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# ON THE BOUNDARY BEHAVIOR OF THE HOLOMORPHIC SECTIONAL CURVATURE OF THE BERGMAN METRIC

ELISABETTA BARLETTA

We obtain a conceptually new differential geometric proof of P. F. Klembeck's result (cf. [9]) that the holomorphic sectional curvature $k_g(z)$ of the Bergman metric of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ approaches $-4/(n+1)$ (the constant sectional curvature of the Bergman metric of the unit ball) as $z \to \partial\Omega$.

## 1. Introduction.

Given a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ C. R. Graham & J. M. Lee studied (cf. [7]) the $C^\infty$ regularity up to the boundary for the solution to the Dirichlet problem $\Delta_g u = 0$ in $\Omega$ and $u = f$ on $\partial\Omega$, where $\Delta_g$ is the Laplace-Beltrami operator of the Bergman metric $g$ of $\Omega$. If $\varphi \in C^\infty(U)$ is a defining function ($\Omega = \{z \in U : \varphi(z) < 0\}$) their approach is to consider the foliation $\mathcal{F}$ of a one-sided neighborhood $V$ of the boundary $\partial\Omega$ by level sets $M_\epsilon = \{z \in V : \varphi(z) = -\epsilon\}$ ($\epsilon > 0$). Then $\mathcal{F}$ is a tangential CR foliation (cf. S. Dragomir & S. Nishikawa, [4]) each of whose leaves is strictly pseudoconvex and one may express $\Delta_g u = 0$ in terms of pseudohermitian invariants of the leaves and the transverse curvature $r = 2\, \partial\overline{\partial}\varphi(\xi, \overline{\xi})$ and

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its derivatives (the meaning of $\xi$ is explained in the next section). The main technical ingredient is an ambient linear connection $\nabla$ on $V$ whose pointwise restriction to each leaf of $\mathcal{F}$ is the Tanaka-Webster connection (cf. S. Webster, [14], and N. Tanaka, [13]) of the leaf. An axiomatic description (and index free proof) of the existence and uniqueness of $\nabla$ (referred to as the *Graham-Lee connection* of $(V, \varphi)$) was provided in [1]. As a natural continuation of the ideas in [7] one may relate the Levi-Civita connection $\nabla^g$ of $(V, g)$ to the Graham-Lee connection $\nabla$ and compute the curvature $R^g$ of $\nabla^g$ in terms of the curvature of $\nabla$. Together with an elementary asymptotic analysis (as $\epsilon \to 0$) this leads to a purely differential geometric proof of the result of P. F. Klembeck, [9], that the sectional curvature of $(\Omega, g)$ tends to $-4/(n+1)$ near the boundary $\partial\Omega$. The Author believes that one cannot overestimate the importance of the Graham-Lee connection (and that the identities (27) and (36) in Section 3 admit other applications as well, e.g. in the study of the geometry of the second fundamental form of a submanifold in $(\Omega, g)$).

## 2. The Levi-Civita versus the Graham-Lee connection.

Let $\Omega$ be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and $K(z, \zeta)$ its Bergman kernel (cf. e.g. [8], p. 364–371). As a simple application of C. Fefferman's asymptotic development (cf. [6]) of the Bergman kernel $\varphi(z) = -K(z, z)^{-1/(n+1)}$ is a defining function for $\Omega$ (and $\Omega = \{\varphi < 0\}$). Cf. A. Korányi & H. M. Reimann, [11], for a proof. Let us set $\theta = \frac{i}{2}(\overline{\partial} - \partial)\varphi$. Then $d\theta = i\ \partial\overline{\partial}\varphi$. Let us differentiate $\log|\varphi| = -(1/(n+1))\log K$ (where $K$ is short for $K(z, z)$) so that to obtain

$$\frac{1}{\varphi}\ \overline{\partial}\varphi = -\frac{1}{n+1}\ \overline{\partial}\log K.$$

Applying the operator $i\ \partial$ leads to

$$\frac{1}{\varphi}\ d\theta - \frac{i}{\varphi^2}\ \partial\varphi \wedge \overline{\partial}\varphi = -\frac{i}{n+1}\ \partial\overline{\partial}\log K. \tag{1}$$

We shall need the Bergman metric $g_{j\overline{k}} = \partial^2 \log K/\partial z^j \partial\overline{z}^k$. This is well known to be a Kähler metric on $\Omega$.

**Proposition 1.** *For any smoothly bounded strictly pseudoconvex domain*

$\Omega \subset \mathbb{C}^n$ *the Bergman metric* $g$ *is given by*

$$g(X,Y) = \frac{n+1}{\varphi}\{\frac{i}{\varphi}\ (\partial\varphi \wedge \overline{\partial}\varphi)(X, JY) - d\theta(X, JY)\}, \tag{2}$$

*for any* $X, Y \in \mathfrak{X}(\Omega)$.

*Proof.* Let $\omega(X,Y) = g(X, JY)$ be the Kähler 2-form of $(\Omega, J, g)$, where $J$ is the underlying complex structure. Then $\omega = -i\ \partial\overline{\partial}\log K$ and (1) may be written in the form (2). Q.e.d.

We denote by $M_\epsilon = \{z \in \Omega : \varphi(z) = -\epsilon\}$ the level sets of $\varphi$. For $\epsilon > 0$ sufficiently small $M_\epsilon$ is a strictly pseudoconvex CR manifold (of CR dimension $n-1$). Therefore, there is a one-sided neighborhood $V$ of $\partial\Omega$ which is foliated by the level sets of $\varphi$. Let $\mathcal{F}$ be the relevant foliation and let us denote by $H(\mathcal{F}) \to V$ (respectively by $T_{1,0}(\mathcal{F}) \to V$) the bundle whose portion over $M_\epsilon$ is the Levi distribution $H(M_\epsilon)$ (respectively the CR structure $T_{1,0}(M_\epsilon)$) of $M_\epsilon$. Note that

$$T_{1,0}(\mathcal{F}) \cap T_{0,1}(\mathcal{F}) = (0),$$

$$[\Gamma^\infty(T_{1,0}(\mathcal{F})), \Gamma^\infty(T_{1,0}(\mathcal{F}))] \subseteq \Gamma^\infty(T_{1,0}(\mathcal{F})).$$

Here $T_{0,1}(\mathcal{F}) = \overline{T_{1,0}(\mathcal{F})}$. For a review of the basic notions of CR and pseudohermitian geometry needed through this paper one may see S. Dragomir & G. Tomassini, [5]. Cf. also S. Dragomir, [3]. By a result of J. M. Lee & R. Melrose, [12], there is a unique complex vector field $\xi$ on $V$, of type $(1,0)$, such that $\partial\varphi(\xi) = 1$ and $\xi$ is orthogonal to $T_{1,0}(\mathcal{F})$ with respect to $\partial\overline{\partial}\varphi$ i.e. $\partial\overline{\partial}\varphi(\xi, \overline{Z}) = 0$ for any $Z \in T_{1,0}(\mathcal{F})$. Let $r = 2\ \partial\overline{\partial}\varphi(\xi, \overline{\xi})$ be the *transverse curvature* of $\varphi$. Moreover let $\xi = \frac{1}{2}(N - iT)$ be the real and imaginary parts of $\xi$. Then

$$(d\varphi)(N) = 2, \quad (d\varphi)(T) = 0,$$

$$\theta(N) = 0, \quad \theta(T) = 1,$$

$$\partial\varphi(N) = 1, \quad \partial\varphi(T) = i.$$

In particular $T$ is tangent to (the leaves of) $\mathcal{F}$. Let $g_\theta$ be the tensor field given by

$$g_\theta(X,Y) = (d\theta)(X, JY), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1, \tag{3}$$

for any $X, Y \in H(\mathcal{F})$. Then $g_\theta$ is a tangential Riemannian metric for $\mathcal{F}$ i.e. a Riemannian metric in $T(\mathcal{F}) \to V$. Note that the pullback of $g_\theta$ to each leaf $M_\epsilon$ of $\mathcal{F}$ is the Webster metric of $M_\epsilon$ (associated to the contact form $j_\epsilon^*\theta$, where $j_\epsilon : M_\epsilon \subset V$). As a consequence of (2), $JT = -N$ and $i_N \; d\theta = r \; \theta$ (see also (8) below)

**Corollary 1.** *The Bergman metric $g$ of $\Omega \subset \mathbb{C}^n$ is given by*

$$g(X, Y) = -\frac{n+1}{\varphi} \; g_\theta(X, Y), \quad X, Y \in H(\mathcal{F}). \tag{4}$$

$$g(X, T) = 0, \quad g(X, N) = 0, \quad X \in H(\mathcal{F}), \tag{5}$$

$$g(T, N) = 0, \quad g(T, T) = g(N, N) = \frac{n+1}{\varphi}\left(\frac{1}{\varphi} - r\right). \tag{6}$$

*In particular $1 - r\varphi > 0$ everywhere in $\Omega$.*

Using (4)-(6) we may relate the Levi-Civita connection $\nabla^g$ of $(V, g)$ to another canonical linear connection on $V$, namely the *Graham-Lee connection* of $\Omega$. The latter has the advantage of staying finite at the boundary (it gives the Tanaka-Webster connection of $\partial\Omega$ as $z \to \partial\Omega$). We proceed to recalling the Graham-Lee connection. Let $\{W_\alpha : 1 \le \alpha \le n-1\}$ be a local frame of $T_{1,0}(\mathcal{F})$, so that $\{W_\alpha, \xi\}$ is a local frame of $T^{1,0}(V)$. We consider as well

$$L_\theta(Z, \overline{W}) \equiv -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(\mathcal{F}).$$

Note that $L_\theta$ and (the $\mathbb{C}$-linear extension of) $g_\theta$ coincide on $T_{1,0}(\mathcal{F}) \otimes T_{0,1}(\mathcal{F})$. We set $g_{\alpha\overline{\beta}} = g_\theta(W_\alpha, W_{\overline{\beta}})$. Let $\{\theta^\alpha : 1 \le \alpha \le n-1\}$ be the (locally defined) complex 1-forms on $V$ determined by

$$\theta^\alpha(W_\beta) = \delta^\alpha_\beta \;, \quad \theta^\alpha(W_{\overline{\beta}}) = 0, \quad \theta^\alpha(T) = 0, \quad \theta^\alpha(N) = 0.$$

Then $\{\theta^\alpha, \; \theta^{\overline{\alpha}}, \; \theta, \; d\varphi\}$ is a local frame of $T(V) \otimes \mathbb{C}$ and one may easily show that

$$d\theta = 2i g_{\alpha\overline{\beta}} \; \theta^\alpha \wedge \theta^{\overline{\beta}} + r \, d\varphi \wedge \theta. \tag{7}$$

As an immediate consequence

$$i_T \, d\theta = -\frac{r}{2} \, d\varphi, \quad i_N \; d\theta = r \; \theta. \tag{8}$$

As an application of (7) we decompose $[T, N]$ (according to $T(V) \otimes \mathbb{C} = T_{1,0}(\mathcal{F}) \oplus T_{0,1}(\mathcal{F}) \oplus \mathbb{C}T \oplus \mathbb{C}N$) and obtain

$$[T, N] = i \;\; W^\alpha(r) W_\alpha - i \;\; W^{\overline{\alpha}}(r) W_{\overline{\alpha}} + 2rT, \tag{9}$$

where $W^\alpha(r) = g^{\alpha\overline{\beta}} W_{\overline{\beta}}(r)$ and $W^{\overline{\alpha}}(r) = \overline{W^\alpha(r)}$.

Let $\nabla$ be a linear connection on $V$. Let us consider the $T(V)$-valued 1-form $\tau$ on $V$ defined by

$$\tau(X) = T_\nabla(T, X), \quad X \in T(V),$$

where $T_\nabla$ is the torsion tensor field of $\nabla$. We say $T_\nabla$ is *pure* if

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2i L_\theta(Z, \overline{W}) T, \tag{10}$$

$$T_\nabla(N, W) = r\ W + i\ \tau(W), \tag{11}$$

for any $Z, W \in T_{1,0}(\mathcal{F})$, and

$$\tau(T_{1,0}(\mathcal{F})) \subseteq T_{0,1}(\mathcal{F}), \tag{12}$$

$$\tau(N) = -\ J\ \nabla^H r - 2r\ T. \tag{13}$$

Here $\nabla^H r$ is defined by $\nabla^H r = \pi_H \nabla r$ and $g_\theta(\nabla r, X) = X(r)$, $X \in T(\mathcal{F})$. Also $\pi_H : T(\mathcal{F}) \to H(\mathcal{F})$ is the projection associated to the direct sum decomposition $T(\mathcal{F}) = H(\mathcal{F}) \oplus \mathbb{R}T$. We recall the following

**Theorem 1.** *There is a unique linear connection $\nabla$ on $V$ such that* i) *$T_{1,0}(\mathcal{F})$ is parallel with respect to $\nabla$,*ii) *$\nabla L_\theta = 0$, $\nabla T = 0$, $\nabla N = 0$, and* iii) *$T_\nabla$ is pure.*

$\nabla$ given by Theorem 1 is the *Graham-Lee connection*. Theorem 1 is essentially Proposition 1.1 in [7], pp. 701–702. The axiomatic description in Theorem 1 is due to [4] (cf. Theorem 2 there). An index-free proof of Theorem 1 was given in [1] relying on the following

**Lemma 1.** *Let $\phi : T(\mathcal{F}) \to T(\mathcal{F})$ be the bundle morphism given by $\phi(X) = JX$, for any $X \in H(\mathcal{F})$, and $\phi(T) = 0$. Then*

$$\phi^2 = -I + \theta \otimes T,$$

$$g_\theta(X, T) = \theta(X),$$

$$g_\theta(\phi X, \phi Y) = g_\theta(X, Y) - \theta(X)\theta(Y),$$

*for any $X, Y \in T(\mathcal{F})$. Moreover, if $\nabla$ is a linear connection on $V$ satisfying the axioms* (i)-(iii) *in Theorem* 1 *then*

$$\phi \circ \tau + \tau \circ \phi = 0 \tag{14}$$

*along $T(\mathcal{F})$. Consequently $\tau$ may be computed as*

$$\tau(X) = -\frac{1}{2}\phi(\mathcal{L}_T\phi)X, \tag{15}$$

*for any* $X \in H(\mathcal{F})$.

A rather lengthy but straightforward calculation (based on Corollary 1) leads to

**Theorem 2.** *Let* $\Omega \subset \mathbb{C}^n$ *be a smoothly bounded strictly pseudoconvex domain,* $K(z,\zeta)$ *its Bergman kernel, and* $\varphi(z) = -K(z,z)^{-1/(n+1)}$*. Then the Levi-Civita connection* $\nabla^g$ *of the Bergman metric and the Graham-Lee connection of* $(\Omega,\varphi)$ *are related by*

$$\nabla^g_X Y = \nabla_X Y + \left\{\frac{\varphi}{1-\varphi r}\; g_\theta(\tau X, Y) + g_\theta(X, \phi Y)\right\}T - \\ -\left\{g_\theta(X,Y) + \frac{\varphi}{1-\varphi r}\; g_\theta(X, \phi\;\tau\;Y)\right\}N, \tag{16}$$

$$\nabla^g_X T = \tau X - \left(\frac{1}{\varphi} - r\right)\phi X - \frac{\varphi}{2(1-r\varphi)}\big\{X(r)T + (\phi X)(r)N\big\}, \tag{17}$$

$$\nabla^g_X N = -\left(\frac{1}{\varphi} - r\right)X + \tau\;\phi\;X + \frac{\varphi}{2(1-r\varphi)}\{(\phi X)(r)T - X(r)\;N\}, \tag{18}$$

$$\nabla^g_T X = \nabla_T X - \left(\frac{1}{\varphi} - r\right)\phi X - \frac{\varphi}{2(1-r\varphi)}\{X(r)T + (\phi X)(r)N\}, \tag{19}$$

$$\nabla^g_N X = \nabla_N X - \frac{1}{\varphi}X + \frac{\varphi}{2(1-r\varphi)}\{(\phi X)(r)T - X(r)N\}, \tag{20}$$

$$\nabla^g_N T = -\frac{1}{2}\phi\;\nabla^H r - \frac{\varphi}{2(1-r\varphi)}\left\{\left(N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi}\right)T + T(r)N\right\}. \tag{21}$$

$$\nabla^g_T N = \frac{1}{2}\phi\nabla^H r - \frac{\varphi}{2(1-r\varphi)}\left\{\left(N(r) + \frac{4}{\varphi^2} - \frac{6r}{\varphi} + 4r^2\right)T + T(r)N\right\}, \tag{22}$$

$$\nabla^g_T T = -\frac{1}{2}\nabla^H r - \frac{\varphi}{2(1-r\varphi)}\left\{T(r)T - \left(N(r) + \frac{4}{\varphi^2} - \frac{6r}{\varphi} + 4r^2\right)N\right\}, \tag{23}$$

(24) $\nabla^g_N N = -\frac{1}{2}\ \nabla^H r + \frac{\varphi}{2(1-r\varphi)}\left\{T(r)T - \left(N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi}\right)N\right\},$

*for any* $X, Y \in H(\mathcal{F})$.

## 3. Klembeck's theorem.

The original proof of the result by P. F. Klembeck (cf. Theorem 1 in [9], p. 276) employs a formula of S. Kobayashi, [10], expressing the components $R_{j\bar{k}r\bar{s}}$ of the Riemann-Christoffel 4-tensor of $(\Omega, g)$ as

$$-\frac{1}{2}R_{j\bar{k}r\bar{s}} = g_{j\bar{k}}g_{r\bar{s}} + g_{j\bar{s}}g_{r\bar{k}} - \frac{1}{K^2}\{K\ \ K_{j\bar{k}r\bar{s}} - K_{jr}K_{\bar{k}\ \bar{s}}\}+$$
$$+\frac{1}{K^4}\sum_{\ell,m} g^{\bar{\ell}m}\{K\ \ K_{jr\bar{\ell}} - K_{jr}K_{\bar{\ell}}\}\{K\ \ K_{\bar{k}\ \bar{s}m} - K_{\bar{k}\ \bar{s}}K_m\}$$

where $K = K(z, z)$ and its indices denote derivatives. However the calculation of the inverse matrix $[g^{j\bar{k}}] = [g_{j\bar{k}}]^{-1}$ turns out to be a difficult problem and [9] only provides an asymptotic formula as $z \to \partial\Omega$. Our approach is to compute the holomorphic sectional curvature of $(\Omega, g)$ by deriving an explicit relation among the curvature tensor fields $R^g$ and $R$ of the Levi-Civita and Graham-Lee connections respectively. We start by recalling a pseudohermitian analog to holomorphic curvature (built by S. M. Webster, [14]).

Let $M$ be a nondegenerate CR manifold of type $(n-1, 1)$ and $\theta$ a contact form on $M$. Let $G_1(H(M))_x$ consist of all 2-planes $\sigma \subset T_x(M)$ such that i) $\sigma \subset H(M)_x$ and ii) $J_x(\sigma) = \sigma$. Then $G_1(H(M))$ (the disjoint union of all $G_1(H(M))_x$) is a fibre bundle over $M$ with standard fibre $\mathbb{C}P^{n-2}$. Let $R^\nabla$ be the curvature of the Tanaka-Webster connection $\nabla$ of $(M, \theta)$. We define a function $k_\theta : G_1(H(M)) \to \mathbb{R}$ by setting

$$k_\theta(\sigma) = -\frac{1}{4}R^\nabla_x(X, J_xX, X, J_xX)$$

for any $\sigma \in G_1(H(M))$ and any linear basis $\{X, J_xX\}$ in $\sigma$ satisfying $G_\theta(X, X) = 1$. It is a simple matter that the definition of $k_\theta(\sigma)$ does not depend upon the choice of orthonormal basis $\{X, J_xX\}$, as a consequence of the following properties

$$R^\nabla(Z, W, X, Y) + R^\nabla(Z, W, Y, X) = 0,$$
$$R^\nabla(Z, W, X, Y) + R^\nabla(W, Z, X, Y) = 0.$$

$k_\theta$ is referred to as the (*pseudohermitian*) *sectional curvature* of $(M, \theta)$.

As mentioned above the notion is due to S. M. Webster, [14], who also gave examples of pseudohermitian space forms (pseudohermitian manifolds $(M, \theta)$ with $k_\theta$ constant). Cf. also [2] for a further study of contact forms of constant pseudohermitian sectional curvature. With respect to an arbitrary (not necessarily orthonormal) basis $\{X, J_x X\}$ of the 2-plane $\sigma$ the sectional curvature $k_\theta(\sigma)$ is also expressed by

$$k_\theta(\sigma) = -\frac{1}{4}\frac{R_x^\nabla(X, J_x X, X, J_x X)}{G_\theta(X, X)^2} \ .$$

To prove this statement one merely applies the definition of $k_\theta(\sigma)$ for the orthonormal basis $\{U, J_x U\}$, with $U = G_\theta(X, X)^{-1/2} X$. As $X \in H(M)_x$ there is $Z \in T_{1,0}(M)_x$ such that $X = Z + \overline{Z}$. Thus

$$k_\theta(\sigma) = \frac{1}{4}\frac{R_x(Z, \overline{Z}, Z, \overline{Z})}{g_\theta(Z, \overline{Z})^2} \ .$$

The coefficient $1/4$ is chosen such that the sphere $S^{2n-1} \subset \mathbb{C}^n$ has constant curvature $+1$. Cf. [5], Chapter 1. With the notations in Section 2 let us set $f = \varphi/(1 - \varphi r)$. Then

$$X(f) = f^2 \ X(r), \quad X \in T(\mathcal{F}).$$

Let $R^g$ and $R$ be respectively the curvature tensor fields of the linear connections $\nabla^g$ and $\nabla$ (the Graham-Lee connection). For any $X, Y, Z \in H(\mathcal{F})$ (by (16))

$$\nabla^g_X \nabla^g_Y Z = \nabla^g_X \big( \nabla_Y Z + \big\{ f \ g_\theta(\tau(Y), Z) + g_\theta(Y, \phi Z) \big\} T -$$

$$- \big\{ g_\theta(Y, Z) + f \ g_\theta(Y, \phi\tau(Z)) \big\} N \big) =$$

by $\nabla_Y Z \in H(\mathcal{F})$ together with (16)

$$= \nabla_X \nabla_Y Z + \big\{ f \ g_\theta(\tau(X), \nabla_Y Z) + g_\theta(X, \phi \nabla_Y Z) \big\} T -$$

$$- \big\{ g_\theta(X, \nabla_Y Z) + f \ g_\theta(X, \phi\tau(\nabla_Y Z)) \big\} N +$$

$$+ \big\{ f \ g_\theta(\tau(Y), Z) + g_\theta(Y, \phi Z) \big\} \nabla^g_X T +$$

$$+ \big\{ X(f) g_\theta(\tau(Y), Z) + f \ X(g_\theta(\tau(Y), Z)) + X(g_\theta(Y, \phi Z)) \big\} T -$$

$$- \big\{ g_\theta(Y, Z) + f \ g_\theta(Y, \phi\tau(Z)) \big\} \nabla^g_X N +$$

$$- \big\{ X(g_\theta(Y, Z)) + X(f) g_\theta(Y, \phi\tau(Z)) + f \ X(g_\theta(Y, \phi\tau(Z))) \big\} N =$$

by (17), (18)

$$= \nabla_X \nabla_Y Z + \{X(\Omega(Y,Z)) + \Omega(X, \nabla_Y Z) +$$

$$+ X(f) A(Y,Z) + f[X(A(Y,Z)) + A(X\ \nabla_Y Z)]\} T -$$

$$- \{X(g_\theta(Y,Z)) + g_\theta(X, \nabla_Y Z) +$$

$$+ X(f)\Omega(Y, \tau(Z)) + f[X(\Omega(Y,\tau(Z))) + \Omega(X, \tau(\nabla_Y Z))]\} N +$$

$$+ \{f\ A(Y,Z) + \Omega(Y,Z)\}\left\{\tau(X) - \frac{1}{f}\ \phi X - \frac{f}{2}\big(X(r)T + (\phi X)(r)N\big)\right\} -$$

$$- \{g_\theta(Y,Z) + f\ \Omega(Y,\tau(Z))\} \times$$

$$\times \left\{ -\frac{1}{f}\ X + \tau(\phi X) + \frac{f}{2}\big((\phi X)(r)T - X(r)N\big)\right\}$$

where we have set as usual $A(X,Y) = g_\theta(\tau(X), Y)$ and $\Omega(X,Y) = g_\theta(X, \phi Y)$. We may conclude that

$$(25)\quad \nabla^g_X \nabla^g_Y Z = \nabla_X \nabla_Y Z + [f\ A(Y,Z) + \Omega(Y,Z)]\left(\tau(X) - \frac{1}{f}\ \phi X\right) +$$

$$+ [g_\theta(Y,Z) + f\ \Omega(Y,\tau(Z))]\left(\frac{1}{f}\ X - \tau(\phi X)\right) +$$

$$+ \{X(\Omega(Y,Z)) + \Omega(X, \nabla_Y Z) + f[X(A(Y,Z)) + A(X, \nabla_Y Z)] +$$

$$+ \frac{f}{2}[X(r)(f\ A(Y,Z) - \Omega(Y,Z)) -$$

$$- (\phi X)(r)(g_\theta(Y,Z) + f\ \Omega(Y,\tau(Z)))]\} T -$$

$$- \{X(g_\theta(Y,Z)) + g_\theta(X, \nabla_Y Z) + f[X(\Omega(Y,\tau(Z))) + \Omega(X, \tau(\nabla_Y Z))] -$$

$$- \frac{f}{2}[X(r)(g_\theta(Y,Z) - f\ \Omega(Y,\tau(Z))) - (\phi X)(r)(f\ A(Y,Z) + \Omega(Y,Z))]\} N$$

for any $X, Y, Z \in H(\mathcal{F})$. Next we use the decomposition $[X,Y] = \pi_H[X,Y] + \theta([X,Y])T$ and (16), (19) to calculate

$$\nabla^g_{[X,Y]} Z = \nabla^g_{\pi_H[X,Y]} Z + \theta([X,Y]) \nabla^g_T Z =$$

$$= \nabla_{\pi_H[X,Y]} Z + \{f\ g_\theta(\tau(\pi_H[X,Y]), Z) + g_\theta(\pi_H[X,Y], \phi Z)\} T -$$

$$- \{g_\theta(\pi_H[X,Y], Z) + f\ g_\theta(\pi_H[X,Y], \phi\tau(Z))\} N +$$

$$+ \theta([X,Y])\left\{\nabla_T Z - \frac{1}{f}\ \phi Z - \frac{f}{2}(Z(r)T + (\phi Z)(r)N)\right\}$$

so that (by $\tau(T) = 0$)

$$\nabla^g_{[X,Y]} Z = \nabla_{[X,Y]} Z - \frac{1}{f}\ \theta([X,Y])\phi Z + \tag{26}$$

$$+\left\{f\ A([X,Y],Z) + \Omega([X,Y],Z) - \frac{f}{2}\theta([X,Y])Z(r)\right\}T -$$

$$-\left\{g_\theta([X,Y],Z) + f\ \Omega([X,Y],\tau(Z)) + \frac{f}{2}\theta([X,Y])(\phi Z)(r)\right\}N$$

for any $X, Y, Z \in H(\mathcal{F})$. Consequently by (25)-(26) (and by $\nabla g_\theta = 0$, $\nabla\Omega = 0$) we may compute

$$R^g(X,Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z$$

so that to obtain

$$R^g(X,Y)Z = R(X,Y)Z + \frac{1}{f}\ \theta([X,Y])\phi Z + \tag{27}$$

$$+(f\,A(Y,Z) + \Omega(Y,Z))\left(\tau(X) - \frac{1}{f}\ \phi X\right) -$$

$$-(f\ A(X,Z) + \Omega(X,Z))\left(\tau(Y) - \frac{1}{f}\phi Y\right) +$$

$$+(g_\theta(Y,Z) + f\ \Omega(Y,\tau(Z))\left(\frac{1}{f}\ X - \tau(\phi X))\right) -$$

$$-(g_\theta(X,Z) + f\,\Omega(X,\tau(Z)))\left(\frac{1}{f}\ Y - \tau(\phi Y)\right) +$$

$$+\big\{f\big[(\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z)\big] +$$

$$+\frac{f}{2}[X(r)(f\ A(Y,Z) - \Omega(Y,Z)) - Y(r)(f\ A(X,Z) - \Omega(X,Z)) -$$

$$-(\phi X)(r)(g_\theta(Y,Z) + f\ \Omega(Y,\tau(Z))) + (\phi Y)(r)(g_\theta(X,Z) +$$

$$+f\Omega(X,\tau(Z))) + Z(r)\theta([X,Y])]\}T - \{f[\Omega(Y,(\nabla_X\tau)Z) - \Omega(X,(\nabla_Y\tau)Z)] -$$

$$-\frac{f}{2}[X(r)(g_\theta(Y,Z) - f\ \Omega(Y,\tau(Z))) - Y(r)(g_\theta(X,Z) - f\ \Omega(X,\tau(Z))) -$$

$$-(\phi X)(r)(f\ A(Y,Z) + \Omega(Y,Z)) + (\phi Y)(r)(f\ A(X,Z) + \Omega(X,Z)) +$$

$$+(\phi Z)(r)\theta([X,Y])]\big\}N$$

for any $X, Y, Z \in H(\mathcal{F})$. Let us take the inner product of (27) with $W \in H(\mathcal{F})$ and use (4)-(5). We obtain

$$g(R^g(X,Y)Z,W) - \frac{n+1}{\varphi}\{g_\theta(R(X,Y)Z,W) - \frac{1}{f}\ \theta([X,Y])\Omega(Z,W)+$$

$$+[f\ A(Y,Z)+\Omega(Y,Z)][A(X,W)+\frac{1}{f}\ \Omega(X,W)]-$$

$$-[f\ A(X,Z)+\Omega(X,Z)][A(Y,W)+\frac{1}{f}\ \Omega(Y,W)]+$$

$$+[g_\theta(Y,Z)+f\ \Omega(Y,\tau(Z))][\frac{1}{f}\ g_\theta(X,W)+\Omega(X,\tau(W))]-$$

$$-[g_\theta(X,Z)+f\ \Omega(X,\tau(Z))][\frac{1}{f}\ g_\theta(Y,W)+\Omega(Y,\tau(W))]\}.$$

In particular for $Z=Y$ and $W=X$ (as $\Omega=-d\theta$)

$$g(R^g(X,Y)Y,X) = -\frac{n+1}{\varphi}\{g_\theta(R(X,Y)Y,X)+$$

$$+\frac{2}{f}\ \Omega(X,Y)^2 + f\ A(X,X)A(Y,Y) - \frac{1}{f}[f^2\ A(X,Y)^2 - \Omega(X,Y)^2]+$$

$$+\frac{1}{f}[g_\theta(X,X)+f\ \Omega(X,\tau(X))][g_\theta(Y,Y)+f\ \Omega(Y,\tau(Y))]-$$

$$-\frac{1}{f}[g_\theta(X,Y)+f\ \Omega(X,\tau(Y))]^2\}.$$

Note that

$$A(\phi X,\phi X) = g_\theta(\tau(\phi X),\phi X) = -g_\theta(\phi\tau X,\phi X) = -A(X,X),$$

$$\Omega(\phi X,\tau(\phi X)) = g_\theta(\phi X,\phi\tau(\phi X)) = g_\theta(X,\tau(\phi X)) =$$
$$= -g_\theta(X,\phi\tau(X)) = -\Omega(X,\tau(X)),$$

$$\Omega(X,\tau(\phi X)) = g_\theta(X,\phi\tau(\phi X)) = -g_\theta(X,\tau(\phi^2 X)) =$$
$$= g_\theta(X,\tau(X)) = A(X,X).$$

Hence

$$g(R^g(X,\phi X)\phi X,X) = -\frac{n+1}{\varphi}\{g_\theta(R(X,\phi X)\phi X,X)+ \tag{28}$$

$$+\frac{4}{f}\ g_\theta(X,X)^2 - 2f[A(X,X)^2 + A(X,\phi X)^2]\}.$$

Let $\sigma \subset T(\mathcal{F})_z$ be the 2-plane spanned by $\{X, \phi_z X\}$ for $X \in H(\mathcal{F})_z$, $X \neq 0$. By (4) if $Y = \phi_z X$ then

$$g_z(X, X)g_z(Y, Y) - g_z(X, Y)^2 =$$

$$= \left(\frac{n+1}{\varphi(z)}\right)^2 \{g_{\theta,z}(X, X)g_{\theta,z}(Y, Y) - g_{\theta,z}(X, Y)\} = \left(\frac{n+1}{\varphi(z)}\right)^2 g_{\theta,z}(X, X)^2$$

so that (by (28)) the sectional curvature $k_g(\sigma)$ of the 2-plane $\sigma$ is expressed by (for $Y = \phi_z X$)

$$k_g(\sigma) = \frac{g_z(R_z^g(X, Y)Y, X)}{g_z(X, X)g_z(Y, Y) - g_z(X, Y)^2} =$$

$$= -\frac{\varphi(z)}{n+1}\{-4k_\theta(\sigma) + \frac{4}{f(z)} - 2f(z)\frac{A_z(X, X)^2 + A_z(X, \phi_z X)^2}{g_{\theta,z}(X, X)^2}\}$$

where $k_\theta$ restricted to a leaf of $\mathcal{F}$ is the pseudohermitian sectional curvature of the leaf. Note that $k_\theta$ and $A$ stay finite at the boundary (and give respectively the pseudohermitian sectional curvature and the pseudohermitian torsion of $(\partial\Omega,\ \theta)$, in the limit as $z \to \partial\Omega$). On the other hand $f(z) \to 0$ and $\varphi(z)/f(z) \to 1$ as $z \to \partial\Omega$. We may conclude that $k_g(\sigma) \to -4/(n+1)$ as $z \to \partial\Omega$. To complete the proof of Klembeck's result we must compute the sectional curvature of the 2-plane $\sigma_0 \subset T_z(\Omega)$ spanned by $\{N_z, T_z\}$ (remember that $JN = T$). Note first that

$$N(f) = f^2\left(\frac{2}{\varphi^2} + N(r)\right).$$

Let us set for simplicity

$$g = N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi}\ , \quad h = N(r) + \frac{4}{\varphi^2} - \frac{6r}{\varphi} + 4r^2\ .$$

We these notations let us recall that (by (23))

$$\nabla_T^g T = -\frac{1}{2}\ X_r - \frac{f}{2}\big\{T(r)T - hN\big\} \tag{29}$$

where $X_r = \nabla^H r$. Using also (20) for $X = X_r$ we obtain

$$-2\nabla_N^g \nabla_T^g T = \nabla_N X_r - \frac{1}{\varphi}\ X_r + \frac{f}{2}\big\{(\phi X_r)(r)T - X_r(r)N\big\} +$$

$$+N(f)\{T(r)T - hN\} + f\big\{N(T(r))T + T(r)\nabla_N^g T - N(h)N - h\nabla_N^g N\big\}.$$

Let us recall that (by (21) and (24))

$$\nabla^g_N T = -\frac{1}{2}\,\phi\; X_r - \frac{f}{2}\big\{gT + T(r)N\big\}, \tag{30}$$

$$\nabla^g_N N = -\frac{1}{2}\; X_r + \frac{f}{2}\big\{T(r)T - gN\big\}. \tag{31}$$

Using these identities and the expression of $N(f)$ gives (after some simplifications)

$$\begin{aligned} -2\nabla^g_N \nabla^g_T T = {} & \nabla_N X_r + \left(\frac{fh}{2} - \frac{1}{\varphi}\right) X_r - \frac{f}{2}\; T(r)\;\phi X_r + \\ & + \frac{f}{2}\left\{2f\left(\frac{2}{\varphi^2} + N(r)\right)T(r) + 2N(T(r)) - f(g+h)T(r)\right\}T - \\ & - \frac{f}{2}\left\{g_\theta(X_r, X_r) + 2fh\left(\frac{2}{\varphi^2} + N(r)\right) + 2N(h) + f[T(r)^2 - gh]\right\}N \end{aligned} \tag{32}$$

because of

$$(\phi X_r)(r) = g_\theta(\nabla r, \phi X_r) = g_\theta(X_r, \phi X_r) = 0,$$

$$X_r(r) = g_\theta(\nabla^H r, X_r) = g_\theta(X_r, X_r).$$

Similarly

$$\begin{aligned} -2\nabla^g_T \nabla^g_N T = {} & \nabla_T \phi X_r + \left(\frac{1}{f} - \frac{fg}{2}\right) X_r + \frac{f}{2}\; T(r)\;\phi X_r + \\ & + \frac{f}{2}\big\{2T(g) + f(g-h)T(r)\big\}T + \\ & + \frac{f}{2}\big\{g_\theta(X_r, X_r) + 2T^2(r) + f[T(r)^2 + gh]\big\}N. \end{aligned} \tag{33}$$

Here $T^2(r) = T(T(r))$. Let us set $\tau(W_\alpha) = A_\alpha^{\overline{\beta}} W_{\overline{\beta}}$. To compute the last term in the right hand member of

$$R^g(N,T)T = \nabla^g_N \nabla^g_T T - \nabla^g_T \nabla^g_N T - \nabla^g_{[N,T]} T \tag{34}$$

note first that $T(f) = f^2\; T(r)$. On the other hand we may use the decomposition (9) so that

$$\nabla^g_{[N,T]} T = rX_r + frT(r)T - \frac{f}{2}\{g_\theta(X_r, X_r) + 2rh\}N +$$

$$+\left(ir^{\overline{\alpha}}A^{\beta}_{\overline{\alpha}}-\frac{1}{f}r^{\beta}\right)W_{\beta}-\left(ir^{\alpha}A^{\overline{\beta}}_{\alpha}+\frac{1}{f}r^{\overline{\beta}}\right)W_{\overline{\beta}}$$

(where $A^{\beta}_{\overline{\alpha}}=\overline{A^{\overline{\beta}}_{\alpha}}$) and by taking into account that

$$\left(ir^{\overline{\alpha}}A^{\beta}_{\overline{\alpha}}-\frac{1}{f}r^{\beta}\right)W_{\beta}-\left(ir^{\alpha}A^{\overline{\beta}}_{\alpha}+\frac{1}{f}r^{\overline{\beta}}\right)W_{\overline{\beta}}=-\frac{1}{f}\;X_r-\tau(\phi X_r)$$

we may conclude that

$$\nabla^g_{[N,T]}T=\left(r-\frac{1}{f}\right)X_r-\tau(\phi X_r)+ \tag{35}$$

$$+frT(r)T-\frac{f}{2}\{g_\theta(X_r,X_r)+2rh\}N.$$

Finally (by plugging into (34) from (32)-(33) and (35))

$$-2R^g(N,T)T=\nabla_N X_r-\nabla_T\phi X_r-fT(r)\phi X_r-2\tau(\phi X_r)+ \tag{36}$$

$$+\left(2r+\frac{f}{2}(g+h)-\frac{1}{\varphi}-\frac{3}{f}\right)X_r+$$

$$+f\left\{f\left(\frac{2}{\varphi^2}+N(r)\right)T(r)+N(T(r))-T(g)+(2r-fg)T(r)\right\}T-$$

$$-f\left\{2\|X_r\|^2+fh\left(\frac{2}{\varphi^2}+N(r)\right)+N(h)+fT(r)^2+T^2(r)+2rh\right\}N.$$

Here $\|X_r\|^2=g_\theta(X_r,X_r)$. Let us take the inner product of (36) with $N$ and use (4)-(6). We obtain

$$2g(R^g(N,T)T,N)=$$

$$=\frac{n+1}{\varphi}\left\{2\|X_r\|^2+fh\left(\frac{2}{\varphi^2}+N(r)\right)+N(h)+fT(r)^2+T^2(r)+2rh\right\}$$

and dividing by

$$g(N,N)g(T,T)-g(N,T)^2=\frac{1}{f^2}\left(\frac{n+1}{\varphi}\right)^2$$

leads to

$$2\frac{g(R^g(N,T)T,N)}{g(N,N)g(T,T)-g(N,T)^2}=$$

$$=\frac{f^2\varphi}{n+1}\left\{2\|X_r\|^2+T^2(r)+fT(r)^2+2hr+N(h)+fhN(r)+2\frac{fh}{\varphi^2}\right\}.$$

It remains that we perform an elementary asymptotic analysis of the right hand member of the previous identity when $z \to \partial\Omega$ (equivalently when $\varphi \to 0$). As $r \in C^\infty(\overline{\Omega})$ (cf. [12]) the terms $\|X_r\|^2$, $T^2(r)$, $T(r)^2$ and $N(r)$ stay finite at the boundary. Also (by recalling the expression of $h$) $f^2\varphi h \to 0$ as $\varphi \to 0$. Moreover

$$2\frac{f^2\varphi}{n+1}\frac{fh}{\varphi^2} = \frac{2}{n+1}\frac{f}{\varphi}\left[f^2N(r) + \frac{4}{(1-r\varphi)^2} - \frac{6f^2r}{\varphi} + 4f^2r^2\right] \to \frac{8}{n+1},$$

$$N(h) = N^2(r) + 4N(r^2) - \frac{16}{\varphi^3} + \frac{12r}{\varphi^2} - \frac{6}{\varphi}\, N(r),$$

$$\frac{f^2\varphi}{n+1}\, N(h) \to -\frac{16}{n+1},$$

as $\varphi \to 0$ hence

$$k_g(\sigma_0) \to -\frac{4}{n+1}, \quad z \to \partial\Omega.$$

Klembeck's theorem is proved.

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*Università degli Studi della Basilicata,*
*Dipartimento di Matematica,*
*Contrada Macchia Romana,*
*85100 Potenza, Italy,*
*e-mail: barletta@unibas.it*