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COEFFICIENT BOUNDS FOR A GENERAL SUBCLASS OF BI-UNIVALENT FUNCTIONS

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In the present investigation, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions defined in the open unit disc. We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class $S_\Sigma(n,h,\lambda)$. The results presented in this paper improve or generalize the recent works of *Jothibasu* [13] and other authors.

1. Introduction and Definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and let S be the class of all functions from A which are univalent in U.

The Koebe one-quarter theorem [8] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z\ ,\ (z \in U)$$

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and

$$f\left(f^{-1}(w)\right)=w\ ,\ \left(|w|<r_0(f)\ ,\ r_0(f)\geq\frac{1}{4}\right),$$

where

$$f^{-1}(w)=w\ -a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots.$$

A function $f(z)\in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples of functions in the class Σ; see [3].

The research into Σ was started by Lewin ([15]). It focused on problems connected with coefficients and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [4], [19]). Thus, following Brannan and Taha [3], a function $f(z)\in A$ is said to be in the class $S_\Sigma^\star(\alpha)$ of strongly bi-starlike functions of order α $(0<\alpha\leq 1)$ if each of the following conditions is satisfied:

$$f\in\Sigma,\quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\quad (0<\alpha\leq 1,\ z\in U)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}\quad (0<\alpha\leq 1,\ w\in U)$$

and is said to be in the class $K_\Sigma(\alpha)$ of strongly bi-convex functions of order α $(0<\alpha\leq 1)$ if each of the following conditions is satisfied:

$$f\in\Sigma,\quad \left|\arg\left(1+\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\quad (0<\alpha\leq 1,\ z\in U)$$

and

$$\left|\arg\left(1+\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}\quad (0<\alpha\leq 1,\ w\in U)$$

where g is the extension of f^{-1} to U. The classes $S_\Sigma^\star(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $S^\star(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_\Sigma^\star(\alpha)$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the initial coefficients. Recently, Srivastava et al. [22] introduced and investigated subclasses of the bi-univalent functions and obtained bounds for the initial coefficients; it was followed by such works as those by Murugunsundaramoorthy et al. [18], Frasin and Aouf [9], Çağlar et al. [6] and others (see, for example, [1], [7], [14], [16], [17], [20], [24]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [5], [10], [11], [12]). The coefficient estimate problem for each of $|a_n|$ $(\, n \in \mathbb{N} \backslash \{1,2\};\ \ \mathbb{N} = \{1,2,3,...\})$ is still an open problem.

Let $f \in A$. We define the differential operator D^n, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ by (see [21])

$$\begin{aligned} D^0 f(z) &= f(z); \\ D^1 f(z) &= Df(z) = zf'(z); \\ &\vdots \\ D^n f(z) &= D^1\left(D^{n-1} f(z)\right). \end{aligned}$$

In this paper, by using the method [23] different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered Jothibasu and get more accurate estimates than that given in [13].

Definition 1.1. Let the functions $h, p : U \to \mathbb{C}$ be so constrained that

$$\min\{Re(h(z)), Re(p(z))\} > 0$$

and

$$h(0) = p(0) = 1.$$

Definition 1.2. A function $f \in \Sigma$ is said to be $S_\Sigma(n,h,\lambda)$, $n \in \mathbb{N}_0$ and $0 \leq \lambda < 1$, if the following conditions are satisfied:

$$\frac{D^{n+1} f(z)}{(1-\lambda) D^n f(z) + \lambda D^{n+1} f(z)} \in h(U) \quad (z \in U) \tag{2}$$

and

$$\frac{D^{n+1} g(w)}{(1-\lambda) D^n g(w) + \lambda D^{n+1} g(w)} \in p(U) \quad (w \in U) \tag{3}$$

where $g(w) = f^{-1}(w)$.

2. Coefficient Estimates

Theorem 2.1. *Let f given by (1) be in the class $S_{\Sigma}(n,h,\lambda)$, $n\in\mathbb{N}_0$ and $0\leq\lambda<1$. Then*

$$|a_2|\leq\min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(1-\lambda)^2 4^n}},\sqrt{\frac{|h''(0)|+|p''(0)|}{4\left|(\lambda^2-1)4^n+2(1-\lambda)3^n\right|}}\right\} \tag{4}$$

and

$$|a_3|\leq\min\left\{\begin{array}{c}\frac{|h'(0)|^2+|p'(0)|^2}{2(1-\lambda)^2 4^n}+\frac{|h''(0)|+|p''(0)|}{8(1-\lambda)3^n},\\ \\ \frac{\left[4(1-\lambda)3^n+\left(\lambda^2-1\right)4^n\right]|h''(0)|+\left(1-\lambda^2\right)4^n|p''(0)|}{8\left|(\lambda^2-1)4^n+2(1-\lambda)3^n\right|(1-\lambda)3^n}\end{array}\right\}. \tag{5}$$

Proof. Let $f\in S_{\Sigma}(n,h,\lambda)$ and $0\leq\lambda<1$. It follows from (2) and (3) that

$$\frac{D^{n+1}f(z)}{(1-\lambda)D^n f(z)+\lambda D^{n+1}f(z)}=h(z) \tag{6}$$

and

$$\frac{D^{n+1}g(w)}{(1-\lambda)D^n g(w)+\lambda D^{n+1}g(w)}=p(w), \tag{7}$$

where $h(z)$ and $p(w)$ satisfy the conditions of Definiton 1. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z)=1+h_1z+h_2z^2+\cdots$$

and

$$p(w)=1+p_1w+p_2w^2+\cdots,$$

respectively. It follows from (6) and (7) that

$$(1-\lambda)2^n a_2=h_1, \tag{8}$$

$$\left(\lambda^2-1\right)4^n a_2^2+2(1-\lambda)3^n a_3=h_2, \tag{9}$$

and

$$-(1-\lambda)2^n a_2=p_1, \tag{10}$$

$$2(1-\lambda)3^n\left(2a_2^2-a_3\right)+\left(\lambda^2-1\right)4^n a_2^2=p_2. \tag{11}$$

From (8) and (10) we obtain

$$h_1=-p_1, \tag{12}$$

and

$$2(1-\lambda)^2 4^n a_2^2 = h_1^2 + p_1^2. \tag{13}$$

By adding (9) to (11), we find that

$$\left[2\left(\lambda^2-1\right)4^n + 4(1-\lambda)3^n\right] a_2^2 = h_2 + p_2, \tag{14}$$

which gives us the desired estimate on $|a_2|$ as asserted in (4).

Next, in order to find the bound on $|a_3|$, by subtracting (11) from (9), we obtain

$$4(1-\lambda)3^n a_3 - 4(1-\lambda)3^n a_2^2 = h_2 - p_2. \tag{15}$$

Then, in view of (13), (14) and (15) , it follows that

$$a_3 = \frac{h_1^2+p_1^2}{2(1-\lambda)^2 4^n} + \frac{h_2-p_2}{4(1-\lambda)3^n}$$

and

$$a_3 = \frac{h_2+p_2}{2(\lambda^2-1)4^n+4(1-\lambda)3^n} + \frac{h_2-p_2}{4(1-\lambda)3^n}.$$

as claimed. This completes the proof of Theorem 2.1. □

3. Corollaries and Consequences

By setting $n=0$ and $\lambda=0$ in Theorem 2.1, we get Corollary 3.1 below.

Corollary 3.1. *Let the function $f(z)$ given by (1) be in the class $S_\Sigma(h)$ $(0 \le \lambda < 1)$. Then*

$$|a_2| \le \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{4}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{2} + \frac{|h''(0)|+|p''(0)|}{8}, \frac{3|h''(0)|+|p''(0)|}{8}\right\}.$$

By setting $n=1$ and $\lambda=0$ in Theorem 2.1, we get Corollary 3.2 below.

Corollary 3.2. *Let the function $f(z)$ given by (1) be in the class $K_\Sigma(h)$ $(0 \le \lambda < 1)$. Then*

$$|a_2| \le \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{8}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{8}}\right\}$$

and

$$|a_3| \leq \min\left\{ \frac{|h'(0)|^2+|p'(0)|^2}{8}+\frac{|h''(0)|+|p''(0)|}{24}, \frac{2|h''(0)|+|p''(0)|}{12} \right\}.$$

Corollary 3.3. *If*

$$h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2z^2+\ldots \quad (0<\alpha\leq 1),$$

then inequalities (4) and (5) become

$$|a_2| \leq \min\left\{ \frac{2\alpha}{(1-\lambda)2^n}, \sqrt{\frac{2}{|(\lambda^2-1)4^n+2(1-\lambda)3^n|}}\alpha \right\}$$

and

$$|a_3| \leq \min\left\{ \frac{4\alpha^2}{(1-\lambda)^2 4^n}+\frac{\alpha^2}{(1-\lambda)3^n}, \frac{2\alpha^2}{|(\lambda^2-1)4^n+2(1-\lambda)3^n|} \right\}.$$

Remark 3.4. Putting $n=0$ and $\lambda=0$ in Corollary 3.3, we obtain the following corollary.

Corollary 3.5 (See also [23]). *Let the function $f(z)$ given by (1) be in the class $S^*_\Sigma(\alpha)$. Then*

$$|a_2| \leq \sqrt{2}\alpha$$

and

$$|a_3| \leq 2\alpha^2.$$

The estimates on the coefficient $|a_2|$ and $|a_3|$ of Corollary 3.5 are improvement of the estimates obtained in [13] and [18].

Remark 3.6. Putting $n=1$ and $\lambda=0$ in Corollary 3.3, we obtain the following corollary.

Corollary 3.7. *Let the function $f(z)$ given by (1) be in the class $K_\Sigma(\alpha)$. Then*

$$|a_2| \leq \alpha$$

and

$$|a_3| \leq \alpha^2.$$

The estimate on the coefficient $|a_3|$ of Corollary 3.7 is improvement of the estimates obtained in [13].

Corollary 3.8. *If*

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} = 1+2(1-\beta)z+2(1-\beta)z^2+\cdots \ (0 \leq \beta < 1),$$

then inequalities (4) and (5) become

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{(1-\lambda)2^n}, \sqrt{\frac{2(1-\beta)}{|(\lambda^2-1)4^n+2(1-\lambda)3^n|}}\right\}.$$

and

$$|a_3| \leq \min\left\{\frac{4(1-\beta)^2}{(1-\lambda)^2 4^n} + \frac{1-\beta}{(1-\lambda)3^n}, \frac{2(1-\beta)}{|(\lambda^2-1)4^n+2(1-\lambda)3^n|}\right\}.$$

Remark 3.9. Putting $n = 0$ and $\lambda = 0$ in Corollary 3.8, we obtain the following corollary.

Corollary 3.10. *Let the function $f(z)$ given by (1) be in the class $K_\Sigma(\beta)$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq 2(1-\beta).$$

The estimate on the coefficient $|a_3|$ of Corollary 3.10 is improvement of the estimates obtained in [13] and [18].

Remark 3.11. Further, taking $n = 1$ and $\lambda = 0$ in Corollary 3.8, we obtain the following corollary.

Corollary 3.12. *Let the function $f(z)$ given by (1) be in the class $K_\Sigma(\beta)$. Then*

$$|a_2| \leq \sqrt{1-\beta}$$

and

$$|a_3| \leq (1-\beta).$$

The estimate on the coefficient $|a_3|$ of Corollary 3.12 is improvement of the estimates obtained in [3] and [13].

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