ON PROPERTIES OF THE NUMBERS COPRIME WITH THE PRIMES UP TO $p_n$

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In this paper we investigate about the effective distribution of the numbers coprime with the primes up to $p_n$. More precisely we prove that these numbers form a periodically monotone sequence $\Psi_{p_n}$. Then we examine some properties of $\Psi_{p_n}$ which, in a certain sense, are transferred to the sequence of primes. Moreover we study the distribution of twin and cousin terms within the sequence $\Psi_{p_n}$. This study also makes furthermore strongly plausible that the set of twin primes as well as the set of cousin primes is infinite.

Introduction.

In this paper we investigate, by elementary method, about the effective distribution of the numbers coprime with the primes up to $p_n$, i.e. the natural numbers that are greater than $p_n$ and are not divisible by $p_1, p_2, \ldots, p_n$ (here $\{p_n\}$ denotes, as usual, the sequence of prime numbers). Some properties of these numbers were studied many years ago by J. Deschamps and H.J.S. Smith (see [2] p. 439).

More precisely we prove that, for every prime number $p_n$, the numbers

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Entrato in redazione il 16 marzo 2006.

Mathematics Subject Classifications: 11A41, 11B25, 40A05, 40A20.
cosequences of the primes up to \( p_n \) form a periodically monotone sequence that we denote by \( \Psi_{p_n} \). Then we examine some properties of \( \Psi_{p_n} \) which, in a certain sense, are transferred to the sequence of primes. From this perspective, in the first section, we prove that the mean distance between two consecutive terms of \( \Psi_{p_n} \) is a property shared by the primes less than \( p_{n+1}^2 \). In the second section we study the distribution of twin and cousin terms of \( \Psi_{p_n} \) (two consecutive terms \( \psi_k \) and \( \psi_{k+1} \) of \( \Psi_{p_n} \) are called twin terms if \( \psi_{k+1} - \psi_k = 2 \) and similarly are called cousin terms if \( \psi_{k+1} - \psi_k = 4 \)). This study and in particular the theorem 2.1 agrees with the conjecture B of Hardy and Littlewood (see [5] p. 19) and extensively explains the experimental fact that the numbers \( \pi_2(x) \) (of the pairs of twin primes less than or equal to \( x \in \mathbb{N} \)) and \( \pi_4(x) \) (of the pairs of cousin primes less than or equal to \( x \in \mathbb{N} \)) are almost the same (see [8]). Moreover this study and the theorem 2.2 makes furthermore strongly plausible that the set of twin primes as well as the set of cousin primes is infinite. In the sequel we put as usual

\[
\mathbb{N} = \{1, 2, 3, \ldots \} \quad \text{and} \quad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \}.
\]

Moreover if \( \Psi \) is a periodically monotone sequence\(^{(1)} \) we denote by \( \mu_d(\Psi) \) the number of couples \((\psi_n, \psi_{n+1})\) of consecutive principal terms of \( \Psi \) such that the difference \( \psi_{n+1} - \psi_n \) is equal to \( d \) (if \( d = 2 \) the terms \((\psi_n, \psi_{n+1})\) are called twin terms and similarly if \( d = 4 \) the terms \((\psi_n, \psi_{n+1})\) are called cousin terms). Finally we denote by \( R(\frac{m}{n}) \) the remainder of the integral division of \( m \) by \( n \) for \( m, n \in \mathbb{N}, \, m \geq n \).

1. The distribution of the terms of \( \Psi_{p_n} \).

Let us begin with the following theorem.

**Theorem 1.1.** For every prime number \( p_n \) there exists a periodically monotone sequence \( \Psi_{p_n} \) (of natural numbers greater than \( p_n \)) with

\(^{(1)} \) A sequence \( \{x_n\} \) in \( \mathbb{R} \) is called periodically monotone if there exist a natural number \( q \) and a real number \( k \) such that

\[
x_{n+q} = x_n + k \quad \forall \, n \in \mathbb{N}
\]

The lowest natural number \( q \) for which \((*)\) holds is called period. The constant \( k \) is called monotony constant. The terms \( x_1, x_2, \ldots, x_q \) are called principal terms of \( \{x_n\} \). The periodically monotone sequences generalize the periodic sequences and the arithmetic progressions (see [3]).
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**period**

$$1 \cdot 2 \cdot 4 \ldots \cdot (p_n - 1),$$

whose monotony constant is

$$2 \cdot 3 \cdot 5 \ldots \cdot p_n$$

and such that every term $p$ of $\Psi_{p_n}$ satisfies the condition

$$R\left(\frac{p}{p_r}\right) \neq 0$$

$\forall \ r = 1, 2, 3, 4, 5, ..., n.$

**Proof.** Let us take as sequence $\Psi_2$ the sequence

$$\{3 + 2k\}_{k \in \mathbb{N}_0},$$

which is an arithmetic progression with difference 2. Now, for finding $\Psi_3$, let us consider the sequence $\{3 + 2k\}_{k \in \mathbb{N}_0}$ and search $k$ such that

(1.1) $$R\left(\frac{3 + 2k}{3}\right) \neq 0$$

Computing

$$R\left(\frac{3 + 2k}{3}\right)$$

for $k = 0, 1, 2$, we obtain respectively the numbers 0, 2, 1; therefore, by observing that the sequence

$$\left\{R\left(\frac{3 + 2k}{3}\right)\right\}$$

is periodic with period 3, it follows that the condition 1.1 is verified if

$$k = 1 + 3h \quad \text{or} \quad k = 2 + 3h \quad (h \in \mathbb{N}_0).$$

In this way we get the 2 sequences

$$\{5 + 6k\}_{k \in \mathbb{N}_0}, \quad \{7 + 6k\}_{k \in \mathbb{N}_0},$$

(which are arithmetic progressions with difference 6). So we obtain as $\Psi_3$ the periodically monotone sequence whose principal terms are

5, 7,

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(2) Let us observe that the period is equal to the value of the Euler's function $\varphi(m)$ calculated for $m = 2 \cdot 3 \cdot 5 \ldots \cdot p_n$. 

whose period is 2 and whose monotony constant is 6. Now, for finding \( \Psi_5 \), let us search \( k \) such that
\[
R \left( \frac{5 + 6k}{5} \right) \neq 0, \quad \text{and} \quad R \left( \frac{7 + 6k}{5} \right) \neq 0.
\]

Computing all the remainders for \( k = 0, 1, 2, 3, 4 \) and taking into account that the sequences
\[
\left\{ R \left( \frac{5 + 6k}{5} \right) \right\}, \quad \left\{ R \left( \frac{7 + 6k}{5} \right) \right\}
\]
are periodic with period 5, we obtain the following 8 sequences (which are arithmetic progressions with difference \( 2 \cdot 3 \cdot 5 = 30 \) and \( k \in \mathbb{N}_0 \):)
\[
\{11 + 30k\}, \quad \{17 + 30k\}, \quad \{23 + 30k\}, \quad \{29 + 30k\}, \\
\{7 + 30k\}, \quad \{13 + 30k\}, \quad \{19 + 30k\}, \quad \{31 + 30k\}.
\]

Thus we obtain as \( \Psi_5 \) the periodically monotone sequence whose principal terms are
\[
7, 11, 13, 17, 19, 23, 29, 31
\]
whose period is \( 8 = 1 \cdot 2 \cdot 4 \) and whose monotony constant is \( 30 = 2 \cdot 3 \cdot 5 \). The reasoning can be iterated so that the proof is completed.

**Remark 1.1.** The terms of the sequence \( \Psi_{pn} \) represent all the natural numbers, greater than \( pn \), that are not divisible by \( p_1, p_2, \ldots, p_n \), thus the terms of the sequence \( \Psi_{pn} \) that are in the interval \( ]p_n, p_{n+1}[ \) or, more generally, that are less than \( p_{n+1}^2 \) are all prime numbers. Moreover it follows easily that the formula
\[
(1.3) \quad \begin{cases}
p_1 = 2, \\
p_{n+1} = \min \Psi_{pn} \quad \forall \ n \geq 1
\end{cases}
\]
is a (very simple) recursive formula for the sequence of primes, that gives also a direct and simple proof of the infinity of primes because, for the theorem 1.1, the set of the numbers coprime with the primes up to \( p_n \) is not empty for all \( n \in \mathbb{N} \).

We have also the following theorems.

**Theorem 1.2.** The recursive formula\(^{(3)}\)
\[
(1.4) \quad \Psi_{pn+1} = \Psi_{pn} - p_{n+1} \Psi_{pn} - \{p_{n+1}\}
\]
\(^{(3)}\) In this formula \( \Psi_{pn} \) as well as \( \Psi_{pn+1} \) represent the ordered (in the natural way) set of the terms of the sequences \( \Psi_{pn} \) and \( \Psi_{pn+1} \).
holds.

Proof. To prove the theorem it is sufficient to prove that the terms of the sequence

\[ p_{n+1} \Psi_{p_n} \]

represent all the terms of \( \Psi_{p_n} \), greater than \( p_{n+1} \), that are divisible by \( p_{n+1} \). Indeed for any term \( p \) of the sequence \( \Psi_{p_n} \), the number \( p_{n+1} \cdot p \) is divisible, obviously, by \( p_{n+1} \), but it is not divisible by \( p_1, p_2, \ldots, p_n \) and so it is a term of the sequence \( \Psi_{p_n} \).

Conversely if \( p \) is a term of the sequence \( \Psi_{p_n} \), greater than \( p_{n+1} \), that is divisible by \( p_{n+1} \), we must have

\[ p = p_{n+1} \cdot q, \]

where \( q \) is not divisible by \( p_1, p_2, \ldots, p_n \); therefore \( p \) is a term of the sequence \( p_{n+1} \Psi_{p_n} \). \( \square \)

Remark 1.2. Taking into account the remark 1.1, the recursive formula 1.4 can be used to generate the sequence of primes more fastly than using the recursive formula 1.1 of [4].

Theorem 1.3. The principal terms of the sequence \( \Psi_{p_n} \) are obtained from the principal terms of \( \Psi_{p_{n-1}} \) by deleting the \( T \) numbers

\[ \psi_1, \; p_n \psi_1, \; p_n \psi_2, \; \ldots, \; p_n \psi_{T-1} \]

in the matrix

\[ A_{p_n} = \begin{pmatrix} \psi_1 & \psi_2 & \ldots & \psi_T \\ \psi_1 + k & \psi_2 + k & \ldots & \psi_T + k \\ \ldots & \ldots & \ldots & \ldots \\ \psi_1 + (p_n - 1)k & \psi_2 + (p_n - 1)k & \ldots & \psi_T + (p_n - 1)k \end{pmatrix} \]

where \( T \) and \( k \) represent, respectively, the period and the monotony constant of \( \Psi_{p_{n-1}} \) and \( \psi_1, \psi_2, \ldots, \psi_T \) are the principal terms of the same sequence.

Proof. By virtue of the theorem 1.1, for obtaining the principal terms of \( \Psi_{p_n} \) we must take the first \( p_n \) terms of each arithmetic progression

\[ \{ \psi_i + h k \}_{h \in \mathbb{N}_0} \quad (i = 1, 2, \ldots, T) \]

and eliminate among them the term such that

\[ R \left( \frac{\psi_i + h k}{p_n} \right) = 0. \]
In this way we delete $T$ terms in the matrix $A_{p_n}$; but for the theorem 1.2 these $T$ terms are the numbers

$$\psi_1, \ p_n\psi_1, \ p_n\psi_2, \ \ldots, \ p_n\psi_{T-1}$$

and the proof is completed. \Box

\textbf{Example 1.1.} For the principal terms of the sequence $\Psi_5$ are the 8 numbers

$$7, 11, 13, 17, 19, 23, 29, 31$$

and the monotony constant of $\Psi_5$ is 30, according to the theorem 1.3 we must eliminate in the matrix

$$A_7 = \begin{pmatrix}
7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \\
37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\
67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\
97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\
127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\
157 & 161 & 163 & 167 & 169 & 173 & 179 & 181 \\
187 & 191 & 193 & 197 & 199 & 203 & 209 & 211
\end{pmatrix}$$

the numbers

$$7, 49, 77, 91, 119, 133, 161, 203.$$ 

Therefore the principal terms of the sequence $\Psi_7$ are the 48 numbers of the table

$$\begin{array}{cccccccc}
11 & 13 & 17 & 19 & 23 & 29 & 31 \\
37 & 41 & 43 & 47 & 53 & 59 & 61 \\
67 & 71 & 73 & 79 & 83 & 89 \\
97 & 101 & 103 & 107 & 109 & 113 & 121 \\
127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\
157 & 161 & 163 & 167 & 169 & 173 & 179 & 181 \\
187 & 191 & 193 & 197 & 199 & 203 & 209 & 211
\end{array}$$

\textbf{Remark 1.3.} If

$$\psi_1^{(n)}, \ \psi_2^{(n)}, \ \ldots, \ \psi_{T_n}^{(n)}$$

denote the principal terms of the sequence $\Psi_{p_n}$, we have

$$\psi_1^{(n)} = p_{n+1} \quad \text{and} \quad \psi_{T_n}^{(n)} = k_n + 1 \quad \forall \ n \in \mathbb{N},$$
where $T_n$ and $k_n$ represent respectively the period and the monotony constant of $\Psi_{p_n}$. Indeed the first equality is a consequence of the remark 1.1 and the second can be proved easily by induction.

Moreover it results

$$p_{n+1}^{(n)} \psi_{T_{n}} > \psi_{T_{n}} + (p_{n+1} - 1)k_n \quad \forall \ n \in \mathbb{N}.$$ 

Therefore the principal terms of the sequence $\Psi_{p_n}$ ($\forall \ n \in \mathbb{N}$) are distributed in the interval $[p_{n+1}, \ p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1]$.

**Remark 1.4.** In the matrix considered in the theorem 1.3 we have $\forall \ n \in \mathbb{N}$

$$\psi_1 + ik - (\psi_T + (i - 1)k) = p_n - 1 \quad \text{for} \ i = 1, 2, ..., p_n - 1,$$

thus the gap $p_n - 1$ between two consecutive terms in the matrix $A_{p_n}$ appears at least $p_n - 1$ times.

**Remark 1.5.** The numbers

$$\psi_1, \ p_n \psi_1, \ p_n \psi_2, \ldots, \ p_n \psi_{T-1}$$

considered in the theorem 1.3 are distributed in the matrix $A_{p_n}$ in a such way that in every column is located one and only one of them. In particular $\psi_1 = p_n$ is located in the first column and in the first row; moreover, for $p_n \geq 11$, $p_n \psi_1 = p_{n}^2$ is located in the first row.

The following definitions will be used in the sequel.

**Definition 1.1.** The number

$$\rho_n = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n}{1 \cdot 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (p_n - 1)},$$

that is the quotient of the monotony constant by the period of the sequence $\Psi_{p_n}$, is called mean distance between two consecutive terms of $\Psi_{p_n}$.

**Definition 1.2.** Let $(a, b)$ an interval ($a \geq p_n$). The number

$$\frac{b - a}{\rho_n}$$

is called mean number of the terms of the sequence $\Psi_{p_n}$ that are in the interval $(a, b)$.

**Definition 1.3.** Let $x \in \mathbb{N}$, $x \geq 3$ and let $\pi(x)$ the number of primes $p$ such that $p \leq x$. The number

$$\frac{x}{\pi(x)}$$
is called mean distance between two consecutive primes less than or equal to \( x \). 

The following theorem allows to consider the quantity \( 2e^{-\gamma} \rho_n \) (\( \gamma \) is the Euler constant) as an approximation of the mean distance between two consecutive primes less than \( p_{n+1}^2 \). Thus \( \rho_n \), the mean distance between two consecutive terms of \( \Psi_{p_n} \), appears as a property which is transferred from the sequence \( \Psi_{p_n} \) to the sequence of primes.

**Theorem 1.4.** The number \( \pi(p_{n+1}^2) \) of prime numbers less than \( p_{n+1}^2 \) is asymptotic, as \( n \) goes to \( \infty \), to

\[
\frac{e^\gamma}{2} \frac{p_{n+1}^2}{\rho_n},
\]

where \( \gamma \) denotes the Euler constant and \( \rho_n \) is the mean distance between two consecutive terms of \( \Psi_{p_n} \).

**Proof.** We have

\[
\frac{\pi(p_{n+1}^2)}{\rho_n} = \frac{\pi(p_{n+1}^2)}{\log p_{n+1}^2} \frac{p_{n+1}^2}{\rho_n} = \frac{\pi(p_{n+1}^2)}{\log p_{n+1}^2} \rho_n = 2 \log p_{n+1} \rho_n
\]

(1.5)

Now for the prime number theorem (see [7] p. 289) it results

\[
\lim_{n \to \infty} \frac{\pi(p_{n+1}^2)}{p_{n+1}^2} = 1
\]

and for the Mertens’ theorem (see [6] p. 351) it results

\[
\lim_{n \to \infty} \frac{\rho_{n+1}}{\log p_{n+1}} = e^\gamma,
\]

therefore from 1.5 the thesis follows. \( \square \)
2. The distribution of twin and cousin terms in the sequence $\Psi_{p_n}$.

Taking into account the theorem 1.3 we are able to prove the following theorem.

**Theorem 2.1.** For $p_n \geq 5$ we have

\[
\mu_2(\Psi_{p_n}) = \mu_4(\Psi_{p_n}) = 3 \cdot 5 \cdot 9 \cdot \ldots \cdot (p_n - 2).
\]

**Proof.** Let us begin by observing that, for $p_n \geq 7$, in the matrix $A_{p_n}$, considered in the theorem 1.3, the number of couples of consecutive terms whose difference is 2 or 4 is given respectively by

\[
\mu_2(\Psi_{p_n-1}) \cdot p_n \quad \text{and} \quad \mu_4(\Psi_{p_n-1}) \cdot p_n.
\]

On the other hand, by effect of the elimination of the $T$ numbers

\[
\psi_1, \quad p_n \psi_1, \quad p_n \psi_2, \quad \ldots, \quad p_n \psi_{T-1}
\]

in the matrix $A_{p_n}$, the number of couples of consecutive terms, whose difference is 2 or 4, that are dropped is given respectively by

\[
2\mu_2(\Psi_{p_n-1}) \quad \text{and} \quad 2\mu_4(\Psi_{p_n-1}).
\]

In fact if $\psi_i$ and $\psi_{i+1}$ are principal twin terms of $\Psi_{p_n-1}$, the element that must be eliminated in the column $i$ of $A_{p_n}$ is of the form $\psi_i + h_1 k$, where $k$ is the monotony constant of $\Psi_{p_n-1}$, $h_1 \in \{0, 1, 2, \ldots, p_n - 1\}$ and $h_1$ is such that

\[
R\left(\frac{\psi_i + h_1 k}{p_n}\right) = 0.
\]

From this it follows

\[
R\left(\frac{\psi_{i+1} + h_1 k}{p_n}\right) = R\left(\frac{2 + \psi_i + h_1 k}{p_n}\right) = 2,
\]

therefore the element that must be deleted in the column $i + 1$ of $A_{p_n}$ is of the form $\psi_{i+1} + h_2 k$, where $h_2 \in \{0, 1, 2, \ldots, p_n - 1\}$ and $h_2 \neq h_1$.

This implies that the elements $\psi_i + h_1 k$ and $\psi_{i+1} + h_2 k$ are located in two different rows of $A_{p_n}$. Therefore the number of couples of twin terms that are dropped in the matrix $A_{p_n}$ is $2\mu_2(\Psi_{p_n-1})$. Obviously the same considerations hold if $\psi_i$ and $\psi_{i+1}$ are cousin terms.

Thus we obtain

\[
\mu_2(\Psi_{p_n}) = \mu_2(\Psi_{p_n-1}) \cdot p_n - 2\mu_2(\Psi_{p_n-1}) = \mu_2(\Psi_{p_n-1})(p_n - 2)
\]
and
\[(2.3) \quad \mu_4(\Psi_{p_n}) = \mu_4(\Psi_{p_{n-1}}) \cdot p_n - 2\mu_4(\Psi_{p_{n-1}}) = \mu_4(\Psi_{p_{n-1}})(p_n - 2).\]

But we also have
\[\mu_2(\Psi_5) = \mu_4(\Psi_5) = 3,\]
therefore from 2.2 and 2.3 the thesis follows easily by induction. □

Remark 2.1. The formula
\[\mu_2(\Psi_{p_n}) = 3 \cdot 5 \cdot 9 \cdot \ldots \cdot (p_n - 2)\]
that holds for \(p_n \geq 5\) can be also proved in the following way. Let us begin by observing that the terms of the sequence \(\Sigma_{p_n}\) considered in the theorem 3.1 of [4] represent all the natural numbers \(p\) greater than \(p_n\) such that \(p\) and \(p + 2\) are not divisible by \(p_1, p_2, ..., p_n\). Therefore, if \(p\) is a principal term of \(\Sigma_{p_n}\) then \(p\) and \(p + 2\) are principal terms of \(\Psi_{p_n}\). Conversely, if \(p\) and \(p + 2\) are principal terms of \(\Psi_{p_n}\) then \(p\) is a principal term of \(\Sigma_{p_n}\). Taking into account the theorem 3.1 of [4] the asserted formula follows easily.

Finally we observe that the computation of \(\mu_2(\Psi_{p_n})\) can be derived from the (more complicated) considerations about twin primes made in [6] on p. 412.

The following definitions are useful to state the next corollary.

Definition 2.1. Let \((a, b)\) an interval \((a \geq p_n)\). The number
\[\frac{b - a}{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n} \mu_2(\Psi_{p_n})\]
is called mean number of the pairs of twin terms of the sequence \(\Psi_{p_n}\) that are in the interval \((a, b)\).

Definition 2.2. Let \((a, b)\) an interval \((a \geq p_n)\). The number
\[\frac{b - a}{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n} \mu_4(\Psi_{p_n})\]
is called mean number of the pairs of cousin terms of the sequence \(\Psi_{p_n}\) that are in the interval \((a, b)\).

The following corollary follows immediately from the theorem 2.1.
Corollary 2.1. The mean number $\sigma_{p_n}$ of pairs of twin terms (or of pairs of cousin terms) of the sequence $\Psi_{p_n}$ that are in the interval $[p_n^2, p_{n+1}^2]$ is given by

$$\sigma_{p_n} = \frac{p_{n+1}^2 - p_n^2}{2 \cdot 3 \cdot 5 \cdot 7 \cdots (p_n - 2)}.$$ 

Remark 2.2. The theorem 2.1 and the corollary 2.1 explain well the experimental fact that the numbers $\pi_2(x)$ (of the pairs of twin primes less than or equal to $x \in \mathbb{N}$) and $\pi_4(x)$ (of the pairs of cousin primes less than or equal to $x \in \mathbb{N}$) are almost the same (see [8]).

The following theorem improves strongly the theorem 3.4 of [4].

Theorem 2.2. There exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that

$$\lim_{k \to \infty} \sigma_{p_{n_k}} = +\infty,$$

where $\sigma_{p_n}$ denotes the mean number of pairs of twin terms (or of pairs of cousin terms) of $\Psi_{p_n}$ that are in the interval $[p_n^2, p_{n+1}^2]$.

Proof. Let us begin by setting

$$d_n = p_{n+1} - p_n \quad \forall \ n \in \mathbb{N}$$

and let be $d_{n_k}$ such that \(^{(4)}\)

$$\frac{d_{n_{k+1}}}{d_{n_k}} \geq 2 \quad \forall \ k \in \mathbb{N}.$$ 

We have

\[(2.4) \quad \sigma_{p_{n_k}} = \frac{p_{n_{k+1}}^2 - p_{n_k}^2}{2 \cdot 3 \cdot 5 \cdot 7 \cdots (p_{n_k} - 2)} \geq \frac{2d_{n_k}p_{n_k}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (p_{n_k} - 2)}.$$

Setting

$$S_k = \frac{2d_{n_k}p_{n_k}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (p_{n_k} - 2)}.$$ 

\(^{(4)}\) We can do this in consequence of the theorem 5 of [6].
it results

\[(2.5) \quad \frac{S_{k+1}}{S_k} = \frac{d_n}{d_{n-1}} \frac{p_{n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n (p_{n+1} - 2)} \geq 2, \]

because the sequence

\[\left\{ \frac{p_n}{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n} \right\}_{n \in \mathbb{N}} \]

is not decreasing for it is a subsequence of the sequence

\[\left\{ \frac{p_n}{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n} \right\}_{n \in \mathbb{N}} \]

which is not decreasing (see theorem 3.4 of [4]).

From 2.5 it follows

\[(2.6) \quad \lim_{k \to \infty} S_k = +\infty, \]

and finally from 2.6 and 2.4 we get the thesis. \(\square\)

**Remark 2.3.** The previous theorems and the theorem 3.4 of [4] make furthermore strongly plausible that the set of twin primes (as well as the set of cousin primes) is infinite.

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