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SECOND HANKEL DETERMINANT FOR BI-STARLIKE FUNCTIONS OF ORDER β

ŞAHSENE ALTINKAYA - SIBEL YALÇIN

Making use of the Hankel determinant, in this work, we consider a general subclass of bi-univalent functions. Moreover, we investigate the bounds of initial coefficients of this class.

1. Introduction

Let A denote the class of functions f which are analytic in the open unit disk $U = \{z : |z| < 1\}$ with the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S be the subclass of A consisting of the form (1) which are also univalent in U. The Koebe one-quarter theorem [9] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

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where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots.$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

For a brief history and interesting examples in the class Σ, see [24]. Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in S such as

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ (see [24]).

Lewin [15] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Netanyahu [18] showed that $max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [6] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. $S^\star(\beta)$ and $K(\beta)$ of starlike and convex function of order β $(0 \leq \beta < 1)$ respectively (see [18]). By definition, we have

$$S^\star(\beta) = \left\{ f \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta; \quad 0 \leq \beta < 1,\ z \in U \right\}$$

and

$$K(\beta) = \left\{ f \in S : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta; \quad 0 \leq \beta < 1,\ z \in U \right\}.$$

It readily follows from the definitions

$$f \in K(\beta) \Leftrightarrow zf' \in S^\star(\beta).$$

The classes $S_\Sigma^\star(\beta)$ and $K_\Sigma(\beta)$ of bi-starlike functions of order α and bi-convex functions of order β, corresponding to the function classes $S^\star(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S_\Sigma^\star(\beta)$ and $K_\Sigma(\beta)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1], [4], [11], [16], [24], [25], [26]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [7], [13], [14]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N}\backslash\{1,2\}$; $\mathbb{N}=\{1,2,3,\ldots\}$) is still an open problem.

The Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ for normalized univalent functions

$$f(z)=z+a_2z^2+\cdots$$

is well known for its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [10]). The functional has since received great attention, particularly in many subclasses of the family of univalent functions. Nowadays, it seems that this topic had become an interest among the researchers (see, for example, [3], [21], [27]).

The q^{th} Hankel determinant for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas ([19]) as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1=1).$$

This determinant has also been considered by several authors. For example, Noor ([20]) determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles ([22], [20]) for different classes of functions.

Note that

$$H_2(1)=\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}=a_3-a_2^2$$

and

$$H_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2.$$

The Hankel determinant $H_2(1)=a_3-a_2^2$ is well-known as Fekete-Szegö functional. Very recently, the upper bounds of $H_2(2)$ for some classes were discussed by Deniz et al. [8].

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $S_{\Sigma}^{\lambda}(\beta)$, if the following conditions are satisfied:

$$\Re\left(\frac{z^{1-\lambda} f'(z)}{[f(z)]^{1-\lambda}}\right) > \beta, \quad 0 \le \beta < 1, \; \lambda \ge 0, \, z \in U \tag{2}$$

and

$$\Re\left(\frac{w^{1-\lambda} g'(w)}{[g(w)]^{1-\lambda}}\right) > \beta, \quad 0 \le \beta < 1, \; \lambda \ge 0, \, w \in U. \tag{3}$$

where $g = f^{-1}$.

In this paper, we get upper bound for the functional $H_2(2) = a_2a_4 - a_3^2$ for functions f belongs to the class $S_{\Sigma}^{\lambda}(\beta)$.

In order to derive our main results, we require the following lemma.

Lemma 1.2 ([23]). *If $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ is an analytic function in U with positive real part, then*

$$|p_n| \le 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \le 2 - \frac{|p_1|^2}{2}.$$

Lemma 1.3 ([12]). *If the function $p \in P$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2)$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

2. Main results

Theorem 2.1. *Let f given by (1) be in the class $B(\alpha, \beta)$, $0 \le \alpha < 1$ and $0 \le \beta < 1$. Then*

$$\left|a_2a_4 - a_3^2\right| \le$$

$$\begin{cases} \frac{8(1-\beta)^2}{1+\lambda}\left[\frac{(\lambda+2)(1-\beta)^2}{3(1+\lambda)^2}+\frac{1}{2(3+\lambda)}\right], \\ \beta\in\left[0,1-\frac{3(1+\lambda)(3+\lambda)+(1+\lambda)\sqrt{9(3+\lambda)^2+24(2+\lambda)(3+\lambda)(5+4\lambda+\lambda^2)}}{4(2+\lambda)^2(3+\lambda)}\right] \\ (1-\beta)^2\left\{\frac{4}{(2+\lambda)^2}\right. \\ -\left\{\frac{3\left[\left(\lambda^2+5\lambda+6\right)(1-\beta)+\left(\lambda^3+5\lambda^2+10\lambda+6\right)\right]^2}{[2(2+\lambda)^2(3+\lambda)(\lambda^2+3\lambda+2)(1-\beta)^2-3(1+\lambda)^2(2+\lambda)(3+\lambda)(1-\beta)-3(1+\lambda)^3(\lambda^2+4\lambda+5)]}\right\} \\ \left.\times\frac{1}{(2+\lambda)^2(3+\lambda)}\right\}, \\ \beta\in\left[1-\frac{3(1+\lambda)(3+\lambda)+(1+\lambda)\sqrt{9(3+\lambda)^2+48(2+\lambda)^3(3+\lambda)}}{8(2+\lambda)^2(3+\lambda)},1\right). \end{cases}$$

Proof. Let $f\in S_\Sigma^\lambda(\beta)$. Then

$$\frac{z^{1-\lambda}f'(z)}{[f(z)]^{1-\lambda}}=\beta+(1-\beta)p(z) \tag{4}$$

$$\frac{w^{1-\lambda}g'(w)}{[g(w)]^{1-\lambda}}=\beta+(1-\beta)q(w) \tag{5}$$

where $p,q\in P$ and $g=f^{-1}$.

It follows from (4) and (5) that

$$(1+\lambda)a_2=(1-\beta)p_1, \tag{6}$$

$$(2+\lambda)a_3-\frac{(1-\lambda)(2+\lambda)}{2}a_2^2=(1-\beta)p_2, \tag{7}$$

$$(3+\lambda)a_4-(1-\lambda)(3+\lambda)a_2a_3-\frac{(1-\lambda)(\lambda-2)(\lambda+3)}{6}a_2^3=(1-\beta)p_3 \tag{8}$$

$$-(1+\lambda)a_2=(1-\beta)q_1, \tag{9}$$

$$\frac{6+5\lambda+\lambda^2}{2}a_2^2-(2+\lambda)a_3=(1-\beta)q_2 \tag{10}$$

$$(12+7\lambda+\lambda^2)a_2a_3-(3+\lambda)a_4+\left[\frac{(1-\lambda)}{6}(30+13\lambda+\lambda^2)-5(3+\lambda)\right]a_2^3 \\ =(1-\beta)q_3. \tag{11}$$

From (6) and (9) we obtain

$$p_1=-q_1. \tag{12}$$

and

$$a_2=\frac{(1-\beta)}{1+\lambda}p_1. \tag{13}$$

Subtracting (7) from (10), we have

$$a_3=\frac{(1-\beta)^2}{(1+\lambda)^2}p_1^2+\frac{(1-\beta)}{2(2+\lambda)}(p_2-q_2). \tag{14}$$

Also, subtracting (8) from (11), we have

$$a_4=\frac{(1-\lambda)(4+\lambda)(1-\beta)^3}{6(1+\lambda)^3}p_1^3+\frac{5(1-\beta)^2}{4(1+\lambda)(2+\lambda)}p_1(p_2-q_2)+\frac{(1-\beta)}{2(3+\lambda)}(p_3-q_3). \tag{15}$$

Then, we can establish that

$$\begin{aligned}\left|a_2a_4-a_3^2\right|=\Bigg|&\left[\frac{(1-\lambda)(4+\lambda)}{6}-1\right]\frac{(1-\beta)^4}{(1+\lambda)^4}p_1^4+\frac{(1-\beta)^3}{4(1+\lambda)^2(2+\lambda)}p_1^2(p_2-q_2)\\&+\frac{(1-\beta)^2}{2(1+\lambda)(3+\lambda)}p_1(p_3-q_3)-\frac{(1-\beta)^2}{4(2+\lambda)^2}(p_2-q_2)^2\Bigg|\end{aligned} \tag{16}$$

According to Lemma 1.1 and (12), we write

$$\left.\begin{array}{l}2p_2=p_1^2+x(4-p_1^2)\\2q_2=q_1^2+y(4-q_1^2)\end{array}\right\}\Rightarrow p_2-q_2=\frac{4-p_1^2}{2}(x-y) \tag{17}$$

and

$$\begin{aligned}4p_3&=p_1^3+2(4-p_1^2)p_1x-p_1(4-p_1^2)x^2+2(4-p_1^2)(1-|x|^2)z\\4q_3&=q_1^3+2(4-q_1^2)q_1y-q_1(4-q_1^2)y^2+2(4-q_1^2)(1-|y|^2)w\end{aligned}$$

$$\begin{aligned}p_3-q_3=\frac{p_1^3}{2}+\frac{p_1(4-p_1^2)}{2}(x+y)-\frac{p_1(4-p_1^2)}{4}(x^2+y^2)\\+\frac{4-p_1^2}{2}\left[(1-|x|^2)z-(1-|y|^2)w\right].\end{aligned} \tag{18}$$

Then, using (17) and (18), in (16),

$$\left|a_2a_4-a_3^2\right|= \tag{19}$$

$$\begin{cases}\Big|\frac{-(2+3\lambda+\lambda^2)}{6}\frac{(1-\beta)^4}{(1+\lambda)^4}p_1^4+\frac{(1-\beta)^3}{4(1+\lambda)^2(2+\lambda)}p_1^2\frac{4-p_1^2}{2}(x-y)\\+\frac{(1-\beta)^2}{4(1+\lambda)(3+\lambda)}p_1^4+\frac{(1-\beta)^2}{2(1+\lambda)(3+\lambda)}p_1^2\frac{4-p_1^2}{2}(x+y)-\frac{(1-\beta)^2}{2(1+\lambda)(3+\lambda)}p_1^2\frac{(4-p_1^2)}{4}(x^2+y^2)\\+\frac{(1-\beta)^2}{2(1+\lambda)(3+\lambda)}p_1\frac{(4-p_1^2)}{2}\left[\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right]-\frac{(1-\beta)^2}{4(2+\lambda)^2}\frac{(4-p_1^2)^2}{4}(x+y)^2\Big|\\\leq\frac{(2+3\lambda+\lambda^2)}{6}\frac{(1-\beta)^4}{(1+\lambda)^4}p_1^4+\frac{(1-\beta)^2}{4(1+\lambda)(3+\lambda)}p_1^4+\frac{(1-\beta)^2}{2(1+\lambda)(3+\lambda)}p_1(4-p_1^2)\\+\left[\frac{(1-\beta)^3}{4(1+\lambda)^2(2+\lambda)}p_1^2\frac{(4-p_1^2)}{2}+\frac{(1-\beta)^2}{2(1+\lambda)(3+\lambda)}p_1^2\frac{(4-p_1^2)}{2}\right](|x|+|y|)\\+\left[\frac{(1-\beta)^2}{2(1+\lambda)(3+\lambda)}p_1^2\frac{(4-p_1^2)}{4}-\frac{(1-\beta)^2}{2(1+\lambda)(3+\lambda)}p_1\frac{(4-p_1^2)}{2}\right](|x|^2+|y|^2)\\+\frac{(1-\beta)^2}{4(2+\lambda)^2}\frac{(4-p_1^2)^2}{4}(|x|+|y|)^2.\end{cases}$$

Since $p \in P$, so $|p_1| \leq 2$. Letting $|p_1| = p$, we may assume without restriction that $p \in [0,2]$. For $\eta = |x| \leq 1$ and $\mu = |y| \leq 1$, we get

$$\left|a_2 a_4 - a_3^2\right| \leq T_1 + (\eta+\mu)\,T_2 + \left(\eta^2+\mu^2\right)T_3 + (\eta+\mu)^2\,T_4 = G(\eta,\mu)$$

where, setting $T_i = T_i(p)$, $i = 1,2,3,4$

$$T_1 = \frac{(1-\beta)^2}{2\,(1+\lambda)}\left[\left(\tfrac{(\lambda^2+3\lambda+2)(1-\beta)^2}{3(1+\lambda)^3} + \frac{1}{2\,(3+\lambda)}\right)p^4 - \frac{1}{3+\lambda}p^3 + \frac{4}{3+\lambda}p\right] \geq 0$$
$$T_2 = \frac{(1-\beta)^2}{4\,(1+\lambda)}p^2(4-p^2)\left[\tfrac{(1-\beta)}{2(1+\lambda)(2+\lambda)} + \tfrac{1}{3+\lambda}\right] \geq 0$$
$$T_3 = \frac{(1-\beta)^2}{8\,(1+\lambda)\,(3+\lambda)}p(4-p^2)(p-2) \leq 0$$
$$T_4 = \frac{(1-\beta)^2}{4\,(2+\lambda)^2}\frac{(4-p^2)^2}{4} \geq 0.$$

We now need to maximize the function $G(\eta,\mu)$ on the closed square $[0,1] \times [0,1]$. We must investigate the maximum of $G(\eta,\mu)$ according to $p \in (0,2)$, $p = 0$ and $p = 2$ taking into account the sign of $G_{\eta\eta}.G_{\mu\mu} - \left(G_{\eta\mu}\right)^2$.

Firstly, let $p \in (0,2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $p \in (0,2)$, we conclude that

$$G_{\eta\eta}.G_{\mu\mu} - \left(G_{\eta\mu}\right)^2 < 0.$$

Thus the function G cannot have a local maximum in the interior of the square. Now, we investigate the maximum of G on the boundary of the square.

For $\eta = 0$ and $0 \leq \mu \leq 1$ (similarly $\mu = 0$ and $0 \leq \eta \leq 1$), we obtain

$$G(0,\mu) = H(\mu) = (T_3+T_4)\mu^2 + T_2\mu + T_1.$$

i. The case $T_3 + T_4 \geq 0$: In this case for $0 < \mu < 1$ and any fixed p with $0 < p < 2$, it is clear that $H'(\mu) = 2(T_3+T_4)\mu + T_2 > 0$, that is, $H(\mu)$ is an increasing function. Hence, for fixed $p \in (0,2)$, the maximum of $H(\mu)$ occurs at $\mu = 1$, and

$$\max H(\mu) = H(1) = T_1 + T_2 + T_3 + T_4.$$

ii. The case $T_3 + T_4 < 0$: Since $T_2 + 2(T_3+T_4) \geq 0$ for $0 < \mu < 1$ and any fixed p with $0 < p < 2$, it is clear that $T_2 + 2(T_3+T_4) < 2(T_3+T_4)\mu + T_2 < T_2$ and so $H'(\mu) > 0$. Hence for fixed $p \in (0,2)$, the maximum of $H(\mu)$ occurs at $\mu = 1$.

Also for $p = 2$ we obtain

$$G(\eta,\mu) = \frac{8(1-\beta)^2}{1+\lambda}\left[\frac{(\lambda+2)\,(1-\beta)^2}{3(1+\lambda)^2} + \frac{1}{2(3+\lambda)}\right] \tag{20}$$

Taking into account the value (20), and the cases **i.** and **ii.**, for $0 \le \mu \le 1$ and any fixed p with $0 \le p \le 2$,

$$\max H(\mu) = H(1) = T_1 + T_2 + T_3 + T_4.$$

For $\eta = 1$ and $0 \le \mu \le 1$ (similarly $\mu = 1$ and $0 \le \eta \le 1$), we obtain

$$G(1,\mu) = F(\mu) = (T_3 + T_4)\mu^2 + (T_2 + 2T_4)\mu + T_1 + T_2 + T_3 + T_4.$$

Similarly to the above cases of $T_3 + T_4$, we get that

$$\max F(\mu) = F(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $H(1) \le F(1)$ for $p \in [0,2]$, $\max G(\eta,\mu) = G(1,1)$ on the boundary of the square. Thus the maximum of G occurs at $\eta = 1$ and $\mu = 1$ in the closed square.

Let $K : [0,2] \to R$

$$K(p) = \max G(\eta,\mu) = G(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{21}$$

Substituting the values of T_1, T_2, T_3 and T_4 in the function K defined by (21), yield

$$K(p) = (1-\beta)^2 \Big\{ \Big(\tfrac{(\lambda^2+3\lambda+2)(1-\beta)^2}{6(1+\lambda)^4} - \tfrac{1-\beta}{4(1+\lambda)^2(2+\lambda)} - \tfrac{1}{2(1+\lambda)(3+\lambda)} + \tfrac{1}{4(2+\lambda)^2} \Big) p^4 \\ + \Big(\tfrac{1-\beta}{(1+\lambda)^2(2+\lambda)} - \tfrac{2}{(2+\lambda)^2} + \tfrac{3}{(1+\lambda)(3+\lambda)} \Big) p^2 + \tfrac{4}{(2+\lambda)^2} \Big\}.$$

Assume that $K(p)$ has a maximum value in an interior of $p \in [0,2]$, by elementary calculation

$$K'(p) = (1-\beta)^2 \Big\{ \Big(\tfrac{2(\lambda^2+3\lambda+2)(1-\beta)^2}{3(1+\lambda)^4} - \tfrac{1-\beta}{(1+\lambda)^2(2+\lambda)} - \tfrac{5+4\lambda+\lambda^2}{(1+\lambda)(2+\lambda)^2(3+\lambda)} \Big) p^3 \\ + \Big(\tfrac{2(1-\beta)}{(1+\lambda)^2(2+\lambda)} + \tfrac{2(6+4\lambda+\lambda^2)}{(1+\lambda)(2+\lambda)^2(3+\lambda)} \Big) p \Big\}.$$

As a result of some calculations we can do the following examine:

Case 1. Let $\Big(\tfrac{2(\lambda^2+3\lambda+2)(1-\beta)^2}{3(1+\lambda)^4} - \tfrac{1-\beta}{(1+\lambda)^2(2+\lambda)} - \tfrac{5+4\lambda+\lambda^2}{(1+\lambda)(2+\lambda)^2(3+\lambda)} \Big) \ge 0$. Therefore $\beta \in \Big[0, 1 - \tfrac{3(1+\lambda)(3+\lambda)+(1+\lambda)\sqrt{9(3+\lambda)^2+24(2+\lambda)(3+\lambda)(5+4\lambda+\lambda^2)}}{4(2+\lambda)^2(3+\lambda)}\Big]$ and $K'(p) > 0$ for $p \in (0,2)$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $p \in [0,2]$, that is, $p = 2$. Thus, we have

$$\max K(p) = K(2) = \frac{8(1-\beta)^2}{1+\lambda} \left[\frac{(\lambda+2)(1-\beta)^2}{3(1+\lambda)^2} + \frac{1}{2(3+\lambda)} \right].$$

Case 2. Let $\left(\frac{2(\lambda^2+3\lambda+2)(1-\beta)^2}{3(1+\lambda)^4}-\frac{1-\beta}{(1+\lambda)^2(2+\lambda)}-\frac{5+4\lambda+\lambda^2}{(1+\lambda)(2+\lambda)^2(3+\lambda)}\right)<0$. that is, $\alpha \in \left(1-\frac{3(1+\lambda)(3+\lambda)+(1+\lambda)\sqrt{9(3+\lambda)^2+24(2+\lambda)(3+\lambda)(5+4\lambda+\lambda^2)}}{4(2+\lambda)^2(3+\lambda)},1\right)$. Then $K'(p)=0$ implies the real critical points $p_{01}=0$ or

$$p_{02}=\sqrt{\frac{-6\left[\left(\lambda^2+5\lambda+6\right)(1-\beta)+(1+\lambda)\left(\lambda^2+4\lambda+6\right)\right](1+\lambda)^2}{2(2+\lambda)^2(3+\lambda)(\lambda^2+3\lambda+2)(1-\beta)^2-3(1+\lambda)^2(2+\lambda)(3+\lambda)(1-\beta)-3(1+\lambda)^3(\lambda^2+4\lambda+5)}}.$$

When

$$\alpha \in \left(1-\frac{(1+\lambda)\left[3(3+\lambda)+\sqrt{9(3+\lambda)^2+24(2+\lambda)(3+\lambda)(5+4\lambda+\lambda^2)}\right]}{4(2+\lambda)^2(3+\lambda)},\right.$$

$$\left.1-\frac{(1+\lambda)\left[3(3+\lambda)+\sqrt{9(3+\lambda)^2+48(2+\lambda)^3(3+\lambda)}\right]}{8(2+\lambda)^2(3+\lambda)}\right],$$

we observe that $p_{02}\geq 2$, that is, p_{02} is out of the interval $(0,2)$. Therefore the maximum value of $K(p)$ occurs at $p_{01}=0$ or $p=p_{02}$ which contradicts our assumption of having the maximum value at the interior point of $p\in[0,2]$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $p\in[0,2]$, that is, $p=2$. Thus, we have

$$\max K(p)=K(2)=\frac{8(1-\beta)^2}{1+\lambda}\left[\frac{(\lambda+2)(1-\beta)^2}{3(1+\lambda)^2}+\frac{1}{2(3+\lambda)}\right].$$

When $\alpha \in \left(1-\frac{3(1+\lambda)(3+\lambda)+(1+\lambda)\sqrt{9(3+\lambda)^2+48(2+\lambda)^3(3+\lambda)}}{8(2+\lambda)^2(3+\lambda)},1\right)$ we observe that $p_{02}<2$, that is, p_{02} is interior of the interval $[0,2]$. Since $K''(p_{02})<0$, the maximum value of $K(p)$ occurs at $p=p_{02}$. Thus, we have

$$K(p_{02})=(1-\beta)^2\left\{\frac{4}{(2+\lambda)^2}-\right.$$
$$\left.\frac{3\left[\left(\lambda^2+5\lambda+6\right)(1-\beta)+(1+\lambda)\left(\lambda^2+4\lambda+6\right)\right]^2}{(1+\lambda)(2+\lambda)^2(3+\lambda)[2(2+\lambda)^3(3+\lambda)(1-\beta)^2-3(1+\lambda)(2+\lambda)(3+\lambda)(1-\beta)-3(1+\lambda)^2(\lambda^2+4\lambda+5)]}\right\}.$$

This completes the proof. □

Remark 2.2. Putting $\lambda=0$ in Theorem 2.1 we have the second Hankel determinant for the well-known class $S_\Sigma^0(\beta)=S_\Sigma^*(\beta)$ as in [8].

Remark 2.3. Let f given by (1) be in the class $S_\Sigma^*(\beta)$ and $0\leq\beta<1$. Then

$$\left|a_2a_4-a_3^2\right|\leq\begin{cases}\frac{4(1-\beta)^2}{3}\left(4\beta^2-8\beta+5\right) & \beta\in\left[0,\frac{29-\sqrt{137}}{32}\right)\\ (1-\beta)^2\left(\frac{13\beta^2-14\beta-7}{16\beta^2-26\beta+5}\right) & \beta\in\left(\frac{29-\sqrt{137}}{32},1\right)\end{cases}.$$

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ŞAHSENE ALTINKAYA
Department of Mathematics
Faculty of Arts and Science,
Uludag University, Bursa, Turkey.
e-mail: `sahsene@uludag.edu.tr`

SIBEL YALÇIN
Department of Mathematics
Faculty of Arts and Science,
Uludag University, Bursa, Turkey.
e-mail: `syalcin@uludag.edu.tr`