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EXISTENCE AND MULTIPLICITY SOLUTIONS FOR $(p(x),q(x))$-KIRCHHOFF TYPE SYSTEMS

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This paper is concerned with the existence and multiplicity solutions for a class of $(p(x),q(x))$-Kirchhoff type systems with Neumann boundary condition. Our technical approach is based on variational methods.

1. Introduction

In this work, we study the existence and multiplicity solutions for the nonlocal elliptic problem under Neumann boundary condition:

$$\begin{cases} -M_1\left(\int_\Omega \frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\right)(\Delta_{p(x)}u-|u|^{p(x)-2}u)=\lambda F_u(x,u,v) & \text{in} \quad \Omega \\ -M_2\left(\int_\Omega \frac{|\nabla v|^{q(x)}+|v|^{q(x)}}{q(x)}dx\right)(\Delta_{q(x)}v-|v|^{q(x)-2}v)=\lambda F_v(x,u,v) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 1)$, with smooth boundary, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $\lambda > 0$ and $p(x)$, $q(x) \in C(\overline{\Omega})$ with $N < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < +\infty$, $N < q^- := \inf_{\overline{\Omega}} q(x) \leq q^+ := \sup_{\overline{\Omega}} q(x) < +\infty$, $F(x,s,t) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is assumed to be continuous in $x \in \overline{\Omega}$ and of class C^1 in $s,t \in \mathbb{R}$, F_u, F_v denote the partial derivatives of F and $M_i : \mathbb{R}^+ \to \mathbb{R}$, $(i = 1,2)$ are continuous functions.

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The $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years, we can for example refer to [1, 4, 23, 26, 31]. This great interest may be justified by their various physical applications. In fact, there are applications concerning elastic mechanics [37], electrorheological fluids [34, 35], image restoration [14], dielectric breakdown, electrical resistivity and polycrystal plasticity [7, 8] and continuum mechanics [5].

As it is well known, problem (1) is related to the stationary problem of a model introduced by Kirchhoff [29]. More precisely, Kirchhoff introduced a model given by the following equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Later (2) was developed to form

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x,u) \text{ in } \Omega. \tag{3}$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$-M\left(\int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x,u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{4}$$

Problems like (4) can be used for modeling several physical and biological systems where u describes a process which depends on the average of it self, such as the population density, see [3]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian

$$-M\left(\int_\Omega |\nabla u|^p dx \right) \Delta_p u = f(x,u) \text{ in } \Omega,$$

see [13, 16, 20, 28]. However, to our knowledge, there is not a great number of papers which have dealt with nonlocal $p(x)$-Laplacian equations. We refer the reader to [15, 18, 27, 30, 32] and the references therein for an overview on this subject.

Hereafter, we state some natural growth hypotheses on $f(x,t)$ and the Kirchhoff function $M_i(t)$, $(i = 1,2)$.

(F_0) $F(x,0,0) = 0$ for all $x \in \Omega$.

(F_1) For all $(x,s,t) \in \Omega \times \mathbb{R}^2$, one has

$$|F(x,s,t)| \le a\left(1+|s|^{\alpha(x)}+|t|^{\beta(x)}\right),$$

where a is a positive constant and $\alpha(x), \beta(x) \in C(\overline{\Omega})$ such that $\alpha^+ = \sup_{\overline{\Omega}} \alpha(x) < p^-$; $\beta^+ = \sup_{\overline{\Omega}} \beta(x) < q^-$ for all $x \in \Omega$.

(F_2) there exist two constants $\mu_1 > \frac{p^+}{1-\theta_1}$, $\mu_2 > \frac{q^+}{1-\theta_2}$ and $R > 0$ such that for all $x \in \Omega$ and all $(s,t) \in \mathbb{R}^2$ with $|s|^{\mu_1}+|t|^{\mu_2} \ge 2R$, one has

$$0 < F(x,s,t) \le \frac{s}{\mu_1}F_s(x,s,t)+\frac{t}{\mu_2}F_t(x,s,t)$$

where θ_i, $(i=1,2)$ comes from (M_1) below.

(M_0) $M_i(t) : \mathbb{R}^+ \to [m_0,+\infty)$, $(i=1,2)$ are continuous and increasing functions such that $m_0 > 0$.

(M_1) there exists $\theta_i \in (0,1)$, $(i=1,2)$ such that

$$\widehat{M_i}(t) \ge (1-\theta_i)M_i(t)t \ \text{ for all } t \ge 0,$$

where $\widehat{M_i}(t) = \int_0^t M_i(\xi)d\xi$.

A typical example of the functions satisfying the conditions (M_1) and (M_2) is given by $M_i(t) = m_0 + b_i t$, $(i=1,2) : \mathbb{R}^+ \to \mathbb{R}$ with b_1, b_2 are two positive constants.

In the present work, by using variational method based on two consequences of a local minimum theorem [10, 12], the existence of at least two, or three solutions for the nonlocal problem (1) is established. .

2. Preliminaries and basic notations

In this section, we state some basic properties of variable exponent Sobolev space, and we recall definitions and theorems to be used in this article. Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais-Smale condition cut off upper at r (in short $(P.S.)^{[r]}$) if any sequence $\{u_n\}$ in X such that $\{I(u_n)\}$ is bounded, $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$ and $\Phi(u_n) < r \quad \forall n \in \mathbb{N}$, has a convergent subsequence.

If $r = +\infty$ it coincides with the classical (PS)-condition.

Now we recall a result of local minimum obtained in [10], which is based on [9, Theorem 5.1].

Theorem 2.1 ([10, Theorem 2.2])**.** *Let X be a real Banach space, and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})} \tag{5}$$

and, for each $\lambda \in \Lambda := \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition. Then, for each $\lambda \in \Lambda := \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[$, there is $u_\lambda \in \Phi^{-1}(]0,r[)$ such that $I_\lambda(u_\lambda) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]0,r[)$ and $I'_\lambda(u_\lambda) = 0$.

We also point out an other result, which insures the existence of at least three critical points, that has been obtained in [12] and it is a more precise version of [11, Theorem 3.2].

Theorem 2.2 ([12, Theorem 3.6])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, moreover*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < r < \Phi(\bar{u})$, such that

(i) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$

(ii) for each $\lambda \in \Lambda := \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$ the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda$, the functional $I_\lambda = \Phi - \lambda\Psi$ has at least three distinct critical points in X.

Remark 2.3. [9, Proposition 2.1] guarantees that if Φ is a sequentially weakly lower semicontinuous, coercive, continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse and Ψ is a Gâteaux differentiable function whose Gâteaux derivative is compact then the functional $\Phi - \Psi$ satisfies the $(P.S.)^{[r]}$ condition for each $r \in \mathbb{R}$.

Here, $p(x) \in C(\overline{\Omega})$ such that $1 < p^- := \min_{\overline{\Omega}} p(x) \leq p^+ := \max_{\overline{\Omega}} p(x) < +\infty$. Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\sigma > 0 : \int_\Omega |\frac{u(x)}{\sigma}|^{p(x)}\, dx \leq 1\},$$

and the variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.4 ([24, 25]). *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, uniformly convex, reflexive Banach spaces. The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $q(x)$ is the conjugate function of $p(x)$; i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

$$\Big| \int_\Omega u(x)v(x)dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

Proposition 2.5 ([24, 25]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p^*(x)$ $(r(x) < p^*(x))$ for all $x \in \overline{\Omega}$, there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega),$$

where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if} \quad p(x) < N \\ +\infty & \text{if} \quad p(x) \geq N. \end{cases}$$

Now, we introduce in $X_p := W^{1,p(x)}(\Omega)$ the norm

$$\|u\|_p := \inf\Big\{\sigma > 0 : \int_\Omega (|\frac{\nabla u(x)}{\sigma}|^{p(x)} + |\frac{u(x)}{\sigma}|^{p(x)})dx \leq 1\Big\},$$

which is equivalent to $\|.\|_{W^{1,p(x)}(\Omega)}$. Set $\rho_{1,p(x)} : X \to \mathbb{R}$ defined by

$$\rho_{1,p(x)}(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx.$$

Proposition 2.6 ([24]). *For $u \in X_p$ we have*

(i) $\|u\|_p < 1 (= 1; > 1) \Leftrightarrow \rho_{1,p(x)}(u) < 1 (= 1; > 1)$;

(ii) If $\|u\|_p < 1 \Rightarrow \|u\|_p^{p^+} \leq \rho_{1,p(x)}(u) \leq \|u\|_p^{p^-}$;

(iii) If $\|u\|_p > 1 \Rightarrow \|u\|_p^{p^-} \leq \rho_{1,p(x)}(u) \leq \|u\|_p^{p^+}$.

From now on, we write $X := X_p \times X_q$ which is a reflexive Banach space endowed with the norm

$$||(u,v)|| = ||u||_p + ||v||_q.$$

Let

$$k := \max \left\{ \sup_{u \in X_p \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{||u||_p} ; \sup_{v \in X_q \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |v(x)|}{||v||_q} \right\}. \tag{6}$$

Since $p^-, q^- > N$, the spaces X_p and X_q are compactly embedded in $C(\overline{\Omega})$ and hence $k < \infty$.

Definition 2.7. We say that $(u,v) \in X$ is a weak solution of problem (1) if

$$\begin{aligned} & M_1 \left(\int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx \right) \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + |u|^{p(x)-2} u \varphi \right) dx \\ & + M_2 \left(\int_\Omega \frac{|\nabla v|^{q(x)} + |v|^{q(x)}}{q(x)} dx \right) \int_\Omega \left(|\nabla v|^{q(x)-2} \nabla v \nabla \psi + |v|^{q(x)-2} v \psi \right) dx \\ & \qquad - \lambda \int_\Omega F_u(x,u,v) \varphi dx - \lambda \int_\Omega F_v(x,u,v) \psi dx = 0, \end{aligned}$$

for all $(\varphi, \psi) \in X$.

We denote by I_λ the energy functional associated with problem (1)

$$I_\lambda(\cdot) := \Phi(\cdot) - \lambda \Psi(\cdot),$$

where $\Phi, \Psi : X \to \mathbb{R}$ are defined as follows

$$\begin{aligned} \Phi(u,v) &= \widehat{M_1} \left(\int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx \right) + \widehat{M_2} \left(\int_\Omega \frac{|\nabla v|^{q(x)} + |v|^{q(x)}}{q(x)} dx \right), \\ \Psi(u,v) &= \int_\Omega F(x,u,v) dx \end{aligned} \tag{7}$$

for all $(u,v) \in X$. It is well known that $I_\lambda \in C^1(X, \mathbb{R})$ and a critical point of I_λ corresponds to a weak solutions of problem (1).

We need the following proposition in the proofs of our main results.

Proposition 2.8 ([30], Proposition 4.2)**.** *If* (M_0) *holds, then*

(i) Φ *is sequentially weakly lower semicontinuous and bounded on each bounded subset;*

(ii) Φ' *is a continuous and strictly monotone operator;*

(iii) Φ' *is a homeomorphism.*

3. Main result

In order to introduce our result, given two positive constants γ and δ, put

$$M_k(\gamma) := \widehat{M_1}\Big(\frac{1}{p^+}\Big(\frac{\gamma}{k}\Big)^{p^-}\Big) + \widehat{M_2}\Big(\frac{1}{q^+}\Big(\frac{\gamma}{k}\Big)^{q^-}\Big),$$

$$M(\delta) := \widehat{M_1}\Big(\frac{\delta^{p^+}}{p^-}|\Omega|\Big) + \widehat{M_2}\Big(\frac{\delta^{q^+}}{q^-}|\Omega|\Big), \text{ and}$$

$$\sigma(\gamma) = k\Big(\frac{p^+}{m_0}M_k(\gamma)\Big)^{\frac{1}{p^-}} + k\Big(\frac{q^+}{m_0}M_k(\gamma)\Big)^{\frac{1}{q^-}}$$

where k is given in (6) and $|\Omega|$ denotes the measure of Ω.

Theorem 3.1. *Assume that* (M_0), (M_1), (F_0), (F_1) *and* (F_2) *hold, and there exist two constants* $\gamma \geq k$ *and* $\delta \geq 1$ *with*

$$\frac{\delta^{p^+}}{p^-}|\Omega| < \frac{1}{p^+}\Big(\frac{\gamma}{k}\Big)^{p^-} \quad \textit{and} \quad \frac{\delta^{q^+}}{q^-}|\Omega| < \frac{1}{q^+}\Big(\frac{\gamma}{k}\Big)^{q^-} \tag{8}$$

such that

$(A1)$
$$\frac{\int_\Omega \max_{|s|+|t|\leq\sigma(\gamma)} F(x,s,t)dx}{M_k(\gamma)} < \frac{\int_\Omega F(x,\delta,\delta)dx}{M(\delta)};$$

$(A2)$ $F(x,\delta,\delta) \geq 0$ *for each* $x \in \overline{\Omega}$.

Then, for each $\lambda \in \Lambda := \Big]\frac{M(\delta)}{\int_\Omega F(x,\delta,\delta)dx}, \frac{M_k(\gamma)}{\int_\Omega \max_{|s|+|t|\leq\sigma(\gamma)} F(x,s,t)dx}\Big[$, *problem* (1) *admits at least two nontrivial weak solutions* $\widetilde{w}_1 := (u_\lambda, v_\lambda)$ *and* $\widetilde{w}_2$ *such that* $|u_\lambda| + |v_\lambda| < \sigma(\gamma)$.

Proof. Let Φ, Ψ be the functionals defined in (7). One has $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$ and since $p^-, q^- > 1$, for each $(u,v) \in X$ such that $\|u\|_p$, $\|v\|_q \geq 1$ we have

$$\Phi(u,v) \geq \frac{m_0}{p^+}\rho_{1,p(x)}(u) + \frac{m_0}{q^+}\rho_{1,q(x)}(v) \geq \frac{m_0}{p^+}\|u\|_p^{p^-} + \frac{m_0}{q^+}\|v\|_q^{q^-} \to \infty \text{ as } \|(u,v)\| \to \infty.$$

So, Φ is a coercive. From proposition 2.8, of course, Φ' admits a continuous inverse on X^*, moreover, Ψ has a compact derivative, it results sequentially weakly continuous. Our aim is to start verify condition (5) of Theorem 2.1. To this end, let $\bar{u}(x) = \delta$ for all $x \in \overline{\Omega}$, and put $r = M_k(\gamma)$. Clearly $\bar{u} \in X$, and

$$\Psi(\bar{u},\bar{u}) = \int_\Omega F(x,\bar{u},\bar{u})dx = \int_\Omega F(x,\delta,\delta)dx, \tag{9}$$

$$\Phi(\bar{u},\bar{u}) = \widehat{M_1}\left(\int_\Omega \frac{|\bar{u}|^{p(x)}}{p(x)}dx\right) + \widehat{M_2}\left(\int_\Omega \frac{|\bar{u}|^{q(x)}}{q(x)}dx\right).$$

Then, in virtue of $\delta \geq 1$ and the strict monotonicity of $\widehat{M_i}$, $(i = 1,2)$, we get

$$\begin{aligned}\widehat{M_1}\left(\frac{\delta^{p^-}}{p^+}|\Omega|\right) + \widehat{M_2}\left(\frac{\delta^{q^-}}{q^+}|\Omega|\right) &\leq \Phi(\bar{u},\bar{u}) \\ &\leq \widehat{M_1}\left(\frac{\delta^{p^+}}{p^-}|\Omega|\right) + \widehat{M_2}\left(\frac{\delta^{q^+}}{q^-}|\Omega|\right) = M(\delta). \quad (10)\end{aligned}$$

Hence, it follows from (8) that

$$0 < \Phi(\bar{u},\bar{u}) < r.$$

Now, let $(u,v) \in X$ such that $(u,v) \in \Phi^{-1}]-\infty, r[$. By (M_0) and Proposition 2.6, we obtain

$$\min\left\{\|u\|_p^{p^+}, \|u\|_p^{p^-}\right\} < \frac{rp^+}{m_0}.$$

Then

$$\|u\|_p < \max\left\{\left(\frac{rp^+}{m_0}\right)^{\frac{1}{p^+}}, \left(\frac{rp^+}{m_0}\right)^{\frac{1}{p^-}}\right\},$$

the fact that $\gamma \geq k$, we get

$$\|u\|_p < \left(\frac{rp^+}{m_0}\right)^{\frac{1}{p^-}}.$$

This together with (6), yields

$$|u(x)| \leq k||u||_p < k\left(\frac{rp^+}{m_0}\right)^{\frac{1}{p^-}} \text{ for all } x \in \Omega. \tag{11}$$

Hence, by same argument, we obtain

$$|u(x)| + |v(x)| < k\left[\left(\frac{rp^+}{m_0}\right)^{\frac{1}{p^-}} + \left(\frac{rq^+}{m_0}\right)^{\frac{1}{q^-}}\right] = \sigma(\gamma) \text{ for all } x \in \Omega.$$

So

$$\Psi(u,v)=\int_\Omega F(x,u,v)dx \le \int_\Omega \max_{\{(u,v)\in X;|u|+|v|\le\sigma(\gamma)\}} F(x,u,v)dx,$$

for all $(u,v)\in X$ such that $(u,v)\in\Phi^{-1}(]-\infty,r[)$. Thus

$$\frac{\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)}\Psi(u,v)}{r}\le\frac{\int_\Omega \max_{|s|+|t|\le\sigma(\gamma)}F(x,s,t)dx}{r}. \tag{12}$$

In view of (9), (10), (12) and taking into account $(A1)$ and $(A2)$, we obtain

$$\begin{aligned}\frac{\sup_{\Phi(u,v)\le r}\Psi(u,v)}{r}&\le\frac{\int_\Omega \max_{|s|+|t|\le\sigma(\gamma)}F(x,s,t)dx}{M_k(\gamma)}\\&<\frac{\int_\Omega F(x,\delta,\delta)dx}{M(\delta)}\le\frac{\Psi(\bar{u},\bar{u})}{\Phi(\bar{u},\bar{u})}.\end{aligned} \tag{13}$$

Therefore, condition (5) of Theorem 2.1 is verified. Now, fixed $\lambda>0$, remark (2.3) and proposition (2.8) assured that I_λ satisfies the $(P.S.)^{[r]}$ condition for all $r>0$. So, for each $\lambda\in\Lambda\subset\left]\frac{\Phi(\bar{u},\bar{u})}{\Psi(\bar{u},\bar{u})},\frac{1}{\sup_{\Phi(u,v)\le r}\Psi(u,v)}\right[$, the functional I_λ admits at least one critical point $\widetilde{w}_1=(u_\lambda,v_\lambda)$ such that $0<\Phi(u_\lambda,v_\lambda)<r$, and so (u_λ,v_λ) is a nontrivial weak solution of problem (1) such that $|u_\lambda|+|v_\lambda|<\sigma(\gamma)$.

Now we prove the existence of a second local minimum distinct from the first one. To this purpose, we verify the hypotheses of the mountain pass theorem for the functional I_λ. Clearly I_λ is of class C^1 and $I_\lambda(0)=0$. The first part of proof guarantees that $\widetilde{w}_1\in X$ is a nontrivial local minimum for I_λ in X. Therefore there is $\rho>0$ such that

$$\inf_{\|u-\widetilde{w}_1\|=\rho} I_\lambda(u)\ge I_\lambda(\widetilde{w}_1),$$

so condition [33, (I_1), Theorem 2.2] is verified. Now, from condition (F_2), by standard computations (see [22]), there is a positive constant c_1 such that

$$F(x,s,t)\ge c_1\left(|s|^{\mu_1}+|t|^{\mu_2}-1\right). \tag{14}$$

By integrating (M_1), we get

$$\widehat{M_i}(t)\le\frac{\widehat{M_i}(t_0)}{t_0^{\frac{1}{1-\theta_i}}}t^{\frac{1}{1-\theta_i}}=c_2^i t^{\frac{1}{1-\theta_i}}\ (i=1,2)\quad\text{for all } t\ge t_0>0. \tag{15}$$

Hence, from (14) and (15), for $(u,v) \in X\backslash\{(0,0)\}$ and $t > 1$, we obtain

$$
\begin{aligned}
I_\lambda(tu,tv) \leq\ & \widehat{M_1}\left(\int_\Omega \frac{|t\nabla u|^{p(x)}+|tu|^{p(x)}}{p(x)}dx\right)+\widehat{M_2}\left(\int_\Omega \frac{|t\nabla v|^{q(x)}+|tv|^{q(x)}}{q(x)}dx\right)\\
& -\lambda\int_\Omega F(x,tu,tv)dx\\
& \leq c_3\Big(\rho_{1,p(x)}(tu)\Big)^{\frac{1}{1-\theta_1}}+c_4\Big(\rho_{1,q(x)}(tv)\Big)^{\frac{1}{1-\theta_2}}\\
& -c_1\lambda t^{\mu_1}\int_\Omega |u(x)|^{\mu_1}dx-c_1\lambda t^{\mu_2}\int_\Omega |v(x)|^{\mu_2}dx-c_5\\
& \leq c_3 t^{\frac{p^+}{1-\theta_1}}\Big(\rho_{1,p(x)}(u)\Big)^{\frac{1}{1-\theta_1}}+c_4 t^{\frac{q^+}{1-\theta_2}}\Big(\rho_{1,q(x)}(v)\Big)^{\frac{1}{1-\theta_2}}\\
& -c_1\lambda t^{\mu_1}\int_\Omega |u(x)|^{\mu_1}dx-c_1\lambda t^{\mu_2}\int_\Omega |v(x)|^{\mu_2}dx-c_5\to-\infty
\end{aligned}
$$

as $t\to\infty$, since $\mu_1 > \frac{p^+}{1-\theta_1}$ and $\mu_2 > \frac{q^+}{1-\theta_2}$. So the condition [33, (I_2), Theorem 2.2] is verified. Finally, we verify the (PS)-condition, it is sufficient to prove that any Palais-Smale sequence is bounded. To this end, let (u_n,v_n) be a Palais-Smale sequence for the functional I_λ, this means that $I_\lambda(u_n,v_n)$ is bounded and $\|I'_\lambda(u_n,v_n)\|_{X^*}\to 0$ as $n\to+\infty$. Using hypotheses (M_0), (M_1) and (F_2), we have

$$
\begin{aligned}
C_0 \geq\ & I_\lambda(u_n,v_n)\\
& \geq (1-\theta_1)M_1\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}\,dx\Big)\int_\Omega \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}\,dx\\
& +(1-\theta_2)M_2\Big(\int_\Omega \frac{|\nabla v_n|^{q(x)}+|v_n|^{q(x)}}{q(x)}\,dx\Big)\int_\Omega \frac{|\nabla v_n|^{q(x)}+|v_n|^{q(x)}}{q(x)}\,dx\\
& -\frac{\lambda}{\mu_1}\int_\Omega F_u(x,u_n,v_n)u_n\,dx-\frac{\lambda}{\mu_2}\int_\Omega F_v(x,u_n,v_n)v_n\,dx-c_6\\
& \geq m_0\Big(\frac{1-\theta_1}{p^+}-\frac{1}{\mu_1}\Big)\rho_{1,p(x)}(u_n)+\frac{1}{\mu_1}I_p(u_n,v_n)(u_n)\\
& +m_0\Big(\frac{1-\theta_2}{q^+}-\frac{1}{\mu_2}\Big)\rho_{1,q(x)}(v_n)+\frac{1}{\mu_2}I_q(u_n,v_n)(v_n)-c_6,\ \text{where}
\end{aligned}
$$

$$
\begin{aligned}
I_p(u,v)\varphi :=\ & M_1\left(\int_\Omega \frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\right)\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi+|u|^{p(x)-2}u\varphi\Big)\,dx\\
& -\lambda\int_\Omega F_u(x,u,v)\varphi dx\\
I_q(u,v)\psi :=\ & M_2\left(\int_\Omega \frac{|\nabla v|^{q(x)}+|v|^{q(x)}}{q(x)}dx\right)\int_\Omega \Big(|\nabla v|^{q(x)-2}\nabla v\nabla\psi+|v|^{q(x)-2}v\psi\Big)\,dx\\
& -\lambda\int_\Omega F_v(x,u,v)\psi dx.
\end{aligned}
$$

Now, suppose that the sequence (u_n,v_n) is not bounded. Without loss of generality, we may assume $\|u_n\|_p \geq \|v_n\|_q$, with $\|.\|_{*,p}$ (respectively $\|.\|_{*,q}$) is the norm of the dual X_p^* (respectively X_q^*), we have

$$C_0' \geq m_0\Big(\frac{1-\theta_1}{p^+}-\frac{1}{\mu_1}\Big)\|u_n\|_p^{p^-}-\Big(\frac{1}{\mu_1}\|I_p(u_n,v_n)\|_{*,p}+\frac{1}{\mu_2}\|I_q(u_n,v_n)\|_{*,q}\Big)\|u_n\|_p,$$

But, this cannot hold true since $p^- > 1$ and $\mu_1 > \frac{p^+}{1-\theta_1}$. Hence, $\{\|(u_n,v_n)\|\}$ is bounded. consequently, the classical theorem of Ambrosetti and Rabinowitz ensures a critical point $\widetilde{w}_2$ such that $I_\lambda(\widetilde{w}_2) > I_\lambda(\widetilde{w}_1)$. So $\widetilde{w}_1$ and $\widetilde{w}_2$ are distinct weak solutions of the problem, and the proof of Theorem 3.1 is achieved. □

Finally, we give an application of Theorem 2.2.

Theorem 3.2. *Assume that* (M_0), (F_0) *and* (F_1) *hold, and there exist two constants* $\gamma \geq k$ *and* $\delta \geq 1$ *with*

$$\frac{\delta^{p^-}}{p^+}|\Omega| > \frac{1}{p^+}\Big(\frac{\gamma}{k}\Big)^{p^-} \quad and \quad \frac{\delta^{q^-}}{q^+}|\Omega| > \frac{1}{q^+}\Big(\frac{\gamma}{k}\Big)^{q^-} \tag{16}$$

such that the assumptions $(A1)$ *and* $(A2)$ *in Theorem 3.1 hold. Then, for each* $\lambda \in \Lambda$, *problem* (1) *admits at least three weak solutions.*

Proof. Let Φ, Ψ be the functionals defined in (7) satisfy all regularity assumptions requested in Theorem 2.2. So, our aim is to verify (i) and (ii). Arguing as in the proof of Theorem 3.1, put $\bar{u}(x) = \delta$ and $r = M_k(\gamma)$, bearing in mind (16) we obtain

$$\Phi(\bar{u},\bar{u}) > r > 0.$$

Therefore, (13) holds and the assumption (i) of Theorem 2.2 is satisfied. Now, we prove that the functional I_λ is coercive. For $(u,v) \in X$ such that $||(u,v)|| \to +\infty$, in fact by using condition (F_1) we have

$$\begin{aligned}
I_\lambda(u,v) &\geq \widehat{M_1}\left(\int_\Omega \frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\right)+\widehat{M_2}\left(\int_\Omega \frac{|\nabla v|^{q(x)}+|v|^{q(x)}}{q(x)}dx\right)\\
&\quad -\lambda\int_\Omega F(x,u,v)dx\\
&\geq \frac{m_0}{p^+}\rho_{1,p(x)}(u)+\frac{m_0}{q^+}\rho_{1,q(x)}(v)-\lambda a\Big(|\Omega|+\int_\Omega |u|^{\alpha(x)}+\int_\Omega |v|^{\beta(x)}\Big),
\end{aligned}$$

On the other hand, there are a constants C_1 and C_2 such that

$$\begin{aligned}
\int_\Omega |u|^{\alpha(x)} &\leq \max\Big\{|u|_{p(x)}^{\alpha^-},|u|_{p(x)}^{\alpha^+}\Big\} \leq C_1||u||_p^{\alpha^+}\\
\int_\Omega |v|^{\beta(x)} &\leq \max\Big\{|v|_{q(x)}^{\beta^-},|v|_{q(x)}^{\beta^+}\Big\} \leq C_2||v||_q^{\beta^+}.
\end{aligned}$$

Thus, for every $\lambda \in \Lambda$ we get

$$I_\lambda(u,v) \geq \frac{m_0}{p^+}||u||_p^{p^-} + \frac{m_0}{q^+}||v||_q^{q^-} - C_1||u||_p^{\alpha^+} - C_2||v||_q^{\beta^+}.$$

Since $p^- > \alpha^+$ and $q^- > \beta^+$, the functional I_λ is coercive, also condition (ii) holds. So, for each $\lambda \in \Lambda$, the functional I_λ has at least three distinct critical points that are weak solutions of system (1). □

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