COEFFICIENT ESTIMATES FOR SOME CERTAIN SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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In this paper, we determine coefficient bounds for functions in certain subclasses of close-to-convex functions, which are introduced here by means of the non-homogeneous Cauchy-Euler equation of order m. Also, we give some corollaries as special cases.

1. Introduction

Let $\mathbb{D}$ be the unit disk $\{z : |z| < 1\}$, $A$ be the class of functions analytic in $\mathbb{D}$, satisfying the conditions

\[ f(0) = 0 \text{ and } f'(0) = 1. \] (1)

Then each function $f$ in $A$ has the Taylor expansion

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \] (2)

because of the conditions (1). Let $S$ denote class of analytic and univalent functions in $\mathbb{D}$ with the normalization conditions (1). Also let $S^*(\gamma)$, $C(\gamma)$, $K(\gamma)$ and $Q(\gamma)$ denote the subclasses of $A$ including of all functions which are starlike, convex, close-to-convex and quasi convex of complex order $\gamma$ ($\gamma \neq 0$) respectively [5], [6] and [8].
**Definition 1.1.** Let \( f(z) \) and \( g(z) \) are analytic functions in \( \mathbb{D} \). We say that \( f(z) \) is subordinate to \( g(z) \) in \( \mathbb{D} \), and we denote

\[
f(z) \prec g(z) \quad (z \in \mathbb{D}),
\]

if there exists a Schwarz function \( w(z) \) analytic in \( \mathbb{D} \), with

\[
w(0) = 0 \text{ and } |w(z)| < 1 \quad (z \in \mathbb{D}),
\]

such that

\[
f(z) = g(w(z)) \quad (z \in \mathbb{D}).
\]

In particular, if the function \( g \) is univalent in \( \mathbb{D} \), the above subordination is equivalent to

\[
f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).
\]

Recently Atıntaş et al. [7] considered following class of functions denoted by \( \text{SC}(\gamma, \lambda, A, B) \) and defined as:

\[
f \in \mathcal{A} \text{ and } 1 + \frac{1}{\gamma} \left( \frac{z[(1 - \lambda)f(z) + \lambda zf'(z)]'}{(1 - \lambda)f(z) + \lambda zf'(z)} - 1 \right) < \frac{1 + Az}{1 + Bz},
\]

where \( 0 \leq \lambda \leq 1, \gamma \in \mathbb{C}\setminus\{0\} \) and \( -1 < B < A < 1 \). Note that

\[
\text{SC}(\gamma, \lambda, 1 - 2\beta, -1) = \text{SC}(\gamma, \lambda, \beta)
\]

which is defined in the article [1]. We can find coefficient estimates for functions in the class \( \text{SC}(\gamma, \lambda, \beta) \) in [2]. Also, \( \text{SC}(\gamma, 0, 1, -1) = S^*(\gamma) \) and \( \text{SC}(\gamma, 1, 1, -1) = C(\gamma) \).

In [7], the following coefficient estimates is obtained as :

**Theorem 1.2.** Let the function \( f(z) \) is given by (2). If \( f(z) \in \text{SC}(\gamma, \lambda, A, B) \), then

\[
|a_n| \leq \frac{\prod_{j=0}^{n-2} \left[ j + \frac{2|\gamma(A-B)|}{1-B} \right]}{(n-1)! \left( 1 + \lambda \frac{n(n-1)}{1} \right)}, \quad n \in \mathbb{N}\setminus\{1\}.
\]

This result is refined by H. M. Srivastava et al. [9]. He state the following theorem.

**Theorem 1.3.** Let the function \( f(z) \) is given by (2). If \( f(z) \in \text{SC}(\gamma, \lambda, A, B) \), then

\[
|a_n| \leq \frac{\prod_{j=0}^{n-2} \left( j + |\gamma| (A-B) \right)}{(n-1)! \left( 1 + \lambda \frac{n(n-1)}{1} \right)}, \quad n \in \mathbb{N}\setminus\{1\}.
\]
Recently, Wasim et al. [10] have obtained upper bounds for Taylor coefficients of functions in the classes $\mathcal{KQ}(\gamma, \lambda, \beta)$ and $\mathcal{BK}(\gamma, \lambda, \beta; \mu)$ of close-to-convex functions of complex order. In fact, the class $\mathcal{KQ}(\gamma, \lambda, \beta)$ introduced first in [3] by the following definition.

**Definition 1.4.** Let $\gamma$ be a non-zero complex number, $0 \leq \beta < 1$ and let $f$ be an univalent function of the form (2). We say that $f$ belongs to $\mathcal{KQ}(\lambda, \gamma, \beta)$ if there exists a function $g(z) \in SC(1, \lambda, \beta)$ such that

$$\Re \left[ 1 + \frac{1}{\gamma} \left( \frac{(1 - \lambda)f(z) + \lambda zf'(z)}{(1 - \lambda)g(z) + \lambda zg'(z)} - 1 \right) \right] > \beta, \; z \in \mathbb{D}. \quad (6)$$

**Definition 1.5.** Let $\gamma$ be a non-zero complex number, $0 \leq \beta < 1$ and let $f$ be an univalent function of the form (2). We say that $f$ belongs to $\mathcal{BK}(\gamma, \lambda, \beta; \mu)$ if it satisfies the following Cauchy-Euler differential equation for $h \in \mathcal{KQ}(\lambda, \gamma, \beta)$

$$z^2 \frac{d^2w}{dz^2} + 2(1 + \mu)z \frac{dw}{dz} + \mu (1 + \mu)w = (2 + \mu)(1 + \mu)h(z),$$

where $w = f(z), \; \mu \in \mathbb{R} \setminus (-\infty, -1]$.

Using Definition 1.4 and Definition 1.5, Wasim et al. [10] obtained the following results.

**Theorem 1.6.** Let $f(z) \in \mathcal{KQ}(\lambda, \gamma, \beta)$ and be defined by (2). Then for $n \in \mathbb{N}^* = \{2, 3, 4, \ldots\}$

$$|a_n| \leq \prod_{j=0}^{k-2} \frac{[j + 2(1 - \beta)]}{n! [1 + \lambda(n - 1)]} + \frac{2|\gamma|(1 - \beta)}{n[1 + \lambda(n - 1)]} \sum_{k=1}^{n-1} \frac{n-k-2}{(n-k-1)!} \prod_{j=0}^{k-2} [j + 2(1 - \beta)]. \quad (7)$$

**Theorem 1.7.** Let $f(z) \in \mathcal{BK}(\lambda, \gamma, \beta; \mu)$ and be defined by (2). Then for $n \in \mathbb{N}^* = \{2, 3, 4, \ldots\}$

$$|a_n| \leq \frac{(1 + \mu)(2 + \mu)}{(n + 1 + \mu)(n + \mu)} \left\{ \prod_{j=0}^{k-2} \frac{[j + 2(1 - \beta)]}{n! [1 + \lambda(n - 1)]} + \frac{2|\gamma|(1 - \beta)}{n[1 + \lambda(n - 1)]} \sum_{k=1}^{n-1} \frac{n-k-2}{(n-k-1)!} \prod_{j=0}^{k-2} [j + 2(1 - \beta)] \right\}. \quad (8)$$
Motivated from results in [10], we define some certain subclasses of close-to-convex functions which are given by the following definitions. In here, the class which is given by Definition 1.9 introduced first in [4]. Also, we get upper bounds for Taylor-Maclaurin coefficients of functions in these classes.

**Definition 1.8.** Let $\gamma$ be a non-zero complex number, $-1 \leq B < A \leq 1$ and let $f$ be an univalent function of the form (2). We say that $f$ belongs to $K_{\lambda}(\gamma, \lambda, A, B)$ if there exists a function $g(z) \in SC(1, \lambda, A, B)$ such that

$$1 + \frac{1}{\gamma} \left( \frac{z[(1-\lambda)f(z) + \lambda zf'(z)]'}{(1-\lambda)g(z) + \lambda zg'(z)} - 1 \right) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{D}. \quad (9)$$

**Definition 1.9.** Let $\gamma$ be a non-zero complex number, $-1 \leq B < A \leq 1$ and let $f$ be an univalent function of the form (2). We say that $f$ belongs to $BK_{\lambda}(\gamma, \lambda, A, B, m; \mu)$ if it satisfies the following Cauchy-Euler differential equation for $h \in K_{\lambda}(\gamma, \lambda, A, B)$

$$z^m \frac{d^m w}{dz^m} + \binom{m}{1}(\mu + m - 1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots + \binom{m}{m}w \prod_{j=0}^{m-1}(\mu + j) = h(z) \prod_{j=0}^{m-1}(\mu + j + 1), \quad (10)$$

where $w = f(z), \mu \in \mathbb{R} \setminus (-\infty, -1]$ and $m \in \mathbb{N} \setminus \{1\}$.

Note that

$$K_{\lambda}(\gamma, \lambda, 1 - 2\beta, -1) = K_{\lambda}(\gamma, \lambda, \beta),$$

$$BK_{\lambda}(\gamma, \lambda, 1 - 2\beta, -1, 2; \mu) = BK_{\lambda}(\gamma, \lambda, \beta; \mu),$$

$$K_{\lambda}(\gamma, 0, 1, -1) = K(\gamma)$$

and

$$K_{\lambda}(\gamma, 1, 1, -1) = Q(\gamma).$$

2. **Coefficient estimates for functions in the classes $K_{\lambda}(\gamma, \lambda, A, B)$ and $BK_{\lambda}(\gamma, \lambda, A, B, m; \mu)$**

We need the following lemma before getting our main results.

**Lemma 2.1.** Let the function $g$ given by

$$g(z) = \sum_{k=1}^{\infty} b_kz^k, \quad z \in \mathbb{D}$$
be convex in $\mathbb{D}$. Also, let the function $f$ given by
\[ f(z) = \sum_{k=1}^{\infty} a_k z^k, \quad z \in \mathbb{D} \]
be holomorphic in $\mathbb{D}$. If
\[ f(z) \prec g(z), \quad z \in \mathbb{D} \]
then
\[ |a_k| \leq |b_1|, \quad k \in \mathbb{N}. \]

Now, we prove our coefficient estimates for functions which belong to the classes $KQ(\lambda, \gamma, A, B)$ and $BK(\gamma, \lambda, A, B, m; \mu)$.

**Theorem 2.2.** Let $f(z) \in KQ(\lambda, \gamma, A, B)$ and be defined by (2). Then for $n \in \mathbb{N}^* = \{2, 3, 4, \ldots\}$
\[ |a_n| \leq \frac{\prod_{j=0}^{k-2} [j + (A - B)]}{n! [1 + \lambda (n - 1)]} + \frac{|\gamma| (A - B)}{n [1 + \lambda (n - 1)]} \sum_{k=1}^{n-1} \frac{\prod_{j=0}^{n-k-2} [j + (A - B)]}{(n-k-1)!}. \quad (11) \]

**Proof.** Since $f(z) \in KQ(\lambda, \gamma, A, B)$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ which belongs to class $SC(1, \lambda, A, B)$ such that
\[ 1 + \frac{1}{\gamma} \left( \frac{z F'(z)}{G(z)} - 1 \right) \prec 1 + \frac{A z}{1 + B z} \quad (12) \]
for $z \in \mathbb{D}$, where $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ and $G(z) = z + \sum_{n=2}^{\infty} B_n z^n$, with
\[ A_n = [1 + \lambda (n - 1)] a_n \quad \text{and} \quad B_n = [1 + \lambda (n - 1)] b_n, \quad n \geq 2. \quad (13) \]

Let us $h(z) = \frac{1 + Az}{1 + Bz}$ and define the function
\[ p(z) = 1 + \frac{1}{\gamma} \left( \frac{z F'(z)}{G(z)} - 1 \right) = 1 + \sum_{n=2}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \quad (14) \]

Therefore we have $p(z) \prec h(z), \quad z \in \mathbb{D}$. Hence, by the definition of the subordination, we deduce that
\[ p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \]

Note that $p(0) = h(0) = 1$ and $p(\mathbb{D}) \subset h(\mathbb{D}), \quad z \in \mathbb{D}$. Also from (14), we get
\[ \gamma G(z) p(z) = G(z) (\gamma - 1) + z F'(z). \quad (15) \]
Then, from (15), we obtain
\[ nA_n = B_n + \gamma \left\{ c_{n-1} + \sum_{k=1}^{n-2} c_k B_{n-k} \right\}. \]  
(16)

On the other hand, according to Lemma 2.1, we get
\[ \left| \frac{p^{(m)}(0)}{m!} \right| \leq A - B, \quad m \in \mathbb{N}. \]  
(17)

Using (17) in (16), we take the following inequality
\[ n |A_n| \leq |B_n| + \gamma (A - B) \left( 1 + \sum_{k=1}^{n-2} |B_{n-k}| \right), \quad n \geq 2. \]

Now using Theorem 1.3, we obtain
\[ |A_n| \leq \frac{\prod_{j=0}^{k-2} [j + (A - B)]}{n!} + \frac{\gamma |(A - B)| \prod_{j=0}^{n-k-2} [j + (A - B)]}{n \sum_{k=1}^{n-1} \prod_{j=0}^{n-k-2} \frac{[j + (A - B)]}{(n-k-1)!}} \]
and hence from the relation between \( F(z) \) and \( f(z) \) as in (13), we obtain the desired result. \( \square \)

**Theorem 2.3.** Let \( f(z) \in \mathcal{BK}(\lambda, \gamma, A, B, m; \mu) \) and be defined by (2). Then for \( n \in \mathbb{N}^* = \{2, 3, 4, \ldots\} \)
\[ |a_n| \leq \frac{\prod_{j=0}^{m-1} (u + j + 1)}{\prod_{j=0}^{m-1} (u + j + n)} \left\{ \frac{\prod_{j=0}^{k-2} [j + (A - B)]}{n! [1 + \lambda (n-1)]} \right\} \]
\[ + \frac{|\gamma| (A - B)}{n [1 + \lambda (n-1)]} \sum_{k=1}^{n-1} \frac{\prod_{j=0}^{n-k-2} [j + (A - B)]}{(n-k-1)!} \]. \( \) \( (18) \)

**Proof.** Since \( f(z) \in \mathcal{BK}(\lambda, \gamma, A, B, m; \mu) \), then there exists \( h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{KQ}(\lambda, \gamma, A, B) \), such that (10) holds true. Thus it follows that
\[ a_n = \frac{\prod_{j=0}^{m-1} (\mu + j + 1)}{\prod_{j=0}^{m-1} (\mu + j + n)} b_n, \quad n \in \mathbb{N}^*, \quad \mu \in \mathbb{R} \setminus (-\infty, -1]. \]

By using Theorem 2.2, we immediately obtain the desired inequality (18). \( \square \)
3. Corollaries and Consequences

By choosing suitable values of the admissible parameters $m, \lambda, \gamma, A$ and $B$ in Theorem 2.2 and Theorem 2.3 above, we deduce the following corollaries.

**Corollary 3.1 ([10]).** Let the function $f(z) \in A$ be given by (2). If $f \in KQ(\gamma,\lambda,1-2\beta,-1) = KQ(\gamma,\lambda,\beta)$, then we get the same result in Theorem 1.6.

**Corollary 3.2 ([10]).** Let the function $f(z) \in A$ be given by (2). If $f \in BK(\lambda,\gamma,1-2\beta,-1,2;\mu) = BK(\lambda,\gamma,\beta)$, then we have the result in Theorem 1.7.

**Corollary 3.3 ([5]).** Let the function $f(z) \in A$ be given by (2). If $f \in KQ(\gamma,0,1,-1) = K(\gamma)$, then

$$|a_n| \leq \frac{1}{\left[1 + \lambda (n-1)\right]} \left\{1 + (n-1) |\gamma|\right\}, n \in \mathbb{N}^*.$$  

**Corollary 3.4 ([6]).** Let the function $f(z) \in A$ be given by (2). If $f \in KQ(\gamma,1,1,-1) = Q(\gamma)$, then

$$|a_n| \leq \frac{1 + (n-1) |\gamma|}{n}, n \in \mathbb{N}^*.$$  

For $\gamma = 1$ in Corollary 3.3 and Corollary 3.4, we obtain the well known coefficient estimates for close-to-convex and quasi convex functions.

REFERENCES


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