LE MATEMATICHE
Vol. LXXI (2016) – Fasc. II, pp. 3–15
doi: 10.4418/2016.71.2.1

GREGUS TYPE FIXED POINT THEOREMS FOR WEAKLY SUBSEQUENTIALLY CONTINUOUS MAPPINGS SATISFYING STRICT CONTRACTIVE CONDITION OF INTEGRAL TYPE

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In the present paper, we prove two common fixed point theorems of Gregus type for two pairs of self mappings satisfying strict contractive condition of integral type by using the weak subsequential continuity property with compatibility of type (E) due to Singh and Mishra [29] in metric spaces. Our results improve and the results of Chauhan et al. [7] and relevant literature.

1. Introduction

Jungck [14] firstly established a common fixed point theorem for a pair of commuting self mappings in metric spaces. In 1982, Sessa [28] defined the weakly commuting mappings which is weaker than commuting mappings. Since then, Jungck [15] investigated the notion of compatible mappings which is more general than commuting and weakly commuting mappings. After that many authors introduced various types of compatibility, compatibility of types (A), (B), (C) and (P) for two self mappings in metric space respectively in [16, 21, 23] and [22]. In 1996, Jungck [17] introduced the notion of weakly compatible mappings which generalizes all the above type of compatibility and is weaker than

Entrato in redazione: 21 aprile 2015

AMS 2010 Subject Classification: 54H25, 47H10.
Keywords: Gregus type fixed point, weakly subsequentially continuous mappings, compatible of type (E).

them. Pant [20] is the first who studied and used non-compatible mappings and replaced them by a new concept which called reciprocal continuity to establish a common fixed point; later Aamri and Moutawakil [2] introduced property (E.A) for two self mappings on metric spaces and they used it with generalized contractions. Since then, Al-Thagafi and Shahzad [3] weakened the weak compatibility; they introduced the notion of occasionally weakly compatible mappings on metric spaces. This was generalized by another concept of subcompatible mappings which was given by Bouhadjera and Godet Thobie [6]; the same authors in their paper [6] generalized reciprocal continuity to subsequential continuity.

On other hand, the Gregus fixed point had been generalized and improved by many authors, as Djoudi and Nisse [11], Djoudi and Aliouche [10, 11], Aliouche [2], Pathak and Shahzad [24]. Recently, Sintunavarat and Kumam [30] proved some results for Gregus type common fixed point of integral type by using the tangential property for two single and self mappings in metric spaces. Also Chauhan et al. [7] used the subsequential continuity with compatibility to prove some results concerning Gregus type fixed point. In the present paper, we will prove two common fixed point theorems of Gregus type for four mappings which satisfy strict contractive condition of integral type in metric spaces, by using the subsequential continuity and compatibility of type (E) due to Singh et al. [29].

2. Preliminaries

Definition 2.1. Two self mappings A and S of a metric space (X,d) are said to be compatible of type (E), if

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At \text{ and } \lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Remark 2.2. If $At = St$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)), however the converse may be not be true. Generally, compatibility of type (E) implies the compatibility of type (B).

Definition 2.3. Two self mappings A and S of a metric space (X,d) are A-compatible of type (E), if

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At,$$

for some $t \in X$. Also, the pair $\{A,S\}$ is said to be S-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$, for some $t \in X$.

Notice that if two self mappings A and S are compatible of type (E), then they are A-compatible and S-compatible of type (E), but the converse is not true.

Definition 2.4 ([20])**.** Two self mappings A and S of a metric space (X,d) are said to be reciprocally continuous, if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Definition 2.5 ([6])**.** Two self mappings A and S of a metric space (X,d) is called to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and satisfy $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n) = St$.

Clearly continuous or reciprocally continuous mappings are subsequentially continuous, but the converse may be not true.

Example 2.6. Let $X = [0,\infty)$ and d is the euclidian metric, we define A, S as follows:

$$Ax = \begin{cases} 2+x, & 0 \le x \le 2 \\ \frac{x+2}{2}, & x > 2 \end{cases}, \quad Sx = \begin{cases} 2-x, & 0 \le x < 2 \\ 2x-2, & x \ge 2 \end{cases}$$

Clearly A and S are discontinuous at 2.

We consider a sequence $\{x_n\}$ such that for each $n \ge 1$: $x_n = \dfrac{1}{n}$, clearly $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2$, also we have:

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(2-\frac{1}{n}) = 4 = A(2),$$

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(2+\frac{1}{n}) = 2 = S(2),$$

then the pair $\{A,S\}$ is subsequentially continuous.

On other hand, let $\{y_n\}$ be a sequence which defined or each $n \ge 1$ by: $y_n = 2+\dfrac{1}{n}$, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = 2,$$

but

$$\lim_{n\to\infty} ASy_n = \lim_{n\to\infty} A(2+\frac{2}{n}) = 2 \neq A(2),$$

$$\lim_{n\to\infty} SAy_n = \lim_{n\to\infty} S(4+\frac{1}{n}) = 6 \neq S(2),$$

then A and S are never reciprocally continuous.

Definition 2.7. Let A and S to be two self mappings of a metric space (X,d), the pair $\{A,S\}$ is said to be weakly subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$, $\lim_{n\to\infty} SAx_n) = Sz$.

Notice that the subsequentially continuous or reciprocally continuous mappings are weakly subsequentially continuous, but the converse may be not true.

Example 2.8. Let $X = [0,8]$ and d is the euclidian metric, we define A, S as follows:

$$Ax = \begin{cases} \frac{x+4}{2}, & 0 \le x \le 4 \\ x+1, & 4 \le x \le 8 \end{cases}, \quad Sx = \begin{cases} 8-x, & 0 \le x \le 4 \\ x-2, & 4 \le x \le 8 \end{cases}$$

We consider a sequence $\{x_n\}$ such that for each $n \ge 1$: $x_n = 4 - e^{-n}$, clearly $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 4$, also we have:

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(4+e^{-n}) = 5,$$

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(4 - \frac{1}{2}e^{-n}) = 4 = S(4),$$

then the pair $\{A,S\}$ is S-subsequentially continuous.

3. Main results

Theorem 3.1. *Let $A,B,S,T : X \to X$, be self mappings of a metric space (X,d) such for all x,y in X we have:*

$$\begin{aligned}
&\left(1 + a\Big(\int_0^{d(Ax,By)} \varphi(t)\Big)^p\right)\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big)^p < \\
&a\left(\begin{array}{c} \Big(\int_0^{d(Ax,Sx)} \varphi(t)dt\Big)^p \Big(\int_0^{d(By,Ty)} \varphi(t)dt\Big)^p \\ + \Big(\int_0^{d(Sx,By)} \varphi(t)dt\Big)^p \Big(\int_0^{d(Ax,Ty)} \varphi(t)dt\Big)^p \end{array}\right) + \alpha\Big(\int_0^{d(Ax,By)} \varphi(t)dt\Big)^p\Big) \\
&\qquad + \beta\tau\left(\begin{array}{c} \Big(\int_0^{d(Ax,Sx)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Ty)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Ax,Ty)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Sx)} \varphi(t)dt\Big)^p \end{array}\right) \qquad (1)
\end{aligned}$$

where a, α, β are non-negative numbers such $\alpha + \beta < 1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$, if the pair $\{A,S\}$ is weakly subsequentially continuous and compatible of type (E) as well as $\{B,T\}$, then A,B,S and T have a unique common fixed point in X.

Proof. Suppose that $\{A,S\}$ is A-subsequentially continuous, there is a sequence $\{x_n\}$in$X$ such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = z$ and $\lim_{n\to\infty} ASx_n = Az$,also the pair is compatible of type (E) implies that $\lim_{n\to\infty} ASx_n = Sz$, also the pair $\{f,S\}$ is compatible implies that $\lim\limits_{n\to\infty} ASx_n = Sz$ and $\lim\limits_{n\to\infty} SAx_n = Sz$, which implies $Sz = fz =$ and z is a coincidence point for f and S.

Similarly for the pair $\{B,T\}$, suppose that $\{B,T\}$ is B-subsequentially continuous, there exists a sequence $\{y_n\} \in X$ such

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n = t,$$

and

$$\lim_{n\to\infty} BTy_n = Bt,$$

also the pair $\{g,T\}$ is compatible of type (E) implies

$$\lim_{n\to\infty} BTy_n = \lim_{n\to\infty} T^2 y_n = Tt$$

and

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} B^2 y_n = Bt$$

which implies that $Bt = Tt$.

Firstly, we prove $Az = Bt$, if not by using (1) we get

$$\begin{aligned}
&\Big(1 + a\Big(\int_0^{d(Az,Bt)} \varphi(t)\Big)^p\Big)\Big(\int_0^{d(Sz,Tt)} \varphi(t)dt\Big)^p < \\
&\quad a\Big(\int_0^{d(Az,Tt)} \varphi(t)dt\Big)^p\Big(\int_0^{d(Bt,Sz)} \varphi(t)dt\Big)^p + \alpha\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p \\
&\qquad + \beta\tau\Big(0,0,\Big(\int_0^{d(Az,Tt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Sz,Bt)} \varphi(t)dt\Big)^p\Big)
\end{aligned}$$

since $Az = Sz$ and $Bt = Tt$, we get

$$\begin{aligned}
\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p &= \Big(\int_0^{d(\{Az\},\{Bt\})} \varphi(t)dt\Big)^p < \alpha\Big(\int_0^{d(Az,Bt)} \varphi(t)\Big)^p \\
&\quad + \beta\tau\Big(0,0,\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p \\
&\quad < (\alpha+\beta)\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p < \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p,
\end{aligned}$$

which is a contradiction, then $Az = Bt$. Now, we prove $z = Az$, if not by using (1), we get

$$\Big(1+a(\int_0^{d(Ax_nBt)}\varphi(t)\Big)^p\Big(\int_0^{d(Sx_n,Tt)}\varphi(t)dt\Big)^p <$$

$$a\left[\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(Bt,Tt)}\varphi(t)dt\Big)^p\\ +\Big(\int_0^{d(Bt,Sx_n)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(Ax_n,Tt)}\varphi(t)dt\Big)^p\end{array}\right]+\alpha\Big(\int_0^{d(Ax_n,Bt)}\varphi(t)dt\Big)^p$$

$$+\beta\tau\left(\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt)^p,0,\Big(\int_0^{d(Ax_n,Tt)}\varphi(t)dt\Big)^p,\\ \Big(\int_0^{d(Sx_n,Bt)}\varphi(t)dt\Big)^p\end{array}\right)$$

Letting $n\to\infty$, we get:

$$\Big(1+a(\int_0^{d(z,Az)}\varphi(t)\Big)^p\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p \le$$

$$a\Big(\int_0^{d(Bt,z)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(z,Tt)}\varphi(t)dt)\Big)^p+\alpha\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p$$

$$+\beta\tau\Big(0,0,\Big(\int_0^{d(z,Tt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Az,t)}\varphi(t)dt\Big)^p\Big),$$

since $Az = Sz = Bt$, we get

$$\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p \le (\alpha+\beta)\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p < \Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p,$$

which is a contradiction, then $z = Az = Sz$. Nextly we prove $z = t$, if not then by using (1), we get

$$\Big(1+a(\int_0^{d(Ax_n,By_n)}\varphi(t))^p\Big)\Big(\int_0^{d(Sx_n,Ty_n)}\varphi(t)dt\Big)^p <$$

$$a\left[\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(By_n,Ty_n)}\varphi(t)dt\Big)^p\\ +\Big(\int_0^{d(Ax_n,Ty_n)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(By_n,Sx_n)}\varphi(t)dt\Big)^p\end{array}\right]+\alpha\Big(\int_0^{d(Ax_n,By_n)}\varphi(t)dt\Big)^p$$

$$+\beta\tau\left(\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By_n,Ty_n)}\varphi(t)dt\Big)^p,\\ \Big(\int_0^{d(Ax_n,Ty_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Sx_n,By_n)}\varphi(t)dt\Big)^p\end{array}\right).$$

Letting $n\to\infty$, we get

$$\Big(1+a\Big(\int_0^{d(z,t)}\varphi(t)\Big)^p\Big)\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p \le$$

$$a\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p+\alpha\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p$$

$$+\beta\tau\left(0,0,\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\right),$$

and so we have

$$\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\leq\alpha\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p$$

$$+\beta\tau\Big(\Big(0,0,\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\Big),$$

which implies that

$$\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\leq(\alpha+\beta)\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p<\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p,$$

which is a contradiction, then z is a common fixed point for A,B,S and T.

For the uniqueness, suppose there is another fixed point w, by using (1) we get

$$\Big(1+a(\int_0^{d(Az,Bw)}\varphi(t))^p\Big)(\int_0^{d(Sz,Tw)}\varphi(t)dt)^p<$$

$$a\Big(\int_0^{d(Az,Tw)}\varphi(t)dt)^p\Big).\Big(\int_0^{d(Bw,Sz)}\varphi(t)dt\Big)^p\Big)+\alpha\Big(\int_0^{d(Az,Bw)}\varphi(t)dt\Big)^p\Big)$$

$$+\beta\tau\Big(0,0,\Big(\int_0^{d(Az,Tw)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Bw,Sz)}\varphi(t)dt\Big)^p\Big),$$

since z and w are fixed points, and so

$$\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p\leq(\alpha+\beta)\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p$$
$$<\left(\int_0^{d(z,w)}\varphi(t)dt\right)^p,$$

which is a contradiction, then $z=w$. □

Theorem 3.1 improves Theorem 2 of Chauhan et al. [7] and some main results of Djoudi and Aliouche [11] and Theorem 2.5 in [25].

If $\alpha=0$, we obtain the following natural result:

Corollary 3.2. *Let $A,B,S,T : X \to X$, be self mappings such that*

$$\Big(\int_0^{d(Sx,Ty)} \varphi(t)\Big)^p < \alpha\Big(\int_0^{d(Ax,By)} \varphi(t)dt\Big)^p\Big) + \beta\tau\left(\begin{array}{c} \Big(\int_0^{d(Ax,Sx)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Ty)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Ax,Ty)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Sx)} \varphi(t)dt\Big)^p \end{array}\right)$$

where α,β are non negative numbers such $\alpha+\beta<1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable function which defined in Theorem 3.1. Suppose that the pairs $\{A,S\}$ and $\{B,T\}$ are A-subsequentially continuous and A-compatible of type(E), then A,B,S and T have a unique common fixed point in X.

Corollary 3.2 improves Corollary 2 of Chauhan et al. [7] and Corollary 2 in [10].

If we take

$$\tau(x_1,x_2,x_3,x_4) = \max\{x_1,x_2,\sqrt{x_1x_3},\sqrt{x_3x_4}\},$$

we get the following corollary:

Corollary 3.3. *Let A,B,S and T self mappings of metric space (X,d) such that*

$$\Big(1+a(\int_0^{d(Ax,By)} \varphi(t))\Big)\int_0^{d(Sx,Ty)} \varphi(t)dt <$$
$$a\left(\begin{array}{c} (\int_0^{d(Ax,Sx)} \varphi(t)dt).(\int_0^{d(By,Ty)} \varphi(t)dt) \\ +(\int_0^{d(Sx,By)} \varphi(t)dt).(\int_0^{d(Ax,Ty)} \varphi(t)dt) \end{array}\right) + \alpha\Big(\int_0^{d(Ax,By)} \varphi(t)dt\Big)$$
$$+(1-\alpha)\max\left\{\begin{array}{c} \int_0^{d(Ax,Sx)} \varphi(t)dt, \int_0^{d(By,Ty)} \varphi(t)dt, \\ \Big(\int_0^{d(Ax,Ty)} \varphi dt\Big)^{\frac{1}{2}}.\Big(\int_0^{d(By,Sx)} \varphi(t)dt\Big)^{\frac{1}{2}}, \\ \Big(\int_0^{d(fx,Ty)} \varphi(t)dt\Big)^{\frac{1}{2}}.\Big(\int_0^{d(By,Sx)} \varphi(t)dt\Big)^{\frac{1}{2}} \end{array}\right\},$$

where $0 \le \alpha < 1$, and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable function which defined in Theorem 3.1. Suppose that

1. *the pair $\{A,S\}$ is A-subsequentially continuous and A-compatible of type (E),*
2. *the pair and $\{B,T\}$ is B-subsequentially continuous and B-compatible of type (E),*

then A,B,S and T have a unique common fixed point in X.

Corollary 3.3 improves and generalizes Theorem 2.5 of Pathak and Shahazad in [25].

Corollary 3.4. *Let A,B,S and T be self mappings such that*

$$(1+ad^p(Ax,By))d^p(Sx,Ty) < a[d^p(Ax,Sx)d^p(Bt,Ty)+d^p(Ax,Sx)d^p(Ax,Sx)] + \alpha d^p(Ax,By)+\beta\tau\begin{pmatrix} d^p(Ax,Sx),d^p(By,Ty), \\ d(Ax,Ty),d^p(By,Sx)\end{pmatrix},$$

if two of the following conditions hold:

1. *the pair $\{A,S\}$ and $\{B,T\}$ are subsequentially continuous,*
2. *the pair $\{A,S\}$ is A-compatible or S-compatible of type (E),*
3. *the pair $\{B,T\}$ is B-compatible or T compatible of type (E),*

then A,B,S and T have a unique common fixed point in X.

Let Λ be a set of all continuous function $\Lambda:\mathbb{R}^5_+\to\mathbb{R}_+$, such $\lambda(0,0,x,x,x)=kx$, where $0<k<1$.

Theorem 3.5. *Let A,B,S and T be self mappings on metric space (X,d) such for all x,y in X we have:*

$$\Big(1+a\Big(\int_0^{d(Ax,By)}\varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(Sx,Ty)}\varphi(t)dt\Big)^p < \lambda\begin{pmatrix} \Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Ty)}\varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Ax,Ty)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Sx)}\varphi(t)dt\Big)^p\end{pmatrix} \tag{2}$$

where $\lambda\in\Lambda$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon>0,\int_0^\varepsilon\varphi(t)dt>0$, assume that the two pairs $\{A,S\}$ and $\{B,T\}$ are weakly subsequentially continuous and compatible of type (E), then $A,B,,S$ and T have a unique common fixed point in X.

Proof. As in proof of Theorem 3.1, z is a coincidence point for A and S and t is a coincidence point for B and T, where

$$\lim_{n\to\infty}By_n=t \text{ and } \lim_{n\to\infty}Ax_n=z,$$

we claim $Az=Bt$, if not by using (2) we get

$$\Big(\int_0^{d(Az,Bt)}\varphi(t)\Big)^p=\Big(\int_0^{d(Sz,Tt)}\varphi(t)dt\Big)^p<$$

$$< \lambda \begin{pmatrix} 0, 0, (\int_0^{d(Az,Tt)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Sz,Bt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p \end{pmatrix}$$

since $d(Sz,Bt) \leq d(fz,gt)$ and $d(Az,Tt) \leq d(Az,Bt)$, we get

$$\begin{aligned} \Big(\int_0^{d(Az,Bt)} \varphi(t)\Big)^p &< \lambda \begin{pmatrix} 0, 0, (\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p \end{pmatrix} \\ &< k\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p \\ &< \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p, \end{aligned}$$

which is a contradiction, then $Az = Bt$. Now, we prove $z = Az$, if not by using (2) we get

$$\Big(\int_0^{d(Sx_n,Tt)} \varphi(t)dt\Big)^p \leq \Big(1 + a\Big(\int_0^{d(Ax_n,Bt)} \varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(Sx_n,Tt)} \varphi(t)dt\Big)^p <$$

$$\lambda \begin{pmatrix} \Big(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Bt,Tt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Bt,Sx_n)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Ax_n,Bt)} \varphi(t)dt\Big)^p \end{pmatrix},$$

letting $n \to \infty$, we get

$$\Big(1 + a\Big(\int_0^{d(z,Bt)} \varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(z,Tt)} \varphi(t)dt\Big)^p \leq$$

$$\lambda \begin{pmatrix} 0, 0, \Big(\int_0^{d(z,Tt)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Bt,z)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(z,Bt)} \varphi(t)dt\Big)^p \end{pmatrix},$$

consequently we get

$$\begin{aligned} \Big(\int_0^{d(z,Az)} \varphi(t)\Big)^p &\leq \lambda \begin{pmatrix} 0, 0, \Big(\int_0^{d(z,Az)} \varphi(t)\Big)^p, \\ \Big(\int_0^{d(z,Az)} \varphi(t)\Big)^p, \Big(\int_0^{d(z,Az)} \varphi(t)\Big)^p \end{pmatrix} \\ &= k\Big(\int_0^{d(z,Az)} \varphi(t)\Big)^p \\ &< \Big(\int_0^{d(z,Az)} \varphi(t)\Big)^p, \end{aligned}$$

which is a contradiction, then $z = Az = Sz$, nextly we claim $z = t$, if not by using (2), we get

$$\Big(1 + a\Big(\int_0^{d(Ax_n,By_n)} \varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(Sx_n,Ty_n)} \varphi(t)dt\Big)^p <$$

$$< \lambda \begin{pmatrix} \left(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt\right)^p, \left(\int_0^{d(By_n,Ty_n)} \varphi(t)dt\right)^p, \left(\int_0^{d(Ax_n,Ty_n)} \varphi(t)dt\right)^p, \\ \left(\int_0^{d(By_n,Sx_n)} \varphi(t)dt\right)^p, \left(\int_0^{d(Ax_n,By_n)} \varphi(t)dt\right)^p \end{pmatrix},$$

letting $n \to \infty$, we get

$$\begin{aligned} \left(1 + a\left(\int_0^{d(z,t)} \varphi(t)dt\right)^p\right)\left(\int_0^{d(z,t)} \varphi(t)dt\right)^p &\leq \lambda \begin{pmatrix} 0, 0, \left(\int_0^{d(z,t)} \varphi(t)\right)^p, \\ \left(\int_0^{d(z,t)} \varphi(t)\right)^p \end{pmatrix} \\ &= k\left(\int_0^{d(z,t)} \varphi(t)\right)^p \\ &< \left(\int_0^{d(z,t)} \varphi(t)\right)^p, \end{aligned}$$

which is a contradiction, then z is a common fixed point for A, B, S and T. For the uniqueness, it is similar as in proof of Theorem 1. □

Corollary 3.6. *For four self-mappings A, B, S and T on metric space (X, d), satisfying for all $x, y \in X$:*

$$(1 + ad^p(Ax, By))d^p(Sx, Ty) < \lambda \begin{pmatrix} d^p(Ax, Sx), d^p(By, Ty), d^p(Ax, Ty), \\ d^p(By, Sx), d^p(Ax, By) \end{pmatrix},$$

if the pairs $\{A, S\}$ and $\{B, T\}$ are A-subsequentially continuous and A-compatible of type (E), then A, B, S and T have a unique common fixed point in X.

Remark 3.7. Corollary 3.6 improves Corollary 4 in [7] and generalizes of [9, Theorem 3.1].

Acknowledgments

The authors would like to express their sincere thanks to Professor Renato Maggioni and the Editorial Board of "Le Matematiche" for supporting this work.

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