpha$, $z \in U$.

**Lemma 1.2.** Let $P(z) = 1 + \sum_{k=1}^{\infty} p_kz^k$ be analytic in $U$. If $\Re\{P(z)\} > \alpha$ in $U$, then

$$\alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{ P(re^{i\theta}) \right\} d\theta < 2\pi(1 - \alpha) + \alpha(\theta_2 - \theta_1),$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$ and $0 \leq r < 1$.

**Proof.** Since

$$\Re\{P(z)\} > \alpha$$

it is easy to see that

$$\left( \Re\{P(z)\} - \alpha \right)_{z=0} = 1 - \alpha.$$

Then by mean value theorem, we have

$$0 \leq \int_{\theta_1}^{\theta_2} \left( \Re\left\{ P(re^{i\theta}) \right\} - \alpha \right) d\theta \leq \int_0^{2\pi} \left( \Re\left\{ P(re^{i\theta}) \right\} - \alpha \right) d\theta = 2\pi(1 - \alpha).$$
or, equivalently
\[ 0 \leq \int_{\theta_1}^{\theta_2} \left( \Re \left\{ P(re^{i\theta}) \right\} \right) d\theta - \alpha(\theta_2 - \theta_1) \leq 2\pi (1 - \alpha), \]
or
\[ \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re \left\{ P(re^{i\theta}) \right\} d\theta < 2\pi(1 - \alpha) + \alpha(\theta_2 - \theta_1). \]

The following lemma is a direct consequence of Lemma 1.2, and improves a result of Patil and Thakare ([10, Lemma 2.2]).

**Lemma 1.3.** If \( f \in S^*(\alpha) \), then
\[ \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} d\theta < 2\pi(1 - \alpha) + \alpha(\theta_2 - \theta_1), \]  \hspace{1cm} (6)
where \( 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta} \) and \( 0 \leq r < 1 \).

In the following lemma, we obtain integral characterization for the class \( C(n, \alpha, \beta) \).

**Lemma 1.4.** If \( f \in C(n, \alpha, \beta) \), then
\[ -\beta\pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} d\theta < \beta\pi + 2\pi(1 - \alpha) + \alpha(\theta_2 - \theta_1), \]  \hspace{1cm} (7)
where \( 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta} \) and \( 0 \leq r < 1 \). Conversely, let \( f \) be analytic and satisfying \( D^n f(z) \neq 0 \) in \( U \), if
\[ \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} d\theta > -\beta\pi + \alpha(\theta_2 - \theta_1) \]
then \( f \in C(n, \alpha, \beta) \).

**Proof.** \( f \in C(n, \alpha, \beta) \) implies that there exists a function \( F \in S^*(\alpha) \) such that
\[ \left| \arg \frac{D^nf(z)}{F(z)} \right| < \frac{\beta\pi}{2}, \quad z \in U. \]
Therefore
\[ -\frac{1}{2}\beta\pi < \arg D^nf(z) - \arg F(z) < \frac{1}{2}\beta\pi. \]

Let \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \). Then with \( z = re^{i\theta_2} \), we have
\[ -\frac{1}{2}\beta\pi < \arg D^nf(re^{i\theta_2}) - \arg F(re^{i\theta_2}) < \frac{1}{2}\beta\pi. \]  \hspace{1cm} (8)
and with $z = re^{i\theta_1}$, we have
\[ -\frac{1}{2}\beta \pi < -\arg D^n f(re^{i\theta_1}) + \arg F(re^{i\theta_1}) < \frac{1}{2}\beta \pi. \quad (9) \]

Combining (8) and (9), we obtain
\[ -\beta \pi + \arg F(re^{i\theta_2}) - \arg F(re^{i\theta_1}) < \arg D^n f(re^{i\theta_2}) - \arg D^n f(re^{i\theta_1}) \]
\[ < \beta \pi + \arg F(re^{i\theta_2}) - \arg F(re^{i\theta_1}), \]
or
\[ -\beta \pi + \int_{\theta_1}^{\theta_2} d \arg F(re^{i\theta}) < \int_{\theta_1}^{\theta_2} d \arg D^n f(re^{i\theta}) < \beta \pi + \int_{\theta_1}^{\theta_2} d \arg F(re^{i\theta}), \]
or
\[ -\beta \pi + \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{zF'(z)}{F(z)} \right\} d\theta \]
\[ < \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} d\theta \]
\[ < \beta \pi + \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{zF'(z)}{F(z)} \right\} d\theta. \quad (10) \]

But $F \in S^*(\alpha)$, then using Lemma 1.3 in (10), we have
\[ -\beta \pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} d\theta < \beta \pi + 2\pi(1 - \alpha) + \alpha(\theta_2 - \theta_1) \]
and this completes the proof of direct part of the lemma.

To prove the converse part, we follow the techniques of Kaplan [3] and Patil and Thakare [10] and can obtain the desired result.

\[ \square \]

**Remark 1.5.** If we put $n = 1$ in Lemma 1.4, we obtain the following result

If $f \in C(\alpha, \beta)$, then
\[ -\beta \pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} d\theta < \beta \pi + 2\pi(1 - \alpha) + \alpha(\theta_2 - \theta_1), \quad (11) \]

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$ and $0 \leq r < 1$. Conversely, let $f$ be analytic and satisfying $f'(z) \neq 0$ in $U$, if
\[ \int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} d\theta > -\beta \pi + \alpha(\theta_2 - \theta_1) \quad (12) \]

then $f \in C(\alpha, \beta)$. 
2. Main Results

**Theorem 2.1.** If \( f \in C(n, \alpha, \beta) \), then \( h \in C(\eta, \gamma) \), provided

\[
-\frac{\gamma}{\beta + 2(1 - \alpha)} \leq \delta \leq \frac{\gamma + 2(1 - \eta)}{\beta + 2(1 - \alpha)}.
\]

(13)

The result is sharp when (i) \( \gamma = 0 \) (ii) \( \eta = 0, \gamma = 1 \).

**Proof.** From relation (3) we have

\[
h'(z) = \left\{ \frac{D^n f(z)}{z} \right\}^\delta.
\]

Applying logarithmic differentiation and then taking real parts of both sides, we obtain

\[
\Re \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} = \delta \Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} + (1 - \delta).
\]

For \( \delta > 0 \), using Lemma 1.4, we get

\[
\int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} d\theta = \delta \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} d\theta + (1 - \delta)(\theta_2 - \theta_1)
\]

\[
> \delta [ -\beta \pi + \alpha(\theta_2 - \theta_1) ] + (1 - \delta)(\theta_2 - \theta_1)
\]

\[
= -\beta \delta \pi + [1 - (1 - \alpha)\delta](\theta_2 - \theta_1).
\]

To prove that \( h \in C(\eta, \gamma) \), we have to show that the right hand side of the above inequality is not less than \( -\gamma \pi + \eta (\theta_2 - \theta_1) \), provided

\[
0 \leq \delta \leq \frac{\gamma + 2(1 - \eta)}{\beta + 2(1 - \alpha)}.
\]

(14)

Similarly, for \( \delta < 0 \), using Lemma 1.4, we get

\[
\int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} d\theta > \delta [ \beta \pi + 2(1 - \alpha) + \alpha(\theta_2 - \theta_1) ] + (1 - \delta)(\theta_2 - \theta_1).
\]

To show that \( h \in C(\eta, \gamma) \), we have to prove that the right-hand side of the above inequality is not less than \( -\gamma \pi + \eta (\theta_2 - \theta_1) \), provided

\[
\frac{-\gamma}{\beta + 2(1 - \alpha)} \leq \delta \leq 0.
\]

(15)

Combining (14) and (15), we obtain (13).

Thus the proof of Theorem 2.1 is established.
To show the sharpness, let us take the function \( f(z) \) defined by the relation

\[
D^n f(z) = \frac{z}{(1-z)^{2(1-\alpha)+\beta}},
\]

then it is easy to see that this function belongs to \( C(n, \alpha, \beta) \) with respect to the function \( \frac{\tilde{z}}{(1-z)^{2(1-\alpha)}} \) belonging to \( S^*(\alpha) \). Then

\[
h(z) = \int_0^z \frac{dt}{(1-t)^{2(1-\alpha)+\beta}\delta}
\]

and from condition (12) this function belongs to \( C(0, 1) \) if and only if

\[
\frac{-1}{2(1-\alpha)+\beta} \leq \delta \leq \frac{3}{2(1-\alpha)+\beta}.
\]

Again for \( \gamma = 0 \), from (17) we have

\[
1 + \frac{zh''(z)}{h'(z)} = 1 + \left[ 1 - 2\left(1 - \frac{2(1-\alpha)+\beta}{2} \delta \right) \right]z
\]

and \( \Re \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} > \eta \) if and only if

\[
1 - \frac{2(1-\alpha)+\beta}{2} \delta \geq \eta \Rightarrow 0 \leq \delta \leq \frac{2(1-\eta)}{\beta + 2(1-\alpha)}.
\]

Remark 2.2. The undermentioned results are particular cases of Theorem 2.1.

(i) If we put \( n = 0 \) and \( n = 1 \) in Theorem 2.1 we obtain the corresponding results of Mishra [7].

(ii) If we put \( n = 1, \beta = 0, \gamma = 0 \) we obtain a result of Patil and Thakare [10].

(iii) If we put \( n = 1, \beta = 0, \eta = 0 \) we obtain a result of Patil and Thakare [10].

(iv) If we put \( n = 1, \alpha = 0, \eta = 0 \) we obtain a result of Patil and Thakare [10].

(v) If we put \( n = 0, \beta = 0, \eta = 0 \) we obtain a result of Patil and Thakare [10].

(vi) If we put \( n = 1, \alpha = 0, \beta = 0, \eta = 0 \) and \( \gamma = 1 \) we obtain a result of Nunokawa [9] as well as that of Merkes and Wright [6].

(vii) If we put \( n = 0, \alpha = 0, \beta = 0, \eta = 0 \) and \( \gamma = 1 \) we obtain a result of Nunokawa [9] as well as that of Merkes and Wright [6].
(viii) If we put $n = 1, \alpha = 0, \beta = 1, \eta = 0$ and $\gamma = 1$ we obtain a result of Nunokawa [9] as well as that of Merkes and Wright [6].

(ix) If we put $n = 0, \alpha = 0, \eta = 0$ we obtain a result of Shukla and Kumar [18].

(x) If we put $n = 0, \alpha = 0, \beta = 1, \eta = 0$ and $\gamma = 1$ we obtain a result of Kim [4].

(xi) If we put $n = 0, \alpha = 1/2, \beta = 0, \eta = 0$ and $\gamma = 1$ we obtain a result of Nunokawa [9] as well as that of Merkes and Wright [6].

**Theorem 2.3.** Let $f \in C(n, \alpha, \beta)$. Then for $|z| = r$

$$\frac{r(1-r)^{\beta}}{(1+r)^{\beta+2(1-\alpha)}} \leq |D^n f(z)| \leq \frac{r(1+r)^{\beta}}{(1-r)^{\beta+2(1-\alpha)}}$$

The result is sharp.

**Proof.** By definition $f \in C(n, \alpha, \beta)$ if and only if there exists a function $P \in P(0)$ and $F(z) \in S^*(\alpha)$ such that

$$\frac{D^n f(z)}{F(z)} = [P(z)]^{\beta}.$$

Therefore

$$|D^n f(z)| = |P(z)|^{\beta} |F(z)|.$$

Now using the well-known inequalities (see [1])

$$\frac{1-r}{1+r} \leq |P(z)| \leq \frac{1+r}{1-r}$$

and

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leq |F(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}},$$

we obtain the required inequalities.

Sharpness follows if we take $f(z)$ connected by the relation

$$D^n f(z) = \frac{z(1+z)^{\beta}}{(1-z)^{\beta+2(1-\alpha)}}$$

and

$$F(z) = \frac{z}{(1-z)^{2(1-\alpha)}}.$$
Theorem 2.4. If \( f \in C(n, \alpha, \beta) \), then
\[
\left| \arg \frac{D^n f(z)}{z} \right| \leq \beta \sin^{-1} \frac{2r}{1+r^2} + 2(1-\alpha) \sin^{-1} r.
\]
The result is sharp.

Proof. If \( f \in C(n, \alpha, \beta) \), then
\[
\frac{D^n f(z)}{F(z)} = [P(z)]^\beta,
\]
for some \( P(z) \in P(0) \) and \( F(z) \in S^*(\alpha) \).

Thus
\[
\left| \arg \frac{D^n f(z)}{z} \right| \leq \beta |\arg P(z)| + \left| \arg \frac{F(z)}{z} \right|.
\]  

(18)

Now using the well-known results
\[
|\arg P(z)| \leq \sin^{-1} \frac{2r}{1+r^2}
\]  

(19)

and a result of Pinchuk [12]
\[
\left| \arg \frac{F(z)}{z} \right| \leq 2(1-\alpha) \sin^{-1} r,
\]  

(20)

using (19) and (20) in (18) we get the required result.

Sharpness follows if we take \( f(z) \) to be the same as in Theorem 2.3. \( \square \)

Theorem 2.5. If \( f \in C(n, \alpha, \beta) \), then \( f \in S(n,0) \) for \( |z| < r_0 \), where
\[
r_0 = \frac{(1+\beta-\alpha)\sqrt{\alpha^2-2\beta\alpha+\beta(2+\beta)}}{1-2\alpha}, \quad \text{when } \alpha \neq \frac{1}{2}
\]
and
\[
r_0 = \frac{1}{1+2\beta}, \quad \text{when } \alpha = \frac{1}{2}.
\]
The result is sharp.

Proof. \( f \in C(n, \alpha, \beta) \) if and only if there there exists a function \( P \in P(0) \) and \( F(z) \in S^*(\alpha) \) such that
\[
\frac{D^n f(z)}{F(z)} = [P(z)]^\beta.
\]
\[
D^n f(z) = [P(z)]^\beta F(z).
\]  

(21)
Logarithmic differentiation of (21) yields
\[
\frac{z(D^n f(z))'}{D^n f(z)} = \beta \frac{zP'(z)}{P(z)} + \frac{zF'(z)}{F(z)}.
\]
Now by a result of MacGregor [5], we know that
\[
\left| \frac{zP'(z)}{P(z)} \right| \leq \frac{2r}{1-r^2}.
\]
Therefore
\[
\Re \left\{ \frac{z(D^n f(z))'}{D^n f(z)} \right\} \geq \Re \left\{ \frac{zF'(z)}{F(z)} \right\} - \beta \left| \frac{zP'(z)}{P(z)} \right|
\geq \frac{1 - (1 - 2\alpha)r}{1 + r} - \beta \left( \frac{2r}{1 - r^2} \right)
= \frac{(1 - 2\alpha)r^2 - 2(1 + \beta - \alpha)r + 1}{1 - r^2}.
\]
The right hand side of the above inequality is not less than or equal to zero provided \(|z| = r < r_0\), where \(r_0\) is as given in the statement of theorem. Sharpness follows if we take \(f(z)\) to be the same as in Theorem 2.3. \(\square\)

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