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THE p-HARMONIC MEASURE OF A SMALL SPHERICAL CAP

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We estimate the p-harmonic measure of a small spherical cap.

1. Introduction

One of the questions studied in the article of Peres et al. (2009) concerned the ∞-harmonic measure of a spherical cap in dimension $d \geq 2$. If A_δ is a spherical cap of radius δ on the unit ball, denote its ∞-harmonic measure (with respect to the point x in the unit ball) by $\omega_\infty^x(A_\delta)$. Those authors proved that for some positive constants C_1 and C_2, independent of the dimension d,

$$C_1\delta^{1/3} \leq \omega_\infty^0(A_\delta) \leq C_2\delta^{1/3}.$$

Using the fact that $\omega_\infty^x(A_\delta)$ is ∞-harmonic in x on the open unit ball, they used a clever comparison argument with a function of Aronsson (1986) that can be regarded as the Martin kernel for the ∞-Laplacian on the half plane with pole at the origin: it is ∞-harmonic on the open half plane, it vanishes continuously on the boundary with the origin deleted, and it has a pole at the origin. The order of the pole determines the rate of decay of $\omega_\infty^x(A_\delta)$ as $\delta \to 0$. As pointed out by the authors, Aronsson has analogous functions for the p-Laplacian ($p > 2$) that might be used in a similar way for p-harmonic measure.

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In fact, Lundström and Vasilis (2013) used this idea to study the problem in two dimensions. They considered p-harmonic measure of small subsets of the boundary of more general domains D than just the disk. Their main theorem gives upper and lower bounds on $\omega_p^x(\partial D \cap B_\delta(y))$ for $y \in \partial D$ and small δ, where $B_\delta(y)$ is the open disk with radius δ and center y. The conditions on ∂D for the upper bound are very simple, while those for the lower bound are lengthy (their own words). As a special case, if D is convex, satisfying a uniform interior ball condition, then for $p \geq 2$,

$$C\,\delta^\alpha \leq \omega_p^x(\partial D \cap B_\delta(y)) \leq C^{-1}\,\delta^\alpha$$

for $y \in \partial D$ and small δ, where α is given explicitly in terms of p. Please note their results do not address the higher dimensional cases with $1 < p < \infty$. The two-dimensional analysis is highly nontrivial and very little is known in higher dimensions.

That is the motivation for our article. Our main result is an upper bound on the p-harmonic measure of a small spherical cap. Observe that the higher dimensional problem for the infinity Laplacian considered by Peres et al. (2009) reduces to the two-dimensional case. Indeed, it is no loss to assume the center of A_δ is the north pole. Clearly by symmetry, $\omega_\infty^x(A_\delta)$ depends only on the radial and azimuthal variables r and θ, respectively. The key observation is that the equation $\Delta_\infty\,\omega_\infty^x(A_\delta) = 0$ reduces to a partial differential equation in r and θ that is independent of the dimension. This is not the case with p-harmonic measure, as we will see below.

Before precisely stating our results, we establish the terminology and notation that we will use. Our primary references for the p-Laplacian and p-harmonic functions will be the notes of Lindqvist (2006) listed in the references, the paper of Granlund et al. (1982) and the monograph of Heinonen et al. (2006). More general degenerate operators are considered in the latter.

For reasonable functions u, the p-Laplacian of u is given by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right).$$

This reduces to the usual Laplacian when $p = 2$. A function u on a domain $\Omega \subseteq \mathbb{R}^d$ ($d \geq 2$) is p-harmonic on Ω iff it satisfies the equation $\Delta_p u = 0$ in the weak sense: $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and for each $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle\,dx = 0,$$

where $\langle\cdot,\cdot\rangle$ is the usual Euclidean inner product. It is known (see DiBenedetto (1983), Lewis (1983), and Ural'tseva (1968)) that p-harmonic functions are $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, where α depends only on p and the dimension d.

A function $v : \Omega \to (-\infty, \infty]$ is p-superharmonic in Ω if

- v is lower semicontinuous in Ω;
- $v \not\equiv \infty$ in Ω;
- for each domain $D \subset\subset \Omega$, if $h \in C(\overline{D})$ is p-harmonic in D and $h \leq v$ on ∂D, then $h \leq v$ in D.

It is known that $v \in C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ is p-superharmonic in Ω iff for each nonnegative $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \, dx \geq 0$$

(see Lindqvist (2006), Section 5.1). Moreover, if v is sufficiently smooth in Ω, then v is p-superharmonic there if $-\Delta_p v \geq 0$.

Given $A \subseteq \partial\Omega$, let $\mathcal{C}(A,\Omega)$ be the set of all nonnegative p-superharmonic functions v on Ω such that for each $\xi \in \partial\Omega$,

$$\liminf_{\substack{x \to \xi \\ x \in \Omega}} v(x) \geq I_A(\xi),$$

where I_A is the indicator function of A: 1 on A and 0 otherwise. The p-harmonic measure of A (relative to Ω) is the function whose value at x is given by

$$\omega_p(x; A, \Omega) = \inf\{v(x) : v \in \mathcal{C}(A,\Omega)\}.$$

For notational simplicity, we will often write this function as

$$\omega_p^x(A).$$

It is known that $\omega_p^x(A)$ is p-harmonic in Ω and

$$0 \leq \omega_p^x(A) \leq 1 \qquad x \in \Omega.$$

Moreover, $\omega_p^x(A)$ has boundary value 1 at each regular point x interior to A and boundary value 0 at each regular point x interior to $\partial\Omega \backslash A$. Note that when $p = 2$, this definition is equivalent to the usual notion of harmonic measure.

The lack of linearity in the p-Laplacian precludes p-harmonic measure from being a true measure. For example, it is not subadditive and it is not even additive on null sets (Llorente et al. (2005)). Even so, it is a useful tool for p-potential theory: it can be still be used to estimate p-harmonic functions and it is a substitute for classical harmonic measure in the theory of quasiregular mappings. Several classical results on harmonic measure, such as Carleman's principle and the Phragmén-Lindelöf principle have analogues for p-harmonic measure (see Heinonen et al. (2006) and Granlund et al. (1982)).

Bennewitz and Lewis (2005) have another definition of p-harmonic measure that truly is a measure. In their article, they explored the Hausdorff dimension of their version of p-harmonic measure. Lewis (2006) improved on these results and Lewis et al. (2011) studied the measure in simply connected domains.

For $x \in \mathbb{R}^d$ and $r > 0$, we will denote the ball with radius r and center x by

$$B_r(x) = \{y \in \mathbb{R}^d : |x-y| < r\}.$$

A δ-spherical cap on the unit sphere is any set of the form

$$B_\delta(y) \cap \partial B_1(0),$$

where $y \in \partial B_1(0)$ and $\delta > 0$. When $d = 2$ we will use the terminology δ-arc.

Our main result is the following theorem. Note our approach can be used in two dimensions, but the results of Lundström and Vasilis (2013) are stronger than those we can obtain.

Theorem 1.1. *Let $d > 2$ and $p \in (1,\infty)$. For $p \le \frac{d+2}{2}$ with $p \neq \frac{d+4}{3}$, set*

$$\alpha = \frac{-(p-2)(2d-p)+(d-p)^2}{2(p-1)(4+d-3p)} + \frac{\sqrt{[(p-2)(2d-p)-(d-p)^2]^2+4(p-1)^2(d-p)(4+d-3p)}}{2(p-1)(4+d-3p)};$$

for $p = \frac{d+4}{3}$, set

$$\alpha = 2\frac{d+1}{d+4};$$

and for $p > \frac{d+2}{2}$, set

$$\alpha = \frac{-(p-2)(2d-p)+(d-p)^2}{2(p-1)(4+d-3p)} - \frac{\sqrt{[(p-2)(2d-p)-(d-p)^2]^2+4(p-1)^2(d-p)(4+d-3p)}}{2(p-1)(4+d-3p)}.$$

Then $\alpha > 0$ and for some positive constant $C(d)$, if $\delta > 0$ is sufficiently small, then the p-harmonic measure of a δ-spherical cap A_δ satisfies

$$\omega_p^0(A_\delta) \le C(d)\,\delta^\alpha.$$

Remark 1.2. (a) By the Harnack inequality for the p-Laplacian, if $0 < r < 1$ and $x \in B_r(0)$, our bounds hold for $\omega_p^x(A_\delta)$, where now the number $C(d)$ depends on r.

(b) Our proof is valid for the case $d = 2$ and recovers the result of Lundström and Vasilis (2013) in the disc.

(c) Specialized to the case $p = 2$, our result yields the power

$$\alpha = \frac{d-2+(d-2)^2+4}{4},$$

which is worse (when $d > 2$) than the correct power of $\alpha = d-1$.

(d) Note that Hirata (2008) has derived estimates for p-harmonic measure in bounded $C^{1,1}$ domains in $\mathbb{R}^d$ when $p = d$. Specialized to our context, his Corollary 2.8 states that $\omega_p^0(A_\delta)$ is comparable to δ, when δ is small. Contrast this with the power $\frac{d}{2(d-1)}$ of δ we obtain in our upper bound. Admittedly our upper bound is crude, but very little seems to be known for $p \neq d$, $d > 2$.

(e) In section two, we explain why there are three cases to distinguish in the main result.

As pointed out above, Aronsson (1986) found a function that can be regarded as the Martin kernel for the ∞-Laplacian on the half plane with pole at the origin: it is ∞-harmonic on the open half plane, it vanishes continuously on the boundary with the origin deleted, and it has a pole at the origin. It is not at all clear how to modify Aronsson's approach to yield the Martin kernel for the p-Laplacian on the half-space with pole at the origin in higher dimensions. See section 2 for more details about this difficulty. Instead, we find a p-superharmonic function on the open half space with the following properties: it is positive on the open half space, continuous on the closed half space with the origin removed, has a pole at the origin and vanishes continuously on the boundary, away from the origin. This allows us to use the comparison method of Peres et al. (2009) to derive our upper bound. The novel feature of our work is the approach we take to find this particular p-superharmonic function. The method could also be used to recover Aronsson's function for $p > 2$ in dimension 2, as well as for those p between 1 and 2.

The article is organized as follows. In Section 2 we find the particular p-superharmonic function we need and give its relevant properties. In Section 3 we prove Theorem 1.1. In section 4 we prove a couple of technical results we use in section 2.

2. A p-Superharmonic Function for the Half-Space

From now on, we will assume the dimension d satisfies $d > 2$. Given a point $x \in \mathbb{R}^d$, we will use $r = r(x)$ to denote its distance to the origin and if $x \neq 0$ we use $\theta = \theta(x)$ to denote the azimuthal angle (i.e., the angle between x and the ray from the origin through the north pole on the unit sphere). Thus we can represent the open upper half space H as

$$H = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\} = \{x \in \mathbb{R}^d : r > 0, \quad \theta < \pi/2\}.$$

We need an expression for $\Delta_p u$ when $u(x)$ is sufficiently well-behaved and depends only on $r = r(x)$ and $\theta = \theta(x)$. It is known that $\Delta_p u$ can be written in the form

$$\Delta_p u = |\nabla u|^{p-4} \left(|\nabla u|^2 \Delta u + (p-2)\, \Delta_\infty u \right), \tag{1}$$

where Δ is the usual Laplacian and the infinity Laplacian $\Delta_\infty u$ is given by

$$\Delta_\infty u = \frac{1}{2} (\nabla u) \cdot \nabla \left(|\nabla u|^2 \right) = \sum_{i,j} \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} \right) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Since u depends only on $r(x)$ and $\theta(x)$, this becomes

$$\begin{aligned} \Delta_\infty u = \left(\frac{\partial u}{\partial r} \right)^2 \frac{\partial^2 u}{\partial r^2} + \frac{2}{r^2} \left(\frac{\partial u}{\partial r} \right) \left(\frac{\partial u}{\partial \theta} \right) \frac{\partial^2 u}{\partial r \partial \theta} \\ + \frac{1}{r^4} \left(\frac{\partial u}{\partial \theta} \right)^2 \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{r^3} \left(\frac{\partial u}{\partial \theta} \right)^2 \frac{\partial u}{\partial r}. \end{aligned} \tag{2}$$

Notice this is independent of the dimension d.

We now provide a brief derivation of this formula modeled on the $d = 2$ case handled in Aronsson (1984). If we denote by $\varphi = (\varphi_1, \ldots \varphi_{d-2})$ the remaining spherical coordinates, then we have the following identities, where we write subscripts to denote differentiation. First,

$$|\nabla u|^2 = u_r^2 + \tfrac{1}{r^2} u_\theta^2,$$

and using operator notation,

$$\begin{pmatrix} \partial_{x_1} \\ \cdot \\ \cdot \\ \cdot \\ \partial_{x_d} \end{pmatrix} = Q \begin{pmatrix} \partial_r \\ \frac{1}{r} \partial_\theta \\ \frac{1}{r} f_1 \, \partial_{\varphi_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{r} f_{d-2} \, \partial_{\varphi_{d-2}} \end{pmatrix},$$

where $Q = Q(r,\theta,\varphi)$ is an orthogonal matrix and the functions $f_1,\dots,f_{d-2}$ depend only on θ and φ. Thus expressing the infinity Laplacian in matrix form and writing $\gamma = |\nabla u|^2$,

$$2\Delta_\infty u = (\nabla u)^T\, \nabla\left(|\nabla u|^2\right)$$

$$= \left(Q \begin{pmatrix} \partial_r u \\ \frac{1}{r}\partial_\theta u \\ \frac{1}{r} f_1\, \partial_{\varphi_1} u \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{r} f_{d-2}\, \partial_{\varphi_{d-2}} u \end{pmatrix} \right)^T Q \begin{pmatrix} \partial_r \gamma \\ \frac{1}{r}\partial_\theta \gamma \\ \frac{1}{r} f_1\, \partial_{\varphi_1} \gamma \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{r} f_{d-2}\, \partial_{\varphi_{d-2}} \gamma. \end{pmatrix}$$

Since Q is orthogonal and u depends only on r and θ, this reduces to

$$2\Delta_\infty u = \begin{pmatrix} \partial_r u \\ \frac{1}{r}\partial_\theta u \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}^T \begin{pmatrix} \partial_r \gamma \\ \frac{1}{r}\partial_\theta \gamma \\ \frac{1}{r} f_1\, \partial_{\varphi_1} \gamma \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{r} f_{d-2}\, \partial_{\varphi_{d-2}} \gamma. \end{pmatrix}$$

$$= u_r\, \gamma_r + \frac{1}{r^2}\, u_\theta\, \gamma_\theta.$$

Upon using the definition of γ and simplifying, we get the desired expression (2) for Δ_∞.

Using the well-known form of the usual Laplacian Δ in spherical coordinates, we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{d-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + (d-2)\,(\cot\theta)\,\frac{\partial u}{\partial \theta}\right].$$

Assuming f is sufficiently smooth, for $u(x) = r^k f(\theta) = r(x)^k f(\theta(x))$ and $x \in \mathbb{R}^d \backslash \{x = c(0,\dots 0,1) : c \geq 0\}$, one can use (1)–(2) to show that

$$\begin{aligned} \Delta_p u(x) = g(r(x),\theta(x)) \Big[& \left[(p-1)\left(f'\right)^2 + k^2 f^2\right] f'' \\ & + [(2p-3)k + d - p]\, k f \left(f'\right)^2 + k^3\, [k(p-1) + d - p]\, f^3 \\ & + (d-2)\left[\left(f'\right)^2 + k^2 f^2\right] f' \cot\theta(x) \Big] \end{aligned} \qquad (3)$$

where we suppress the $\theta(x)$ in f and its derivatives and

$$g(r,\theta)=r^{k(p-1)-p}\left[\left(f'(\theta)\right)^2+k^2f^2(\theta)\right]^{(p-4)/2}.$$

Remark 2.1. When $d=2$ the p-harmonic equation $\Delta_p u=0$ reduces to

$$\begin{aligned}&\left[(p-1)\left(f'\right)^2+k^2f^2\right]f''\\ &\qquad+\left[(2p-3)k+2-p\right]kf\left(f'\right)^2\\ &\qquad\qquad+k^3\left[k(p-1)+2-p\right]f^3=0\end{aligned}$$

(cf. equation (5) in Aronsson (1986)). This equation is autonomous and Aronsson's trick was to make a functional transformation $H(f)=f'$, converting it into a first order ordinary differential equation in H that he was able to solve. But for $d>2$, the presence of $\cot\theta$ keeps the equation from being autonomous and using a functional transformation will no longer work.

The main result of this section is the following lemma giving the special p-superharmonic function described in the introduction.

Lemma 2.2. *Let $d>2$ and $p\in(1,\infty)$. For $p\le\dfrac{d+2}{2}$ with $p\neq\dfrac{d+4}{3}$, set*

$$\begin{aligned}k=&\frac{(p-2)(2d-p)-(d-p)^2}{2(p-1)(4+d-3p)}\\ &-\frac{\sqrt{[(p-2)(2d-p)-(d-p)^2]^2+4(p-1)^2(d-p)(4+d-3p)}}{2(p-1)(4+d-3p)};\end{aligned}$$

for $p=\dfrac{d+4}{3}$, set

$$k=-2\frac{d+1}{d+4};$$

and for $p>\dfrac{d+2}{2}$, set

$$\begin{aligned}k=&\frac{(p-2)(2d-p)-(d-p)^2}{2(p-1)(4+d-3p)}\\ &+\frac{\sqrt{[(p-2)(2d-p)-(d-p)^2]^2+4(p-1)^2(d-p)(4+d-3p)}}{2(p-1)(4+d-3p)}.\end{aligned}$$

Then $k<0$ and there is a function $f\in C[0,\pi/2]\cap C^\infty([0,\pi/2))$ such that:

- *f is positive on $[0,\pi/2)$;*

- *f is decreasing on $[0,\pi/2)$;*
- *for some positive constant C,*

$$f(\theta) \leq C\,(\pi/2-\theta)\,, \qquad \theta \in [0,\pi/2];$$

- *the function*

$$G(x) = r(x)^k f(\theta(x)), \qquad x \in \overline{H}\backslash\{0\}$$

is in $C\left(\overline{H}\backslash\{0\}\right) \cap C^\infty(H)$ and

$$-\Delta_p G \geq 0 \quad \textit{on } H.$$

Before giving the proof, we explain our reasoning.

Now if $f(\theta)$ is a function on $[0,\pi/2]$ satisfying the equation

$$\begin{aligned}\left[(p-1)\left(f'\right)^2 + k^2 f^2\right] f'' \\ + \left[(2p-3)k + d - p\right] kf\left(f'\right)^2 \\ + k^3\left[k(p-1) + d - p\right] f^3 = 0\end{aligned} \tag{4}$$

on $[0,\pi/2)$ (notice the left hand side is the expression in the outer square brackets in (3), less the cotangent term), along with the properties described in the Lemma 2.2 (smoothness, positivity on $[0,\pi/2)$ and vanishing at the endpoint $\pi/2$), then by the positivity and smoothness, it must be true that

$$f'(\theta)\cot\theta \leq 0, \theta \in (0,\pi/2).$$

Consequently by (3), for $x \in H\backslash\{x = c(0,\dots 0,1) : c \geq 0\}$,

$$\Delta_p\left(r(x)^k f(\theta(x))\right) \leq 0.$$

But by the smoothness of f and that $f(0) = 0$, $f'(\theta)\cot\theta$ is smooth in a neighborhood of $\theta = 0$ and so this inequality holds on all of H. The bottom line is that in order to prove Lemma 2.2, we need to find a function f on $[0,\pi/2]$ satisfying the stated properties in the Lemma and the equation (4) for the value of k as specified in the Lemma.

Our new idea is to write the desired function f in the form

$$f(\theta) = \exp\left(\int_0^\theta a(z)\,dz\right)$$

and find the function a. Substituting this into (4), we get

$$a' + \frac{k^3[k(p-1)-(p-d)] + k[2k(p-1)-(p-d)]a^2 + (p-1)a^4}{k^2 + (p-1)a^2} = 0. \tag{5}$$

By separating variables and using a partial fraction decomposition, this equation can be solved in a useful implicit form.

Now let us explain why three cases are needed in the statements of Theorem 1.1 and Lemma 2.2. Indeed, for our idea to work, it is necessary for the implicit solution $a(\theta)$ to be defined for $\theta \in [-\varepsilon, \pi/2]$, where $\varepsilon > 0$ is small (the extension to the left of $\theta = 0$ is needed to ensure $f(\theta)$ is smooth at $\theta = 0$). For the case $p = d$, this requires

$$k = -\frac{d}{2(d-1)}$$

(see the proof of part (b) in Lemma 2.5 below).

When $p \neq d$, we will see below (cf. Lemma 2.4 and the proof of part (b) in Lemma 2.5) that this will happen if there is a negative root of the following equation in k:

$$k(p-2)-(p-d)+[(p-2)+(p-d)]\sqrt{k^2-k\frac{p-d}{p-1}}=0 \tag{6}$$

Aside from trivial cases, this forces

$$\frac{k(p-2)-(p-d)}{(p-2)+(p-d)}<0. \tag{7}$$

Upon eliminating the radical, we see k is a negative root of the quadratic equation

$$(p-1)(4+d-3p)k^2-\left[(p-2)(2d-p)-(d-p)^2\right]k-(d-p)(p-1)=0$$

(it is interesting to note that when $p = d$, this quadratic reduces to

$$-2(d-1)k^2-dk=0$$

and we see the proper value for k in the $p = d$ case described above is exactly the negative root of this quadratic—this is why we do not need to separate out the case $p = d$ in the statement of Lemma 2.2).

When $p = (d+4)/3$, the coefficient of k^2 in the quadratic is 0 and so the quadratic reduces to a linear equation in k. Thus it is easy to see why there would be two cases to specify k: $p \neq (d+4)/3$ and $p = (d+4)/3$. So the obvious question is why does $p \neq (d+4)/3$ split into two cases? The answer is that in this case, there are two situations for the roots of the quadratic equation: exactly one negative root and two negative roots. The factor

$$\sqrt{k^2-k\frac{p-d}{p-1}}$$

appearing in (6) requires a positive radicand; moreover, (7) must hold. These are automatically true for the situation of exactly one negative root for the quadratic, while in the case of two negative roots, only one of the roots works. The two cases for $p \neq (d+4)/3$ occur because sometimes only the smaller negative root satisfies (7) and sometimes only the larger one works. To show (6) has a negative root for which the radicand there is nonnegative and satisfies (7) is technically demanding, so we state the essential conclusions in the next two Lemmas and defer the tricky proofs to the final section.

Lemma 2.3. *The number k in Lemma 2.2 is negative and satisfies $k < \frac{p-d}{p-1}$. Moreover,*

a) if $\quad 2 < p < \dfrac{d+2}{2} \quad$ *then* $\quad k > \dfrac{p-d}{p-2}$*;*

b) if $\quad \dfrac{d+2}{2} < p < d \quad$ *then* $\quad k < \dfrac{p-d}{p-2}$*.*

Lemma 2.4. *When $p \neq d$, with k from Lemma 2.2 and*

$$\beta = \sqrt{k^2 - k\frac{p-d}{p-1}}.$$

we have

$$\frac{k(p-2)-(p-d)}{\beta} + (p-2) + (p-d) = 0.$$

To specify the implicit solution $a(\theta)$ to (5), let k be from from Lemma 2.2 and let β be from Lemma 2.4. By Lemma 2.3, β is positive. Next, define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(\theta,a) = \begin{cases} -\dfrac{k(p-2)+d-p}{\beta}\tan^{-1}\dfrac{a}{\beta} \\ \qquad\qquad +(p-2)\tan^{-1}\dfrac{a}{k} + (p-d)\theta, & p \neq d \\ -\dfrac{d-2}{2}\dfrac{a}{a^2+k^2} + \dfrac{d}{2k}\tan^{-1}\dfrac{a}{k} + (d-1)\theta, & p = d. \end{cases}$$

Using the definition of β, we have

$$\frac{\partial F}{\partial a} = \begin{cases} \dfrac{p-d}{p-1}\cdot\dfrac{k^2+a^2(p-1)}{(\beta^2+a^2)(k^2+a^2)}, & p \neq d \\ \dfrac{k^2+(d-1)a^2}{(k^2+a^2)^2}, & p = d. \end{cases} \tag{8}$$

Since $F(0,0)=0$ and $\frac{\partial F}{\partial a}(0,0)\neq 0$, we can apply the Implicit Function Theorem to get a maximal open interval $(-N,M)$—where M and N are both positive and extended reals—and a unique C^1 function $a(\theta)$, $\theta\in(-N,M)$, such that on that interval,

$$\begin{cases} F(\theta,a(\theta))=0 \\ \\ \dfrac{\partial F}{\partial a}(\theta,a(\theta))\neq 0. \end{cases}$$

By (8) it follows that

$$a'(\theta)=-\frac{\dfrac{\partial F}{\partial\theta}}{\dfrac{\partial F}{\partial a}}=\begin{cases} -\dfrac{(p-1)(\beta^2+a^2)(k^2+a^2)}{k^2+a^2(p-1)}, & p\neq d \\ \\ -\dfrac{(p-1)(k^2+a^2)^2}{k^2+(p-1)a^2}, & p=d. \end{cases} \tag{9}$$

Using the definition of β when $p\neq d$, (9) implies that $a(\theta)$ solves (5) on $(-N,M)$. Now we list properties of $a(\theta)$.

Lemma 2.5. *(a) The function $a(\theta)$ is decreasing and C^∞ on its domain and $a(0)=0$;*

(b) $M=\frac{\pi}{2}$ and so the domain of a is $\left(-N,\frac{\pi}{2}\right)$;

(c) $a(\theta)\left(\dfrac{\pi}{2}-\theta\right)=-1+O\left(\left(\dfrac{\pi}{2}-\theta\right)^2\right)$, *as* $\theta\to\left(\dfrac{\pi}{2}\right)^-$.

Remark 2.6. It is not hard to show a extends uniquely to an odd function on $(-\pi/2,\pi/2)$, but we do not need this fact.

Proof. (a): Since $F(0,0)=0$, by unicity, we must have $a(0)=0$. That a is decreasing on $(-N,M)$ is immediate from (9). By repeatedly differentiating (9), it follows that $a\in C^\infty(-N,M)$.

(b): By the monotonicity of a on $(-N,M)$, the limit

$$\lim_{\theta\to M^-}a(\theta)$$

exists as an extended real number. The inverse tangent and the function $x\mapsto x/(x^2+k^2)$ are bounded, and since

$$\begin{aligned} 0&=F(\theta,a(\theta))\\ &=\begin{cases} -\dfrac{k(p-2)+d-p}{\beta}\tan^{-1}\dfrac{a(\theta)}{\beta} \\ \quad +(p-2)\tan^{-1}\dfrac{a(\theta)}{k}+(p-d)\theta, & p\neq d \\ -\dfrac{d-2}{2}\dfrac{a(\theta)}{a^2+k^2}+\dfrac{d}{2k}\tan^{-1}\dfrac{a(\theta)}{k}+(d-1)\theta, & p=d, \end{cases} \end{aligned}$$

it follows that M must be finite.

Write

$$L := \lim_{\theta\to M^-} a(\theta) \in [-\infty, 0).$$

To get a contradiction, assume $L \neq -\infty$. Then

$$\begin{aligned} 0 &= \lim_{\theta\to M^-} F(\theta, a(\theta)) \\ &= \begin{cases} -\dfrac{k(p-2)+d-p}{\beta} \tan^{-1}\dfrac{L}{\beta} + (p-2)\tan^{-1}\dfrac{L}{k} + (p-d)M, & p \neq d \\[2ex] -\dfrac{d-2}{2}\dfrac{L}{L^2+k^2} + \dfrac{d}{2k}\tan^{-1}\dfrac{L}{k} + (d-1)M, & p = d, \end{cases} \\ &= F(M, L). \end{aligned}$$

Also, by (8),

$$\begin{aligned} \frac{\partial F}{\partial a}(M, L) &= \begin{cases} \dfrac{p-d}{p-1} \cdot \dfrac{k^2 + L^2(p-1)}{(\beta^2+L^2)(k^2+L^2)}, & p \neq d \\[2ex] \dfrac{k^2+(d-1)L^2}{(k^2+L^2)^2}, & p = d. \end{cases} \\ &\neq 0. \end{aligned}$$

Then by the Implicit Function Theorem, there exists an interval $(M-\varepsilon, M+\varepsilon)$ and a unique C^1 function $\widetilde{a}(\theta)$, $\theta \in (M-\varepsilon, M+\varepsilon)$, such that $F(\theta, \widetilde{a}(\theta)) = 0$ on that interval. But then maximality of the interval $(-N, M)$ is violated. Thus we must have $L = -\infty$.

Since $k < 0 < \beta$, this implies that

$$\begin{aligned} 0 &= \lim_{\theta\to M^-} F(\theta, a(\theta)) \\ &= \begin{cases} -\dfrac{k(p-2)+d-p}{\beta}\left(-\dfrac{\pi}{2}\right) + (p-2)\left(\dfrac{\pi}{2}\right) + (p-d)M, & p \neq d \\[2ex] \dfrac{d}{2k}\left(\dfrac{\pi}{2}\right) + (d-1)M, & p = d. \end{cases} \end{aligned}$$

Upon using Lemma 2.4 when $p \neq d$ and that $k = -\dfrac{d}{2(d-1)}$ when $p = d$, this reduces to

$$0 = \begin{cases} -(p-d)\dfrac{\pi}{2} + (p-d)M, & p \neq d \\[2ex] -(d-1)\dfrac{\pi}{2} + (d-1)M, & p = d. \end{cases}$$

Thus $M = \pi/2$ and we have proved that

$$\lim_{\theta \to (\pi/2)^-} a(\theta) = -\infty, \tag{10}$$

a fact we will use below.

(c): For $\theta \in (-N, \pi/2)$, the equation $F(\theta, a(\theta)) = 0$ implies that

$$\theta = \begin{cases} \dfrac{1}{p-d}\left[\dfrac{k(p-2)+d-p}{\beta}\tan^{-1}\dfrac{a}{\beta} - (p-2)\tan^{-1}\dfrac{a}{k}\right], & p \neq d \\ \dfrac{1}{d-1}\left[\dfrac{d-2}{2}\cdot\dfrac{a}{a^2+k^2} - \dfrac{d}{2k}\tan^{-1}\dfrac{a}{k}\right], & p = d. \end{cases}$$

By (10), $a(\theta) \to -\infty$ as $\theta \to (\pi/2)^-$, and since $k < 0 < \beta$, we have

$$\frac{\pi}{2} = \begin{cases} \dfrac{1}{p-d}\left[\dfrac{k(p-2)+d-p}{\beta}\left(-\dfrac{\pi}{2}\right) - (p-2)\left(\dfrac{\pi}{2}\right)\right], & p \neq d \\ \dfrac{1}{d-1}\left[-\dfrac{d}{2k}\cdot\dfrac{\pi}{2}\right], & p = d. \end{cases}$$

Thus

$$\frac{\pi}{2} - \theta = \begin{cases} \dfrac{1}{p-d}\left[-\dfrac{k(p-2)+d-p}{\beta}\left(\dfrac{\pi}{2} + \tan^{-1}\dfrac{a}{\beta}\right)\right. \\ \qquad \left. -(p-2)\left(\dfrac{\pi}{2} - \tan^{-1}\dfrac{a}{k}\right)\right], & p \neq d \\ \dfrac{1}{d-1}\left[-\dfrac{d}{2k}\left(\dfrac{\pi}{2} - \tan^{-1}\dfrac{a}{k}\right) - \dfrac{d-2}{2}\dfrac{a}{a^2+k^2}\right], & p = d. \end{cases}$$

Using the expansions

$$\begin{aligned} \frac{\pi}{2} - \tan^{-1} x &= \frac{1}{x} + O\left(|x|^{-3}\right) \quad \text{as} \quad x \to \infty, \\ \frac{\pi}{2} + \tan^{-1} x &= -\frac{1}{x} + O\left(|x|^{-3}\right) \quad \text{as} \quad x \to -\infty, \\ \frac{x}{x^2+k^2} &= \frac{1}{x} + O\left(|x|^{-3}\right) \quad \text{as} \quad x \to -\infty, \end{aligned}$$

together with (10), as $\theta \to (\pi/2)^-$, we get

$$\frac{\pi}{2} - \theta = \begin{cases} \dfrac{1}{p-d}\Bigg[-\dfrac{k(p-2)+d-p}{\beta}\left(-\dfrac{\beta}{a(\theta)} + O\left(|a(\theta)|^{-3}\right)\right) \\ \qquad -(p-2)\left(\dfrac{k}{a(\theta)} + O\left(|a(\theta)|^{-3}\right)\right)\Bigg], & p \neq d \\ \dfrac{1}{d-1}\Bigg[-\dfrac{d}{2k}\left(\dfrac{k}{a(\theta)} + O\left(|a(\theta)|^{-3}\right)\right) \\ \qquad -\dfrac{d-2}{2}\left(\dfrac{1}{a(\theta)} + O\left(|a(\theta)|^{-3}\right)\right)\Bigg], & p = d. \end{cases}$$
$$= -\frac{1}{a(\theta)} + O\left(|a(\theta)|^{-3}\right).$$

Rearranging this yields

$$a(\theta)\left(\frac{\pi}{2} - \theta\right) = -1 + O\left(|a(\theta)|^{-2}\right) \qquad \text{as } \theta \to \left(\frac{\pi}{2}\right)^-. \tag{11}$$

In particular,

$$a^2(\theta)\left(\frac{\pi}{2} - \theta\right)^2 \to 1 \qquad \text{as } \theta \to \left(\frac{\pi}{2}\right)^-.$$

Then (11) becomes

$$a(\theta)\left(\frac{\pi}{2} - \theta\right) = -1 + O\left(\left(\frac{\pi}{2} - \theta\right)^2\right) \qquad \text{as } \theta \to \left(\frac{\pi}{2}\right)^-.$$

This gives (c) and the proof of the Lemma 2.5 is complete. □

Proof of Lemma 2.2. Define

$$f(\theta) = \begin{cases} \exp\left(\int_0^\theta a(z)\,dz\right) & \theta \in [0, \pi/2), \\ 0 & \theta = \pi/2. \end{cases}$$

Then f is positive on $[0, \pi/2)$. Since a is smooth on $(-N, \pi/2)$, we have that f is smooth on $([0, \pi/2))$, and since $a(\cdot)$ satisfies (9), f satisfies (4). Since a is decreasing and $a(0) = 0$, $a \leq 0$ on $[0, \pi/2)$, and so f is decreasing there too.

All that remains is to show $f \in C[0, \pi/2]$ and for some $C > 0$,

$$f(\theta) \leq C(\pi/2 - \theta), \qquad \theta \in [0, \pi/2]. \tag{12}$$

In light of the definition of f and that $f \in C^\infty([0,\pi/2))$, is it enough to prove (12). By part (c) in Lemma 2.5, choose $\delta \in (0,\pi/2)$ so small that

$$M_\delta := \sup_{|\theta-\pi/2|<\delta} \frac{a(\theta)(\pi/2-\theta)+1}{(\pi/2-\theta)^2} < \infty.$$

Then for $0 < \pi/2-\theta < \delta$,

$$\begin{aligned}
\frac{f(\theta)}{\pi/2-\theta} &= (\pi/2-\theta)^{-1} \exp\left(\int_0^{\pi/2-\delta} a(z)\,dz\right) \exp\left(\int_{\pi/2-\delta}^{\theta} a(z)\,dz\right) \\
&\leq (\pi/2-\theta)^{-1} C_\delta \exp\left(\int_{\pi/2-\delta}^{\theta} -\frac{1}{\pi/2-z}\,dz\right. \\
&\qquad \left. + \int_{\pi/2-\delta}^{\theta} M_\delta(\pi/2-z)\,dz\right) \\
&= (\pi/2-\theta)^{-1} C_\delta \exp\Bigg(\ln(\pi/2-\theta) - \ln\delta \\
&\qquad -\frac{1}{2} M_\delta \left[(\pi/2-\theta)^2 - \delta^2\right]\Bigg) \\
&\leq \widetilde{C}_\delta,
\end{aligned}$$

as desired. □

3. Proof of Theorem 1.1

Once we have the special p-superharmonic function from Section 2, we can use some of the ideas of Peres et al. (2009) from the case $p=\infty$, replacing their use of comparison with cones for the ∞-Laplacian with the Comparison Principle for the p-Laplacian. If we denote the unit ball in $\mathbb{R}^d$ centered at the origin by Ω, then it is no loss to assume that

$$A_\delta = B_\delta(\mathbf{S}) \cap \partial\Omega,$$

where $\mathbf{S}$ is the south pole of the unit sphere in $\mathbb{R}^d$. For notational simplicity, we will write

$$u(x) = \omega_p^x(A_\delta). \tag{13}$$

It is known (Heinonen et al. (2006)) that u is p-harmonic on Ω and has boundary value 0 at all regular points interior to $(\partial\Omega)\backslash A_\delta$.

Define $h : \overline{\Omega} \to [0,1]$ by

$$h(y) = \begin{cases} 0, & d(y,A_\delta) > \delta \\ 1 - \frac{1}{\delta} d(y,A_\delta), & d(y,A_\delta) \leq \delta. \end{cases}$$

Then h is continuous and there exists a p-harmonic function $v \in C(\overline{\Omega})$ with

$$v = h \quad \text{on } \partial\Omega$$

(see Theorem 2.16 in Lindqvist (2006)).

Apply Lemma 2.2 to get the p-superharmonic function

$$G(x) = G(r(x), \theta(x)) = r(x)^k f(\theta(x)), \quad x \in \overline{H},$$

where H is the upper half space, f enjoys the properties given in the Lemma and k is as specified there. Notice α from the statement of Theorem 1.1 satisfies

$$k = -\alpha$$

For $x = (x_1, \ldots, x_d)$ define

$$G_\delta(x) = \delta^\alpha G(x - (1+2\delta)\mathbf{S}), \quad x_d \geq -1$$

and note that G_δ is p-superharmonic on Ω. By Lemma 2.2 f is strictly positive on the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and so for some $C_1 > 0$,

$$G(x) = r(x)^{-\alpha} f(\theta(x)) \geq C_1\, r(x)^{-\alpha}, \qquad \theta(x) \leq \tfrac{\pi}{4}.$$

Now for each $x \in A_{2\delta}$, if we denote the azimuthal angle of $x - (1+2\delta)\mathbf{S}$ by θ, then for $\delta < 1/2$ and $\tilde{x} = (x_1, \ldots, x_{d-1})$ we have

$$(x_d + 1)^2 + |\tilde{x}|^2 = 4\delta^2$$

and

$$|\tan\theta| = \frac{|\tilde{x}|}{x_d + 1 + 2\delta} = \frac{\sqrt{4\delta^2 - (x_d+1)^2}}{x_d + 1 + 2\delta} \leq \frac{2\delta}{2\delta} = 1.$$

Thus $\theta \in \left[0, \frac{\pi}{4}\right]$ and so for $x \in A_{2\delta}$, we have

$$\begin{aligned} h(x) \leq 1 &\leq \frac{1}{C_1} |x - (1+2\delta)\mathbf{S}|^\alpha\, G(x - (1+2\delta)\mathbf{S}) \\ &= \frac{1}{C_1} \left[\frac{|x - (1+2\delta)\mathbf{S}|}{\delta}\right]^\alpha G_\delta(x). \end{aligned}$$

But

$$x \in A_{2\delta} \quad \Longrightarrow \quad |x - \mathbf{S}| = 2\delta$$

and so

$$\begin{aligned} |x - (1+2\delta)\mathbf{S}| &\leq |x - \mathbf{S}| + 2\delta \\ &= 4\delta. \end{aligned}$$

It follows that for some $C_3 > 0$, independent of $0 < \delta < \frac{1}{2}$,

$$h(x) \le C_3\, G_\delta(x), \quad x \in A_{2\delta}.$$

Since $h \equiv 0$ on $(\partial\Omega)\backslash A_{2\delta}$, we get that

$$h(x) \le C_3\, G_\delta(x), \quad x \in \partial\Omega, \quad \delta < \tfrac{1}{2}.$$

Since $v = h$ on $\partial\Omega$, by an extension of the Comparison Principle in Lindqvist (2006)–see the remark after Theorem 2.15 there–we have

$$v(x) \le C_3\, G_\delta(x), \quad x \in \Omega, \quad \delta < \tfrac{1}{2}.$$

In particular, for $\delta < \frac{1}{2}$,

$$\begin{aligned} v(0) &\le C_3\, G_\delta(0) \\ &= C_3\, \delta^\alpha\, G(-(1+2\delta)\mathbf{S}) \\ &= C_3\, \delta^\alpha\, (1+2\delta)^k f(0) \\ &\le C_4\, \delta^\alpha, \end{aligned}$$

where $C_4 > 0$ is independent of δ. By the definition of harmonic measure,

$$\omega_p^0(A_\delta) = u(0) \le v(0) \le C_4\, \delta^\alpha,$$

as desired. □

4. Proof of Lemmas 2.3 and 2.4

Proof of Lemma 2.3. We distinguish several cases and simultaneously prove the inequality

$$k < \frac{p-d}{p-1}$$

and part a) or b) of the Lemma, according to the situation of the particular case under consideration.

First note that since $d > 2$,

$$2 < \frac{d+4}{3} < \frac{d+2}{2} < d. \tag{14}$$

Write

$$f(x) = (p-1)(4+d-3p)x^2 + \left[(d-p)^2 - (p-2)(2d-p)\right]x - (p-1)(d-p)$$

and notice, after some simplification,

$$f\left(\frac{p-d}{p-1}\right) = \frac{p-d}{p-1}. \tag{15}$$

It is easy to see that when $p \neq (d+4)/3$, k solves the equation

$$f(k) = 0.$$

For notational simplicity, we write this as

$$ak^2 + bk + c = 0.$$

- The cases $p = (d+4)/3$ and $p = d$ follow easily by direct computation.
- $p < (d+4)/3$: Then $a > 0$ and $c < 0$. Thus the root

  $$k = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

  is negative and the other root is positive.

  Since the graph of $y = f(x)$ is an upward parabola, it intersects the half line $y = x$, $x < 0$, in exactly one point. By (15) and that $p < d$, the point of intersection is $\left(\frac{p-d}{p-1}, \frac{p-d}{p-1}\right)$. It follows that $k < \frac{p-d}{p-1}$, as desired.

  Moreover, if also $2 < p$, then since $p < (d+4)/3 < (d+2)/2 < d$, we are in the situation of part a) and we have $(p-d)/(p-2) < 0$. Hence to show $k > (p-d)/(p-2)$–that is, to verify part a) of the lemma–inspection of the graph of $y = f(x)$ reveals it is enough to show $f((p-d)/(p-2)) > 0$. For this, we have

  $$\begin{aligned} f\left(\frac{p-d}{p-2}\right) &= (p-1)(4+d-3p)\left(\frac{p-d}{p-2}\right)^2 \\ &\qquad + \left[(d-p)^2 - (p-2)(2d-p)\right]\left(\frac{p-d}{p-2}\right) - (p-1)(d-p) \\ &= \frac{d-p}{(p-2)^2}\Bigg[\left[(p-2)+1\right]\left[(d-p)-2(p-2)\right](d-p) \\ &\qquad - \left[(d-p)^2 - (p-2)(d-p) - d(p-2)\right](p-2) \\ &\qquad - (p-1)(p-2)^2\Bigg], \end{aligned}$$

  and after some manipulation, keeping the $(d-p)$ and $(p-2)$ as units, we end up with

  $$f\left(\frac{p-d}{p-2}\right) = \frac{d-p}{(p-2)^2}\left[(p-d)+(p-2)\right]^2 > 0, \qquad \frac{p-d}{p-2} < 0. \tag{16}$$

  Notice the derivation of (16) only required $2 < p < d$.

- $(d+4)/3 < p \le (d+2)/2$: It is easy to check that as a function of $p \in (1, d+1/2)$, b is decreasing. Thus for $p \in ((d+4)/3, d]$,

$$b(p) \le b((d+4)/3) = -(d-2)(d+4)/9 < 0.$$

We have $a < 0$ and since $p < d$, we also have $c < 0$. Thus there are two negative roots k and r, with the larger one being

$$k = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{|b| - \sqrt{b^2 - 4ac}}{2a}.$$

Since the graph of $y = f(x)$ is a downward parabola with two negative roots, its intersection with the line $y = x$ consists of two points (x_1, x_1) and (x_2, x_2) with $x_1 < r < k < x_2 < 0$. Thus if we can show $r \le (p-d)/(p-1)$, then since $f((p-d)/(p-1)) = (p-d)/(p-1)$, it follows that $x_2 = (p-d)/(p-1)$ and we would then have $k < (p-d)/(p-1)$, as desired. To this end, observe that after some simplification,

$$\begin{aligned} r \le \frac{p-d}{p-1} &\iff \frac{-b + \sqrt{b^2 - 4ac}}{2a} \le \frac{p-d}{p-1} \\ &\iff [(d-p) - (p-2)]^2 + 2(p-2) + \sqrt{b^2 - 4ac} \ge 0, \end{aligned}$$

and this is true because

$$p > \frac{d+4}{3} > \frac{2+4}{3} = 2.$$

Since $2 < p < d$, (16) holds and by looking at the graph of $y = f(x)$, we see this forces $k > \dfrac{p-d}{p-2}$. Thus part a) holds.

- $(d+2)/2 < p < d$: Exactly as in the previous case, the graph of $y = f(x)$ is a downward parabola with two negative roots, this time the smaller one given by

$$k = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

From (16), we have $f((p-d)/(p-2)) > 0$, and so it follows that $k < (p-d)/(p-2)$, giving part b). Since $(p-d)/(p-2) < (p-d)/(p-1)$, this immediately yields that $k < (p-d)/(p-1)$.

- $d < p$: Here parts (a) and (b) of the lemma are irrelevant. We have $c > 0$ and $a < 0$. Thus $\sqrt{b^2 - 4ac} > |b|$ and so

$$k = \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0 < \frac{p-d}{p-1} < \frac{p-d}{p-2}. \qquad \square$$

Proof of Lemma 2.4. We distinguish the cases $p = 2$, $p = (d+2)/2$, $p < d$ with $p \neq 2$ or $(d+2)/2$, and $d < p$.

- $p = 2$: The first expression for k in Lemma 2.2 applies in this situation and it is easy to show $\beta^2 = 1$. Then direct calculation yields the desired conclusion.
- $p = (d+2)/2$: The radicand in the definition of k reduces to 0 and then k in turn reduces to $k = -1$. Direct calculation yields the desired conclusion.
- $p < d$ with $p \neq 2$ or $(d+2)/2$: Notice $(p-2)+(p-d) \neq 0$. We first show

$$\frac{k(p-2)+(d-p)}{(p-2)+(p-d)} < 0. \tag{17}$$

Indeed, if $p < (d+2)/2$, then the denominator in (17) is negative and so the inequality (17) is equivalent to

$$k(p-2)-(p-d) > 0.$$

But this is equivalent to

$$k > \frac{p-d}{p-2} \quad \text{if} \quad p > 2$$

or

$$k < \frac{p-d}{p-2} \quad \text{if} \quad p < 2.$$

The first follows from part a) of Lemma 2.3 and the second follows from Lemma 2.3 because $\dfrac{p-d}{p-2} > 0 > k$.

On the other hand, if $p > (d+2)/2$, then the denominator in (17) is positive and so the inequality (17) is equivalent to

$$k(p-2)-(p-d) < 0.$$

Since $p > (d+2)/2 > 2$, this is the same as $k < \dfrac{p-d}{p-2}$, which follows from part b) of Lemma 2.3.

In any case, we have that (17) holds. So rewriting the conclusion of Lemma 2.4 as

$$\beta = -\frac{k(p-2)-(p-d)}{(p-2)+(p-d)},$$

we see that it suffices to show

$$\beta^2 = \left(-\frac{k(p-2)-(p-d)}{(p-2)+(p-d)}\right)^2.$$

For this, clear the denominator to get

$$\beta^2\,[(p-2)+(p-d)]^2 = [k(p-2)-(p-d)]^2.$$

Using the definition of β in terms of k, expand, keeping $(p-2)$ and $(p-d)$ as units. Then collect coefficients of like powers of k. We end up with the equation

$$\begin{aligned}(p-1)(4+d-3p)k^2 \\ +[(d-p)^2-(p-2)(2d-p)]k \\ -(p-1)(d-p) \\ =0.\end{aligned}$$

Since k solves this equation, the conclusion of Lemma holds

- $d < p$: As in the previous case, it is enough to verify (17). Since $p > d > (d+2)/2$, the denominator appearing in (17) is positive and so (17) is equivalent to negativity of the numerator there. But this is immediate since $p > d > 2$ and $k < 0$. □

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REFERENCES

[1] G. Aronsson, *On certain singular solutions of the partial differential equation* $u_x^2u_{xx}+2\,u_x\,u_y\,u_{xy}+u_y^2u_{yy}=0$, Manuscripta Math. 47 (1984), 135–158.

[2] G. Aronsson, *Construction of singular solutions to the p-harmonic equation and its limit equation for* $p=\infty$, Manuscripta Math. 56 (1986), 133–151.

[3] B. Bennewitz - J. L. Lewis *On the dimension of p-harmonic measure*, Ann. Acad. Sci. Fenn. Math. 30 (2005), 459–505.

[4] E. DiBenedetto, $C^{1+\alpha}$*-local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. 7 (1983), 827–850.

[5] S. Granlund - P. Lindquist - O. Martio, *F-Harmonic measure in space*, Ann. Acad. Sci. Fenn. Math. 7 (1982), 233–247.

[6] J. Heinonen - T. Kilpeläinen - O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations,* Dover, New York, 2006.

[7] K. Hirata, *Global estimates for non-symmetric Green type functions with applications to the p-Laplace equation*, Potential Analysis 29 (2008), 221–239.

[8] J. L. Lewis, *Regularity of the derivatives of solutions to certain elliptic equations*, Indiana Univ. Math. J. 32 (1983), 849–856.

[9] J. L. Lewis, *Note on p-harmonic measure*, Comput. Methods Funct. Theory 6 (2006), 109–144.

[10] J. L. Lewis - K. Nystrm - P. Poggi-Corradini, *p-harmonic measure in simply connected domains*, Ann. Inst. Fourier (Grenoble) 61 (2011), 689–715.

[11] J. G. Llorente - J. J. Manfredi - J. -M. Wu, *p-harmonic measure is not additive on null sets*, Ann Scuola Norm. Sup. Pisa Cl. Sci. IV (2005), 357–373.

[12] P. Lindqvist, *Notes on the p-Laplace equation*, (2006), http://www.math.ntnu.no/ lqvist/p-laplace.pdf

[13] N. L. P. Lundström - J. Vasilis, *Decay of p-harmonic measure in the plane*, Annales Academiæ Scientiarum Fennicæ Mathematica 38 (2013), 351–356.

[14] Y. Peres - O. Schramm - S. Sheffield - D. B. Wilson, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc. 22 (2009), 167–210.

[15] N. N. Ural'tseva, *Degenerate quasilinear elliptic systems*, Zap. Naučn. Sem. (Otdel. Math. Inst. Steklov, Leningrad) 7 (1968), 1184–222.

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