LE MATEMATICHE
Vol. LXXI (2016) – Fasc. I, pp. 127–134
doi: 10.4418/2016.71.1.10

COEFFICIENT ESTIMATES FOR SOME CERTAIN SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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In this paper, we determine coefficient bounds for functions in certain suclasses of close-to-convex functions, which are introduced here by means of the non-homogeneous Cauchy-Euler equation of order m. Also, we give some corollaries as special cases.

1. Introduction

Let $\mathbb{D}$ be the unit disk $\{z : |z| < 1\}$, $\mathcal{A}$ be the class of functions analytic in $\mathbb{D}$, satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1. \tag{1}$$

Then each function f in $\mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

because of the conditions (1). Let S denote class of analytic and univalent functions in $\mathbb{D}$ with the normalization conditions (1). Also let $S^*(\gamma)$, $C(\gamma)$, $\mathcal{K}(\gamma)$ and $Q(\gamma)$ denote the subclasses of $\mathcal{A}$ including of all functions which are starlike, convex, close-to-convex and quasi convex of complex order γ $(\gamma \neq 0)$ respectively [5], [6] and [8].

Entrato in redazione: 25 marzo 2015
AMS 2010 Subject Classification: 30C45, 33C45.
Keywords: Analytic Functions, Close-to-Convex Functions, Coefficient Estimates, Non Homogeneous Cauchy-Euler Differential Equation.

Definition 1.1. Let $f(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$. We say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{D}$, and we denote

$$f(z) \prec g(z) \ \ (z \in \mathbb{D}),$$

if there exists a Schwarz function $w(z)$ analytic in $\mathbb{D}$, with

$$w(0) = 0 \text{ and } |w(z)| < 1 \ \ (z \in \mathbb{D}),$$

such that

$$f(z) = g(w(z)) \ \ (z \in \mathbb{D}).$$

In particular, if the function g is univalent in $\mathbb{D}$, the above subordination is equivalent to

$$f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

Recently Atıntaş et al. [7] considered following class of functions denoted by $SC(\gamma,\lambda,A,B)$ and defined as:

$$f \in \mathcal{A} \text{ and } 1 + \frac{1}{\gamma}\left(\frac{z[(1-\lambda)f(z)+\lambda zf'(z)]'}{(1-\lambda)f(z)+\lambda zf'(z)} - 1\right) \prec \frac{1+Az}{1+Bz}, \tag{3}$$

where $0 \le \lambda \le 1$, $\gamma \in \mathbb{C}\setminus\{0\}$ and $-1 \le B < A \le 1$. Note that

$$SC(\gamma,\lambda,1-2\beta,-1) = SC(\gamma,\lambda,\beta)$$

which is defined in the article [1]. We can find coefficient estimates for functions in the class $SC(\gamma,\lambda,\beta)$ in [2]. Also, $SC(\gamma,0,1,-1) = S^*(\gamma)$ and $SC(\gamma,1,1,-1) = C(\gamma)$.

In [7], the following coefficient estimates is obtained as :

Theorem 1.2. *Let the function $f(z)$ is given by (2). If $f(z) \in SC(\gamma,\lambda,A,B)$, then*

$$|a_n| \le \frac{\prod_{j=0}^{n-2}\left[j + \frac{2|\gamma|(A-B)}{1-B}\right]}{(n-1)!\,(1+\lambda(n-1))}, \ n \in \mathbb{N}\setminus\{1\}. \tag{4}$$

This result is refined by H. M. Srivastava et al. [9]. He state the following theorem.

Theorem 1.3. *Let the function $f(z)$ is given by (2). If $f(z) \in SC(\gamma,\lambda,A,B)$, then*

$$|a_n| \le \frac{\prod_{j=0}^{n-2}[j + |\gamma|(A-B)]}{(n-1)!\,(1+\lambda(n-1))}, \ n \in \mathbb{N}\setminus\{1\}. \tag{5}$$

Recently, Wasim et al. [10] have obtained upper bounds for Taylor coefficients of functions in the classes $\mathcal{KQ}(\gamma,\lambda,\beta)$ and $\mathcal{BK}(\gamma,\lambda,\beta;\mu)$ of close-to-convex functions of complex order. In fact, the class $\mathcal{KQ}(\gamma,\lambda,\beta)$ introduced first in [3] by the following definition.

Definition 1.4. Let γ be a non-zero complex number, $0 \leq \beta < 1$ and let f be an univalent function of the form (2). We say that f belongs to $\mathcal{KQ}(\lambda,\gamma,\beta)$ if there exists a function $g(z) \in SC(1,\lambda,\beta)$ such that

$$\Re\left[1+\frac{1}{\gamma}\left(\frac{z[(1-\lambda)f(z)+\lambda zf'(z)]'}{(1-\lambda)g(z)+\lambda zg'(z)}-1\right)\right] > \beta,\ z\in\mathbb{D}. \tag{6}$$

Definition 1.5. Let γ be a non-zero complex number, $0 \leq \beta < 1$ and let f be an univalent function of the form (2). We say that f beolngs to $\mathcal{BK}(\gamma,\lambda,\beta;\mu)$ if it satisfies the following Cauchy-Euler differential equation for $h \in \mathcal{KQ}(\lambda,\gamma,\beta)$

$$z^2\frac{d^2w}{dz^2}+2(1+\mu)z\frac{dw}{dz}+\mu(1+\mu)w=(2+\mu)(1+\mu)h(z),$$

where $w = f(z)$, $\mu \in \mathbb{R}\backslash(-\infty,-1]$.

Using Definition 1.4 and Definition 1.5, Wasim et al. [10] obtained the following results.

Theorem 1.6. *Let $f(z) \in \mathcal{KQ}(\lambda,\gamma,\beta)$ and be defined by (2). Then for $n \in \mathbb{N}^* = \{2,3,4,\ldots\}$*

$$|a_n| \leq \frac{\prod_{j=0}^{k-2}[j+2(1-\beta)]}{n![1+\lambda(n-1)]}+\frac{2|\gamma|(1-\beta)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}[j+2(1-\beta)]}{(n-k-1)!}. \tag{7}$$

Theorem 1.7. *Let $f(z) \in \mathcal{BK}(\lambda,\gamma,\beta;\mu)$ and be defined by (2). Then for $n \in \mathbb{N}^* = \{2,3,4,\ldots\}$*

$$|a_n| \leq \frac{(1+\mu)(2+\mu)}{(n+1+\mu)(n+\mu)}\left\{\frac{\prod_{j=0}^{k-2}[j+2(1-\beta)]}{n![1+\lambda(n-1)]}\right.$$
$$\left.+\frac{2|\gamma|(1-\beta)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}[j+2(1-\beta)]}{(n-k-1)!}\right\} \tag{8}$$

Motivated from results in [10], we define some certain subclasses of close-to-convex functions which are given by the following definitions. In here, the class which is given by Definition 1.9 introduced first in [4]. Also, we get upper bounds for Taylor-Maclaurin coefficients of functions in these classes.

Definition 1.8. Let γ be a non-zero complex number, $-1 \leq B < A \leq 1$ and let f be an univalent function of the form (2). We say that f belongs to $\mathcal{KQ}(\gamma,\lambda,A,B)$ if there exists a function $g(z) \in SC(1,\lambda,A,B)$ such that

$$1+\frac{1}{\gamma}\left(\frac{z\left[(1-\lambda)f(z)+\lambda zf'(z)\right]'}{(1-\lambda)g(z)+\lambda zg'(z)}-1\right) \prec \frac{1+Az}{1+Bz},\ z\in\mathbb{D}. \tag{9}$$

Definition 1.9. Let γ be a non-zero complex number, $-1 \leq B < A \leq 1$ and let f be an univalent function of the form (2). We say that f beolngs to $\mathcal{BK}(\gamma,\lambda,A,B,m;\mu)$ if it satisfies the following Cauchy-Euler differential equation for $h \in \mathcal{KQ}(\lambda,\gamma,A,B)$

$$z^m\frac{d^mw}{dz^m}+\binom{m}{1}(\mu+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}}+\cdots+\binom{m}{m}w\prod_{j=0}^{m-1}(\mu+j) = h(z)\prod_{j=0}^{m-1}(\mu+j+1), \tag{10}$$

where $w=f(z)$, $\mu\in\mathbb{R}\backslash(-\infty,-1]$ and $m\in\mathbb{N}\backslash\{1\}$.

Note that

$$\mathcal{KQ}(\gamma,\lambda,1-2\beta,-1)=\mathcal{KQ}(\gamma,\lambda,\beta),$$
$$\mathcal{BK}(\gamma,\lambda,1-2\beta,-1,2;\mu)=\mathcal{BK}(\lambda,\gamma,\beta;\mu),$$
$$\mathcal{KQ}(\gamma,0,1,-1)=\mathcal{K}(\gamma)$$

and

$$\mathcal{KQ}(\gamma,1,1,-1)=\mathcal{Q}(\gamma).$$

2. Coefficient estimates for functions in the classes $\mathcal{KQ}(\lambda,\gamma,A,B)$ and $\mathcal{BK}(\gamma,\lambda,A,B,m;\mu)$

We need the following lemma before getting our main results.

Lemma 2.1. *Let the function g given by*

$$g(z)=\sum_{k=1}^{\infty}b_kz^k,\ z\in\mathbb{D}$$

be convex in $\mathbb{D}$. *Also, let the function* f *given by*

$$f(z) = \sum_{k=1}^{\infty} a_k z^k,\ z \in \mathbb{D}$$

be holomorphic in $\mathbb{D}$. *If*

$$f(z) \prec g(z),\ z \in \mathbb{D}$$

then

$$|a_k| \le |b_1|,\ k \in \mathbb{N}.$$

Now, we prove our coefficient estimates for functions which belong to the classes $\mathcal{KQ}(\lambda,\gamma,A,B)$ and $\mathcal{BK}(\gamma,\lambda,A,B,m;\mu)$.

Theorem 2.2. *Let* $f(z) \in \mathcal{KQ}(\lambda,\gamma,A,B)$ *and be defined by* (2). *Then for* $n \in \mathbb{N}^* = \{2,3,4,\ldots\}$

$$|a_n| \le \frac{\prod\limits_{j=0}^{k-2}[j+(A-B)]}{n!\,[1+\lambda(n-1)]} + \frac{|\gamma|(A-B)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod\limits_{j=0}^{n-k-2}[j+(A-B)]}{(n-k-1)!}. \tag{11}$$

Proof. Since $f(z) \in \mathcal{KQ}(\lambda,\gamma,A,B)$, $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ which belongs to class $SC(1,\lambda,A,B)$ such that

$$1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{G(z)} - 1\right) \prec \frac{1+Az}{1+Bz} \tag{12}$$

for $z \in \mathbb{D}$, where $F(z) = z + \sum\limits_{n=2}^{\infty} A_n z^n$ and $G(z) = z + \sum\limits_{n=2}^{\infty} B_n z^n$, with

$$A_n = [1+\lambda(n-1)]\,a_n \text{ and } B_n = [1+\lambda(n-1)]\,b_n,\ n \ge 2. \tag{13}$$

Let us $h(z) = \frac{1+Az}{1+Bz}$ and define the function

$$p(z) = 1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{G(z)} - 1\right) = 1 + \sum_{n=2}^{\infty} c_n z^n,\ z \in \mathbb{D}. \tag{14}$$

Therefore we have $p(z) \prec h(z)$, $z \in \mathbb{D}$. Hence, by the definition of the subordination, we deduce that

$$p(z) = \frac{1+Aw(z)}{1+Bw(z)}.$$

Note that $p(0) = h(0) = 1$ and $p(\mathbb{D}) \subset h(\mathbb{D})$, $z \in \mathbb{D}$. Also from (14), we get

$$\gamma G(z)p(z) = G(z)(\gamma - 1) + zF'(z). \tag{15}$$

Then, from (15), we obtain

$$nA_n = B_n + \gamma \left\{ c_{n-1} + \sum_{k=1}^{n-2} c_k B_{n-k} \right\}. \tag{16}$$

On the other hand, according to Lemma 2.1, we get

$$\left| \frac{p^{(m)}(0)}{m!} \right| \leq A - B,\ m \in \mathbb{N}. \tag{17}$$

Using (17) in (16), we take the following inequality

$$n\left|A_n\right| \leq \left|B_n\right| + \gamma (A-B) \left\{ 1 + \sum_{k=1}^{n-2} \left|B_{n-k}\right| \right\},\ n \geq 2.$$

Now using Theorem 1.3, we obtain

$$\left|A_n\right| \leq \frac{\prod_{j=0}^{k-2} [j + (A-B)]}{n!} + \frac{|\gamma| (A-B)}{n} \sum_{k=1}^{n-1} \frac{\prod_{j=0}^{n-k-2} [j + (A-B)]}{(n-k-1)!}$$

and hence from the relation between $F(z)$ and $f(z)$ as in (13), we obtain the desired result. $\square$

Theorem 2.3. *Let* $f(z) \in \mathcal{BK}(\lambda, \gamma, A, B, m; \mu)$ *and be defined by* (2). *Then for* $n \in \mathbb{N}^* = \{2, 3, 4, \ldots\}$

$$\left|a_n\right| \leq \frac{\prod_{j=0}^{m-1} (u + j + 1)}{\prod_{j=0}^{m-1} (u + j + n)} \left\{ \frac{\prod_{j=0}^{k-2} [j + (A-B)]}{n!\left[1 + \lambda (n-1)\right]} + \frac{|\gamma| (A-B)}{n\left[1 + \lambda (n-1)\right]} \sum_{k=1}^{n-1} \frac{\prod_{j=0}^{n-k-2} [j + (A-B)]}{(n-k-1)!} \right\}. \tag{18}$$

Proof. Since $f(z) \in \mathcal{BK}(\lambda, \gamma, A, B, m; \mu)$, then there exists $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{KQ}(\lambda, \gamma, A, B)$, such that (10) holds true. Thus it follows that

$$a_n = \frac{\prod_{j=0}^{m-1} (\mu + j + 1)}{\prod_{j=0}^{m-1} (\mu + j + n)} b_n,\ n \in \mathbb{N}^*,\ \mu \in \mathbb{R} \backslash (-\infty, -1].$$

By using Theorem 2.2, we immediately obtain the desired inequality (18). $\square$

3. Corollaries and Consequences

By choosing suitable values of the admissible parameters m, λ, γ, A and B in Theorem 2.2 and Theorem 2.3 above, we deduce the following corollaries.

Corollary 3.1 ([10]). *Let the function $f(z) \in \mathcal{A}$ be given by (2). If $f \in \mathcal{KQ}(\gamma,\lambda,1-2\beta,-1) = \mathcal{KQ}(\gamma,\lambda,\beta)$, then we get the same result in Theorem 1.6.*

Corollary 3.2 ([10]). *Let the function $f(z) \in \mathcal{A}$ be given by (2). If $f \in \mathcal{BK}(\lambda,\gamma,1-2\beta,-1,2;\mu) = \mathcal{BK}(\lambda,\gamma,\beta)$, then we have the result in Theorem 1.7.*

Corollary 3.3 ([5]). *Let the function $f(z) \in \mathcal{A}$ be given by (2). If $f \in \mathcal{KQ}(\gamma,0,1,-1) = \mathcal{K}(\gamma)$, then*

$$|a_n| \leq \frac{1}{[1+\lambda(n-1)]}\{1+(n-1)|\gamma|\},\ n \in \mathbb{N}^*.$$

Corollary 3.4 ([6]). *Let the function $f(z) \in \mathcal{A}$ be given by (2). If $f \in \mathcal{KQ}(\gamma,1,1,-1) = \mathcal{Q}(\gamma)$, then*

$$|a_n| \leq \frac{1+(n-1)|\gamma|}{n},\ n \in \mathbb{N}^*.$$

For $\gamma = 1$ in Corollary 3.3 and Corollary 3.4, we obtain the well known coefficient estimates for close-to-convex and quasi convex functions.

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