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REDUCED SECOND ZAGREB INDEX OF BICYCLIC GRAPHS WITH PENDENT VERTICES

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Reduced second Zagreb index has been defined recently. In this paper we characterized extremal bicyclic graphs with pendent vertices with respect to this novel index.

1. Introduction

Let G be a simple connected graph with n vertices and m edges. d_v is the number of edges incident to the vertex v. A vertex of degree one is said to be a pendent vertex. Unicyclic graphs are connected graphs with n vertices and n edges. Bicyclic graphs are connected graphs with n vertices and $n+1$ edges.We write Δ and δ for the largest and the smallest of all degrees of vertices of G, respectively. The first Zagreb and the second Zagreb index of the graph G are defined as:

$$M_1 = M_1(G) = \sum_{v \in V(G)} d_v^2 \tag{1}$$

and

$$M_2 = M_2(G) = \sum_{uv \in E(G)} d_u d_v \tag{2}$$

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respectively. In formula (2), uv denotes an edge connecting the vertices u and v. In 1972, the quantities M_1 and M_2 were found to occur within certain approximate expressions for the total π-electron energy [19]. The first Zagreb index satisfies the identity

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) \tag{3}$$

where the notation is same as in equation(2) [20]. In view of the extensive research on the two Zagreb indices, in particular, their difference $M_2 - M_1$, the difference between the equations (2) and (3), has been never examined. In [17], Furtula *et al.* examined $M_2 - M_1$ and proposed a new degree based topological index, named it 'Reduced second Zagreb index' and characterized the maximum trees with respect to reduced second Zagreb index. Reduced second Zagreb index is defined [17] as follows;

$$RM_2 = RM_2(G) = \sum_{uv \in E(G)} (d_u - 1)(d_v - 1) = M_2(G) - M_1(G) + m \tag{4}$$

where m denotes the number of edges. Zagreb indices of bicyclic graphs are investigated in [2–4, 9]. For other topological indices of bicyclic graphs see in [1, 5–8, 10–16]. RM_2 index of unicyclic graphs were investigated in [18]. In this paper we investigate maximum and minimum bicyclic graphs with respect to RM_2 index.

2. Minimum and maximum RM_2 index of bicyclic graphs

Let DC denote all bicyclic graphs with n vertex , $n+1$ edges and k pendent vertices (here, DC stands for double cycle). The arrangement of cycles of DC has at most three possible cases.

Case 1: $DC_{a,b}(k_1, k_2, \ldots, k_a, s_2, s_3, \ldots, s_b)$ is the set of $G \in DC$ in which the cycles C_a and C_b have only one common vertex. Here,

$$k_1, k_2, \ldots, k_a, s_2, s_3, \ldots, s_b$$

denote the number of pendent vertices of corresponding

$$v_1, v_2, \ldots, v_a, v_2', v_3', \ldots, v_b'$$

vertices. See Figure 1.

Case 2: $DC^l_{a,b}(k_1, k_2, \ldots, k_a, r_1, r_2, \ldots, r_l, s_1, s_2, \ldots, s_b)$ is the set of $G \in DC$ in which the cycles C_a and C_b have no common vertex for $l \geq 0$. Here

$$k_1, k_2, \ldots, k_a, r_1, r_2, \ldots, r_l, s_1, s_2, \ldots, s_b$$

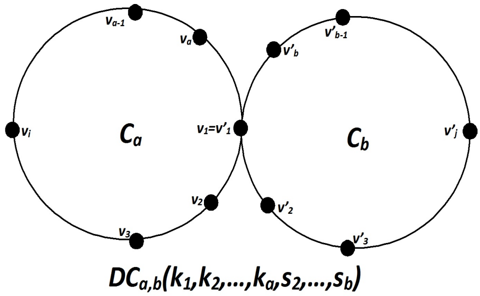

Figure 1: The first class of bicyclic graphs with k pendent vertices: $DC_{a,b}\left(k_1,k_2,\ldots,k_a,s_2,s_3,\ldots,s_b\right)$

denote the number of pendent vertices of corresponding

$$v_1,v_2,\ldots,v_a,n_1,n_2,\ldots,n_l,u_1,u_2,\ldots,u_b$$

vertices. See Figure 2.

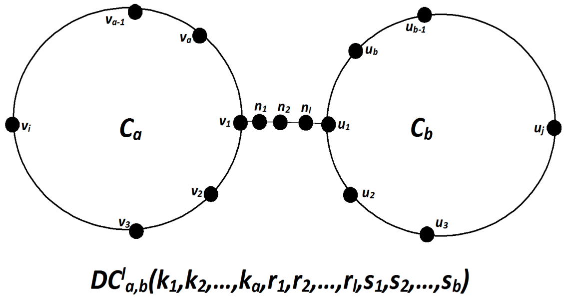

Figure 2: The second class of bicyclic graphs with k pendent vertices: $DC^l_{a,b}\left(k_1,k_2,\ldots,k_a,r_1,r_2,\ldots,r_l,s_1,s_2,\ldots,s_b\right)$

Case 3: $DC^l_{a+b}\left(k_1,k_2,\ldots,k_{a-l},r_1,r_2,\ldots,r_l,s_2,\ldots,s_{b-l-1}\right)$ is the set of $G\in DC$ in which the cycles C_a and C_b have a common path of length $l+1$ for $l\geq 0$. Here

$$k_1,k_2,\ldots,k_{a-l},r_1,r_2,\ldots,r_l,s_2,\ldots,s_{b-l-1}$$

denote the number of pendent vertices of corresponding

$$v_1,v_2,\ldots,v_{a-l},n_1,n_2,\ldots,n_l,{v_2}',{v_3}',\ldots,v'_{b-l-1}$$

vertices. See Figure 3.

With direct calculations, we get the following propositions.

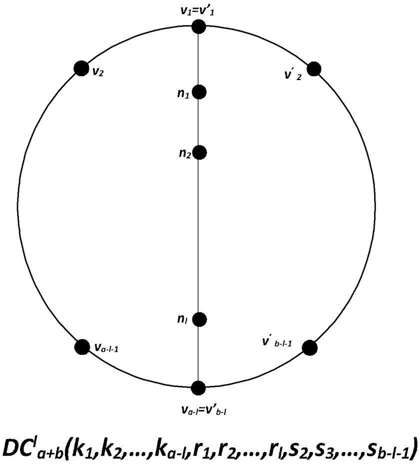

Figure 3: The third class of bicyclic graphs with k pendent vertices: $DC^l_{a+b}(k_1,k_2,\dots,k_a,r_1,r_2,\dots,r_l,s_2,\dots,s_{b-l-1})$

Proposition 2.1. *Let $DC_{a,b}(k_1,k_2,\dots,k_a,s_2,s_3,\dots,s_b)$ be the set of $G\in DC$. Then*

$$RM_2(G)=(k_1+3)(k_2+k_a+s_2+s_b+4)+(k_2+1)(k_3+1)+\dots$$
$$+(k_{a-1}+1)(k_a+1)+(s_2+1)(s_3+1)+\dots+(s_{b-1}+1)(s_b+1).$$

Proposition 2.2. *Let $DC^l_{a,b}(k_1,k_2,\dots,k_a,r_1,r_2,\dots,r_l,s_1,s_2,\dots,s_b)$ be the set of $G\in DC$ then*

$$(a)\qquad RM_2(G)=(k_a+k_2+2)+(k_1+2)(s_1+2)+(k_2+1)(k_3+1)+\dots$$
$$+(k_{a-1}+1)(k_a+1)+(s_2+1)(s_3+1)+\dots+(s_{b-1}+1)(s_b+1)$$

for $l=0$.

$$(b)\qquad RM_2(G)=(k_1+2)(k_2+k_a+r_1+3)+(k_2+1)(k_3+1)+\dots$$
$$+(k_{a-1}+1)(k_a+1)+(r_1+1)(r_2+1)+\dots+(r_{l-1}+1)(r_l+1)$$
$$+(s_1+2)(s_2+s_b+r_l+3)+(s_2+1)(s_3+1)+\dots+(s_{b-1}+1)(s_b+1)$$

for $l\geq 1$.

Proposition 2.3. *Let $DC^l_{a+b}(k_1,k_2,\dots,k_{a-l},r_1,r_2,\dots,r_l,s_2,\dots,s_{b-l-1})$ be the* *set of $G \in DC$ then*

$$
\begin{aligned}
(a)\quad RM_2(G) = {} & (k_1+2)(k_2+s_2+r_1+3) \\
& + (k_{a-l}+2)(k_{a-l-1}+s_{b-l-1}+r_l+3) \\
& + (k_2+1)(k_3+1)+\cdots+(k_{a-l-2}+1)(k_{a-l-1}+1)+ \\
& (s_2+1)(s_3+1)+\cdots+(s_{b-l-2}+1)(s_{b-l-1}+1) \\
& + (r_1+1)(r_2+1)+\ldots+(r_{l-1}+1)(r_l+1)
\end{aligned}
$$

for $1 \le l \le n-4$.

$$
\begin{aligned}
(b)\quad RM_2(G) = {} & (k_1+2)(k_2+s_2+2)+(k_2+1)(k_3+1)+\ldots \\
& + (k_{a-2}+1)(k_{a-1}+1)+(k_a+2)(k_{a-1}+s_{b-1}+2) \\
& + (s_2+1)(s_3+1)+\cdots+(s_{b-2}+1)(s_{b-1}+1)+(k_1+2)(k_a+2)
\end{aligned}
$$

for $l=0$.

Lemma 2.4. *Let $G_1 \in DC_{a,b}$ with no pendent vertex and n vertices. Let $G_2 \in DC_{a,b}$ with $k \ge 1$ pendent vertices and n vertices. Then $RM_2(G_1) < RM_2(G_2)$.*

Proof. Let $k=1$. Let uvl be a path of G_1 where all degrees are 2. Then, we obtain G_2 from G_1 by taking u attached to v as a pendent vertex. In this case $RM_2(G_2)-RM_2(G_1)=1>0$. On the other hand, let uvl be a path of G_1 and $v=v_1$ so that $d_v=4$. Then, we obtain G_2 from G_1 by taking u attached to v as a pendent vertex. In this case $RM_2(G_2)-RM_2(G_1)=6$. The other cases for $k \ge 2$ are similar. □

Now, we give the following lemmas whose proofs are similar to that of Lemma 2.4.

Lemma 2.5. *Let $G_1 \in DC^l_{a,b}$ with no pendent vertex and n vertices. Let $G_2 \in DC^l_{a,b}$ with $k \ge 1$ pendent vertices and n vertices. Then $RM_2(G_1) < RM_2(G_2)$.*

Lemma 2.6. *Let $G_1 \in DC^l_{a+b}$ with no pendent vertex and n vertices. Let $G_2 \in DC^l_{a+b}$ with $k \ge 1$ pendent vertices and n vertices. Then $RM_2(G_1) < RM_2(G_2)$.*

Corollary 2.7. *(a) Let $G \in DC_{a,b}(k_1,k_2,\ldots,k_a,s_2,s_3,\ldots,s_b)$. Then the minimum RM_2 index of G is $DC_{a,b}(0,0,\ldots,0)$.*
(b) Let $G \in DC^l_{a,b}(k_1,k_2,\ldots,k_a,r_1,r_2,\ldots,r_l,s_1,s_2,\ldots,s_b)$. Then the minimum RM_2 index of G is $DC^l_{a,b}(0,0,\ldots,0)$.
(c) Let $G \in DC^l_{a+b}(k_1,k_2,\ldots,k_{a-l},r_1,r_2,\ldots,r_l,s_1,s_2,\ldots,s_{b-l-1})$. Then the minimum RM_2 index of G is $DC^l_{a+b}(0,0,\ldots,0)$.
Notice that in all three cases G has no pendent vertices.

Proposition 2.8. *Let $G \in DC_{a,b}(0,0,\ldots,0)$. Then the minimum RM_2 index is $RM_2(G) = n+9$.*

Proof. From Proposition 2.1 and Corollary 2.7, $RM_2(G) = a+b+8$. Since $a+b = n+1$, the desired result is acquired. □

Proposition 2.9. *Let $G \in DC^{l}_{a,b}(0,0,\ldots,0)$. Then the minimum RM_2 index is $RM_2(DC^{l}_{a,b}) = n+7$ for $l \geq 1$.*

Proof. From Proposition 2.2b and Corollary 2.7, $RM_2(G) = a+b+l+7$. Since $n = a+b+l$, the desired result is acquired. □

Proposition 2.10. *Let $G \in DC^{l}_{a+b}(0,0,\ldots,0)$. Then the minimum RM_2 index is $RM_2(DC^{l}_{a+b}) = n+7$ for $l \geq 1$.*

Proof. From Proposition 2.3a and Corollary 2.7, $RM_2(G) = a+b-l+5$. Since $a+b-l = n+2$, the desired result is acquired. □

Definition 2.11. Let Ξ be a family of the set $DC_{3,3}(k_1,k_2,k_3,s_2,s_3)$ such that $s_2 = s_3 = 0$ and $k_i - k_j = 0$ or $k_i - k_j = 1$ for $1 \leq i \leq j \leq 3$. Or by symmetry, $k_2 = k_3 = 0$ and $s_i - s_j = 0$ or $s_i - s_j = 1$ for $1 \leq i \leq j \leq 3$. See Figure 4 .

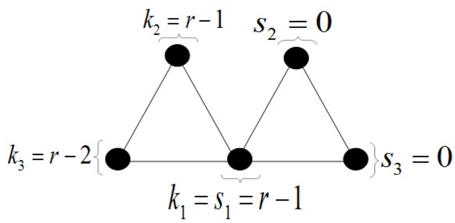

Figure 4: $G = DC_{3,3}(r-1,r-1,r-2,0,0) \in \Xi$ for $n = 3r+1$, $r \geq 2$.

Proposition 2.12. *Let $G \in \Xi$ with n vertices and $n-5$ pendent vertices. Then*

$$(a) \quad RM_2(G) = 3r^2+2r+2 \qquad for \quad n = 3r, r \geq 2.$$
$$(b) \quad RM_2(G) = 3r^2+4r+3 \qquad for \quad n = 3r+1, r \geq 2.$$
$$(c) \quad RM_2(G) = 3r^2+6r+1 \qquad for \quad n = 3r+2, r \geq 2.$$

Proof. We only prove the case (b), the other cases are similar. From Proposition 2.1 and Figure 4, we can directly write

$$\begin{aligned} RM_2(G) &= (r+2)(r+r-1+2)+r(r-1)+1 \\ &= (r+2)(2r+1)+r^2-r+1 = 3r^2+4r+3 \end{aligned}$$ □

Lemma 2.13. *Let a,b,c,d,e be non-negative integers and $a+b+c+d=e$. Then $ab+cd$ takes its maximum value when $a=\left\lceil \frac{e}{2}\right\rceil$, $b=\left\lfloor \frac{e}{2}\right\rfloor$ and $c=d=0$. Or by symmetry $c=\left\lceil \frac{e}{2}\right\rceil$, $d=\left\lfloor \frac{e}{2}\right\rfloor$ and $a=b=0$.*

Lemma 2.14. *Let a,b,c,d,e,f be non-negative integers and $a+b+c+d+e=f$. Then $ab+ac+de$ takes its maximum value when $e=\left\lceil \frac{f}{2}\right\rceil$, $d=\left\lfloor \frac{f}{2}\right\rfloor$ and $a=b=c=0$. Or by symmetry $a=\left\lceil \frac{f}{2}\right\rceil$, $b+c=\left\lfloor \frac{f}{2}\right\rfloor$ and $d=e=0$.*

Proof. Let $b+c=x$. Then $ab+ac+ed=a(b+c)+ed=ax+ed$. From Lemma 2.13, the desired is result acquired. □

Proposition 2.15. *Let $G\in DC_{a,b}(k_1,k_2,\dots,k_a,s_2,s_3,\dots,s_b)$ with $a,b\geq 4$ and $k_2+k_a+s_2+s_b=\Omega$. Then, the maximum RM_2 index of G is $DC_{a,b}(k_1,k_2,\dots,k_a,s_2,s_3,\dots,s_b)$ such that $k_3=k_4=\dots=k_{a-1}=0$, $s_3=s_4=\dots=s_{b-1}=0$ and $k_1=\Omega$ or $|k_1-\Omega|=1$.*

Proof. By Proposition 2.1,

$$\begin{aligned}
RM_2(G) &= (k_1+3)(k_2+k_a+s_2+s_b+4)+(k_2+1)(k_3+1)+\dots\\
&\quad+(k_{a-1}+1)(k_a+1)+(s_2+1)(s_3+1)+\dots+(s_{b-1}+1)(s_b+1)\\
&= k_1k_2+k_1k_a+k_1s_2+k_1s_b+4k_1+3k_2+3k_a+3s_2+3s_b+12\\
&\quad+k_2k_3+k_2+k_3+1+\dots+k_{a-1}k_a+k_{a-1}+k_a+1+s_2s_3+s_2+s_3+1+\dots\\
&\quad+s_{b-1}s_b+s_{b-1}+s_b+1\\
&= k_1k_2+k_1k_a+k_1s_2+k_1s_b+k_2k_3+k_3k_4+\dots+k_{a-2}k_{a-1}+k_{a-1}k_a\\
&\quad+s_2s_3+s_3s_4+\dots+s_{b-2}s_{b-1}+s_{b-1}s_b+4k_1+4k_2+2k_3+\dots+2k_{a-1}\\
&\quad+4k_a+4s_2+2s_3+\dots+2s_{a-1}+4s_b+a+b+8=\\
&= k_1k_2+k_1k_a+k_1s_2+k_1s_b+k_2k_3+k_3k_4+\dots+k_{a-2}k_{a-1}+k_{a-1}k_a\\
&\quad+s_2s_3+s_3s_4+\dots+s_{b-2}s_{b-1}+s_{b-1}s_b\\
&\quad+2(k_1+k_2+\dots+k_a+s_2+s_3+\dots+s_b)\\
&\quad+2(k_1+k_2+k_a+s_2+s_b)+a+b+8
\end{aligned}$$

Since $k_1+k_2+\dots+k_a+s_2+s_3+\dots+s_b=n-a-b+1$, then

$$\begin{aligned}RM_2(G) = {} & k_1k_2 + k_1k_a + k_1s_2 + k_1s_b + k_2k_3 + k_3k_4 + \cdots + k_{a-2}k_{a-1} + k_{a-1}k_a \\ & + s_2s_3 + s_3s_4 + \cdots + s_{b-2}s_{b-1} + s_{b-1}s_b + 2n - 2a - 2b + 2 \\ & + 2(k_1 + k_2 + k_a + s_2 + s_b) + a + b + 8 \\ = {} & k_1(k_2 + k_a + s_2 + s_b) + k_2k_3 + k_3k_4 + \cdots + k_{a-2}k_{a-1} + k_{a-1}k_a \\ & + s_2s_3 + s_3s_4 + \cdots + s_{b-2}s_{b-1} + s_{b-1}s_b \\ & + 2(k_1 + k_2 + k_a + s_2 + s_b) + 2n - a - b + 10 \\ = {} & k_1\Omega + k_2k_3 + k_3k_4 + \cdots + k_{a-2}k_{a-1} + k_{a-1}k_a \\ & + s_2s_3 + s_3s_4 + \cdots + s_{b-2}s_{b-1} + s_{b-1}s_b + 2(k_1 + \Omega) + 2n - a - b + 10.\end{aligned}$$

Clearly from the last equality by using Lemma 2.13, RM_2 takes its maximum value when $k_3 = k_4 = \cdots = k_{a-1} = 0$, $s_3 = s_4 = \cdots = s_{b-1} = 0$ and $k_1 = \Omega$ or $|k_1 - \Omega| = 1$. □

Theorem 2.16. *Let $G \in \Xi$ with n vertices and $n-5$ pendent vertices. Then G has maximum RM_2 value among the all graphs belong to $DC_{a,b}$ with n vertices.*

Proof. We only consider $n = 3r + 1$ for $r \geq 2$. The other cases are similar. Firstly, we show that G has maximum RM_2 value among all the graphs belonging to $DC_{3,3}(k_1, k_2, k_3, s_2, s_3)$. From the definition of RM_2 index,

$$RM_2(G) = (k_1 + 3)(k_2 + k_3 + s_2 + s_3 + 4) + (k_2 + 1)(k_3 + 1) + (s_2 + 1)(s_3 + 1)$$

$$= (k_1 + 3)(k_2 + k_3 + s_2 + s_3 + 4) + (k_2 + k_3 + s_2 + s_3) + k_2k_3 + s_2s_3 + 2$$

Since $k_2 + k_3 + s_2 + s_3 = n - k_1 - 5 = 3r - k_1 - 4$ then

$$RM_2(G) = (k_1 + 3)(3r - k_1) + 3r - k_1 - 4 + k_2k_3 + s_2s_3 + 2.$$

By Lemma 2.13, $k_2k_3 + s_2s_3$ takes its maximum value when $k_2 = k_3$ or $k_2 = k_3 + 1$ and $s_2 = s_3 = 0$. Or by symmetry $s_2 = s_3$ or $s_2 = s_3 + 1$ and $k_2 = k_3 = 0$. We only consider the first part of the Lemma 2.13. The second part can be handled similarly. Then

$$RM_2(G) = f(k_1, k_2, k_3) = (k_1 + 3)(3r - k_1) + 3r - k_1 - 4 + k_2k_3 + 2.$$

Since $k_1 + k_2 + k_3 = 3r - 4$, then

$$g(k_1, k_2, k_3) = 3r - 4 - k_1 - k_2 - k_3 = 0 \tag{5}$$

can be written. By using the Lagrange multipliers method, we obtain; $3r - 2k_1 - 4 = k_2$ and $3r - 2k_1 - 4 = k_3$. Thus, $k_2 + k_3 = 6r - 4k_1 - 8$. From Equation 5,

$k_2+k_3=3r-k_1-4$. And from these last two equalities $k_1=r-1$. By Definition 2.11, $k_2=r-1$ and $k_3=r-2$.
Secondly, we show that G has maximum RM_2 value among all the graphs belonging to $DC_{a,b}(k_1,\dots,k_a,s_2,\dots,s_b)$ for $a+b\geq 7$ with n vertices. There are two cases in this situation.
Case 1: Let $a=4$ and $b=3$. See Figure 5. From the definition of RM_2 index,

$$RM_2(G)=(k_1+3)(k_2+k_4+s_2+s_3+4)+(k_2+1)(k_3+1)$$
$$+(k_3+1)(k_4+1)+(s_2+1)(s_3+1).$$

Since $k_2+k_4+s_2+s_3=n-6-k_1=3r-5-k_1$, then

$$\begin{aligned}RM_2(G)&=(k_1+3)(3r-k_1-1)+k_2+k_4+s_2+s_3\\&\quad+k_3+k_2k_3+k_3k_4+s_2s_3+3\\&=(k_1+3)(3r-k_1-1)+3r-5-k_1+k_3+k_2k_3+k_3k_4+s_2s_3+3.\end{aligned}$$

By Lemma 2.14, $k_2k_3+k_3k_4+s_2s_3$ takes its maximum value when $k_2=k_3=k_4=0$ and $s_2=s_3$ or $s_2=s_3+1$. Therefore

$$RM_2(G)=f(k_1,s_2,s_3)=(k_1+3)(3r-k_1-1)+3r-5-k_1+s_2s_3+3.$$

Since $s_2+s_3=3r-5-k_1$, then

$$g(k_1,s_2,s_3)=3r-5-k_1-s_2-s_3=0 \tag{6}$$

can be written. By using the Lagrange multipliers method, we get $s_2=3r-k_1-5$ and $s_3=3r-k_1-5$. Thus, $s_2+s_3=6r-4k_1-10$. From Equation 6, $s_2+s_3=3r-k_1-5$. And from these last two equalities $k_1=r-1$, $s_2=s_3=r-2$ can be found. Thus,

$$RM_2(G)=(r+2)(r-2+r-2+4)+(r-1)^2+2$$
$$=r^2+2r+r^2-2r+1+2=2r^2+3.$$

This last value is smaller than that of Proposition 2.12 (b). For $a\geq 5$ and $b=3$ the proof is similar.

Case 2: Let $a\geq 4$ and $b\geq 4$. By Proposition 2.15 the proof is clear. □

Now, we begin to investigate the maximum RM_2-index of the second class of bicyclic graphs with k pendent vertices.

Proposition 2.17. *Let $G\in DC^0_{3,3}(k_1,k_2,k_3,s_1,s_2,s_3)$ with n vertices and $n-6$ pendent vertices. Then $G=\Psi=DC^0_{3,3}(k_1,0,0,s_1,0,0)$, with $k_1=s_1$ or $|k_1-s_1|=1$, has maximum RM_2 index among all the graphs belonging to $DC^0_{3,3}(k_1,k_2,k_3,s_1,s_2,s_3)$.*

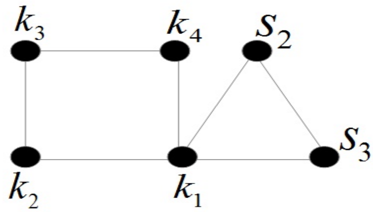

Figure 5: $G = DC_{4,3}(k_1, k_2, k_3, s_2, s_3)$ for the Case 1 of Theorem 2.16

Proof. From Proposition 2.2 (a),

$$\begin{aligned} RM_2(G) &= (k_1+2)(s_1+2) + (k_1+2)(k_2+k_3+2) + (k_2+1)(k_3+1) \\ &\quad + (s_1+2)(s_2+s_3+2) + (s_2+1)(s_3+1) \\ &= k_1s_1 + k_1(k_2+k_3) + s_1(s_2+s_3) + k_2k_3 + s_2s_3 + 14. \end{aligned}$$

$$f(k_1,k_2,k_3,s_1,s_2,s_3) = k_1s_1 + k_1(k_2+k_3) + s_1(s_2+s_3) + k_2k_3 + s_2s_3 + 14.$$

Since $k_1+k_2+k_3+s_1+s_2+s_3 = n-6$ then,

$$g(k_1,k_2,k_3,s_1,s_2,s_3) = n-k_1-k_2-k_3-s_1-s_2-s_3-6 = 0.$$

And by using the Lagrange multipliers method $k_2 = k_3 = 0$, $s_2 = s_3 = 0$, $k_1 = s_1$ or $|k_1 - s_1| = 1$. □

Corollary 2.18. *Let $\Psi \in DC^0_{3,3}(k_1,k_2,k_3,s_1,s_2,s_3)$. Then*

$$RM_2(\Psi) = \left\lceil \frac{n-2}{2} \right\rceil \left\lfloor \frac{n-2}{2} \right\rfloor.$$

Proof. Without loss of generality, from Proposition 2.17, let $k_1 = \left\lceil \frac{n-6}{2} \right\rceil$ and $s_1 = \left\lfloor \frac{n-6}{2} \right\rfloor$. Then with direct calculations the desired result is acquired. □

Theorem 2.19. *Let $\Psi \in DC^0_{3,3}(k_1,k_2,k_3,s_1,s_2,s_3)$ with n vertices and $n-6$ pendent vertices. Then Ψ has maximum RM_2 value among all the graphs belonging to $DC^l_{a,b}$ with n vertices and k pendent vertices.*

Proof. From Proposition 2.2, Lemma 2.14, Proposition 2.17 and Corollary 2.18 the desired result is acquired. □

Now, we begin to investigate the maximum RM_2-index of the third class of bicyclic graphs with k pendent vertices.

Proposition 2.20. *Let $G \in DC^0_{3+3}(k_1,k_2,k_3,s_2)$ with n vertices and $n-4$ pendent vertices. Then $G = Z = DC^0_{3+3}(k_1,k_2,k_3,s_2)$, with $|k_1-k_3| \le 1$ and $\left|k_j-(k_2+s_2)\right| \le 1$ ($j=1$ or $j=3$), has maximum RM_2 index among all the graphs belonging to $DC^0_{3+3}(k_1,k_2,k_3,s_2)$.*

Proof. From Proposition 2.3 (b),

$$RM_2(G) = (k_1+k_3+4)(k_2+s_2+2)+(k_1+2)(k_3+2).$$

If we put $k_2+s_2 = x$ then

$$RM_2(G) = f(k_1,k_3,x) = k_1x+k_3x+4x+k_1k_3+2k_1+2k_3+12.$$

Since $k_1+k_3+x = n-4$ then $g(k_1,k_3,x) = n-k_1-k_3-x-4 = 0$ can be written. And by using the Lagrange multipliers method we get $k_1 = k_3 = x = \frac{n-4}{3}$. Thus, $|k_1-k_3| \le 1$ and $|k_{1,3}-(k_2+s_2)| \le 1$. □

Proposition 2.21. *Let $G \in Z$ with n vertices and $n-4$ pendent vertices. Then*

$$\begin{array}{lll} (a) & RM_2(G) = 3r^2+4r+1 & for \quad n=3r, r\ge 2. \\ (b) & RM_2(G) = 3r^2+6r+3 & for \quad n=3r+1, r\ge 2. \\ (c) & RM_2(G) = 3r^2+8r+5 & for \quad n=3r+2, r\ge 2. \end{array}$$

Proof. By Proposition 2.3 and Proposition 2.20 we get the desired result. □

Theorem 2.22. *Let $Z \in DC^0_{3+3}(k_1,k_2,k_3,s_2)$ with n vertices and $n-4$ pendent vertices. Then Z has maximum RM_2 value among all the graphs belonging to DC^l_{a+b} with n vertices and k pendent vertices.*

Proof. From Proposition 2.3, Lemma 2.14, Proposition 2.20 and Proposition 2.21 the desired result is obtained. □

And now, from Theorem 2.16 , Theorem 2.19 and Theorem 2.22 we can state the following corollary.

Corollary 2.23. *Among all the bicyclic graphs with n vertices and k pendent vertices $Z = DC^0_{3+3}(k_1,k_2,k_3,s_2)$, with $|k_1-k_3| \le 1$ and $\left|k_j-(k_2+s_2)\right| \le 1$ ($j=1$ or $j=3$), has maximum RM_2 index.*

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