LE MATEMATICHE
Vol. LXI (2006) - Fasc. II, pp. 317-329

ON PROPERTIES OF THE NUMBERS COPRIME WITH THE PRIMES UP TO p_n

GIOVANNI FIORITO

In this paper we investigate about the effective distribution of the numbers coprime with the primes up to p_n.

More precisely we prove that these numbers form a periodically monotone sequence Ψ_{p_n}. Then we examine some properties of Ψ_{p_n} which, in a certain sense, are transferred to the sequence of primes. Moreover we study the distribution of twin and cousin terms within the sequence Ψ_{p_n}. This study also makes furthermore strongly plausible that the set of twin primes as well as the set of cousin primes is infinite.

Introduction.

In this paper we investigate, by elementary method, about the effective distribution of the numbers coprime with the primes up to p_n, i.e. the natural numbers that are greater than p_n and are not divisible by $p_1, p_2, \dots, p_n$ (here $\{p_n\}$ denotes, as usual, the sequence of prime numbers). Some properties of these numbers were studied many years ago by J. Deschamps and H.J.S. Smith (see [2] p. 439).

More precisely we prove that, for every prime number p_n, the numbers

Entrato in redazione il 16 marzo 2006.
Mathematics Subject Classifications: 11A41, 11B25, 40A05, 40A20.

coprime with the primes up to p_n form a periodically monotone sequence that we denote by Ψ_{p_n}. Then we examine some properties of Ψ_{p_n} which, in a certain sense, are transferred to the sequence of primes. From this perspective, in the first section, we prove that the mean distance between two consecutive terms of Ψ_{p_n} is a property shared by the primes less than p_{n+1}^2. In the second section we study the distribution of twin and cousin terms of Ψ_{p_n} (two consecutive terms ψ_k and ψ_{k+1} of Ψ_{p_n} are called twin terms if $\psi_{k+1} - \psi_k = 2$ and similarly are called cousin terms if $\psi_{k+1} - \psi_k = 4$). This study and in particular the theorem 2.1 agrees with the conjecture B of Hardy and Littewood (see [5] p. 19) and extensively explains the experimental fact that the numbers $\pi_2(x)$ (of the pairs of twin primes less than or equal to $x \in \mathbb{N}$) and $\pi_4(x)$ (of the pairs of cousin primes less than or equal to $x \in \mathbb{N}$) are almost the same (see [8]). Moreover this study and the theorem 2.2 makes furthermore strongly plausible that the set of twin primes as well as the set of cousin primes is infinite. In the sequel we put as usual

$$\mathbb{N} = \{1, 2, 3, ...\} \quad \text{and} \quad \mathbb{N}_0 = \{0, 1, 2, 3, ...\}.$$

Moreover if Ψ is a periodically monotone sequence[(1)] we denote by $\mu_d(\Psi)$ the number of couples (ψ_n, ψ_{n+1}) of consecutive principal terms of Ψ such that the difference $\psi_{n+1} - \psi_n$ is equal to d (if $d = 2$ the terms (ψ_n, ψ_{n+1}) are called twin terms and similarly if $d = 4$ the terms (ψ_n, ψ_{n+1}) are called cousin terms). Finally we denote by $R(\frac{m}{n})$ the remainder of the integral division of m by n for $m, n \in \mathbb{N},\ m \geq n$.

1. The distribution of the terms of Ψ_{p_n}.

Let us begin with the following theorem.

Theorem 1.1. *For every prime number p_n there exists a periodically monotone sequence Ψ_{p_n} (of natural numbers greater than p_n) with*

---

(1) A sequence $\{x_n\}$ in $\mathbb{R}$ is called periodically monotone if there exist a natural number q and a real number k such that

$$(*) \qquad x_{n+q} = x_n + k \quad \forall\, n \in \mathbb{N}$$

The lowest natural number q for which $(*)$ holds is called period. The constant k is called monotony constant. The terms $x_1, x_2, ..., x_q$ are called principal terms of $\{x_n\}$. The periodically monotone sequences generalize the periodic sequences and the arithmetic progressions (see [3]).

period[2]

$$1 \cdot 2 \cdot 4 \cdot \ldots \cdot (p_n - 1),$$

whose monotony constant is

$$2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_n$$

and such that every term p *of* Ψ_{p_n} *satisfies the condition*

$$R\left(\frac{p}{p_r}\right) \neq 0$$

$\forall\ r = 1, 2, 3, 4, 5, ..., n.$

Proof. Let us take as sequence Ψ_2 the sequence

$$\{3 + 2k\}_{k \in \mathbb{N}_0},$$

which is an arithmetic progression with difference 2. Now, for finding Ψ_3, let us consider the sequence $\{3 + 2k\}_{k \in \mathbb{N}_0}$ and search k such that

$$R\left(\frac{3 + 2k}{3}\right) \neq 0 \tag{1.1}$$

Computing

$$R\left(\frac{3 + 2k}{3}\right)$$

for $k = 0, 1, 2$, we obtain respectively the numbers $0, 2, 1$; therefore, by observing that the sequence

$$\left\{R\left(\frac{3 + 2k}{3}\right)\right\}$$

is periodic with period 3, it follows that the condition 1.1 is verified if

$$k = 1 + 3h \quad \text{or} \quad k = 2 + 3h \quad (h \in \mathbb{N}_0).$$

In this way we get the 2 sequences

$$\{5 + 6k\}_{k \in \mathbb{N}_0}, \quad \{7 + 6k\}_{k \in \mathbb{N}_0},$$

(which are arithmetic progressions with difference 6). So we obtain as Ψ_3 the periodically monotone sequence whose principal terms are

$$5, \quad 7,$$

---

[2] Let us observe that the period is equal to the value of the Eulero's function $\varphi(m)$ calculated for $m = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_n$.

whose period is 2 and whose monotony constant is 6. Now, for finding Ψ_5, let us search k such that

$$R\left(\frac{5+6k}{5}\right) \neq 0, \quad \text{and} \quad R\left(\frac{7+6k}{5}\right) \neq 0. \tag{1.2}$$

Computing all the remainders for $k = 0, 1, 2, 3, 4$ and taking into account that the sequences

$$\left\{R\left(\frac{5+6k}{5}\right)\right\}, \quad \left\{R\left(\frac{7+6k}{5}\right)\right\}$$

are periodic with period 5, we obtain the following 8 sequences (which are arithmetic progressions with difference $2 \cdot 3 \cdot 5 = 30$ and $k \in \mathbb{N}_0$):

$$\{11+30k\}, \quad \{17+30k\}, \quad \{23+30k\}, \quad \{29+30k\},$$

$$\{7+30k\}, \quad \{13+30k\}, \quad \{19+30k\}, \quad \{31+30k\}.$$

Thus we obtain as Ψ_5 the periodically monotone sequence whose principal terms are

$$7, 11, 13, 17, 19, 23, 29, 31$$

whose period is $8 = 1\cdot2\cdot4$ and whose monotony constant is $30 = 2\cdot3\cdot5$. The reasoning can be iterated so that the proof is completed. □

Remark 1.1. The terms of the sequence Ψ_{p_n} represent all the natural numbers, greater than p_n, that are not divisible by $p_1, p_2, ..., p_n$, thus the terms of the sequence Ψ_{p_n} that are in the interval $]p_n^2, p_{n+1}^2[$ or, more generally, that are less than p_{n+1}^2 are all prime numbers. Moreover it follows easily that the formula

$$\begin{cases} p_1 = 2, \\ p_{n+1} = \min \Psi_{p_n} \quad \forall\ n \geq 1 \end{cases} \tag{1.3}$$

is a (very simple) recursive formula for the sequence of primes, that gives also a direct and simple proof of the infinity of primes because, for the theorem 1.1, the set of the numbers coprime with the primes up to p_n is not empty for all $n \in \mathbb{N}$.

We have also the following theorems.

Theorem 1.2. *The recursive formula*[3]

$$\Psi_{p_{n+1}} = \Psi_{p_n} - p_{n+1}\Psi_{p_n} - \{p_{n+1}\} \tag{1.4}$$

[3] In this formula Ψ_{p_n} as well as $\Psi_{p_{n+1}}$ represent the ordered (in the natural way) set of the terms of the sequences Ψ_{p_n} and $\Psi_{p_{n+1}}$.

holds.

Proof. To prove the theorem it is sufficient to prove that the terms of the sequence

$$p_{n+1}\Psi_{p_n}$$

represent all the terms of Ψ_{p_n}, greater than p_{n+1}, that are divisible by p_{n+1}. Indeed for any term p of the sequence Ψ_{p_n}, the number $p_{n+1} \cdot p$ is divisible, obviously, by p_{n+1}, but it is not divisible by $p_1, p_2, ..., p_n$ and so it is a term of the sequence Ψ_{p_n}.

Conversely if p is a term of the sequence Ψ_{p_n}, greater than p_{n+1}, that is divisible by p_{n+1}, we must have

$$p = p_{n+1} \cdot q,$$

where q is not divisible by $p_1, p_2, ..., p_n$; therefore p is a term of the sequence $p_{n+1}\Psi_{p_n}$. □

Remark 1.2. Taking into account the remark 1.1, the recursive formula 1.4 can be used to generate the sequence of primes more fastly than using the recursive formula 1.1 of [4].

Theorem 1.3. *The principal terms of the sequence Ψ_{p_n} are obtained from the principal terms of $\Psi_{p_{n-1}}$ by deleting the T numbers*

$$\psi_1, \quad p_n\psi_1, \quad p_n\psi_2, \quad \ldots, \quad p_n\psi_{T-1}$$

in the matrix

$$A_{p_n} = \begin{pmatrix} \psi_1 & \psi_2 & \ldots & \psi_T \\ \psi_1 + k & \psi_2 + k & \ldots & \psi_T + k \\ \ldots & \ldots & \ldots & \ldots \\ \psi_1 + (p_n - 1)k & \psi_2 + (p_n - 1)k & \ldots & \psi_T + (p_n - 1)k \end{pmatrix}$$

where T and k represent, respectively, the period and the monotony constant of $\Psi_{p_{n-1}}$ and $\psi_1, \psi_2, \ldots, \psi_T$ are the principal terms of the same sequence.

Proof. By virtue of the the theorem 1.1, for obtaining the principal terms of Ψ_{p_n} we must take the first p_n terms of each arithmetic progression

$$\{\psi_i + hk\}_{h\in\mathbb{N}_0} \quad (i = 1, 2, \ldots, T)$$

and eliminate among them the term such that

$$R\Big(\frac{\psi_i + hk}{p_n}\Big) = 0.$$

In this way we delete T terms in the matrix A_{p_n}; but for the theorem 1.2 these T terms are the numbers

$$\psi_1, \quad p_n\psi_1, \quad p_n\psi_2, \quad \dots, \quad p_n\psi_{T-1}$$

and the proof is completed. □

Example 1.1. For the principal terms of the sequence Ψ_5 are the 8 numbers

$$7, 11, 13, 17, 19, 23, 29, 31$$

and the monotony constant of Ψ_5 is 30, according to the theorem 1.3 we must eliminate in the matrix

$$A_7 = \begin{pmatrix} 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \\ 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 179 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{pmatrix}$$

the numbers

$$7, 49, 77, 91, 119, 133, 161, 203.$$

Therefore the principal terms of the sequence Ψ_7 are the 48 numbers of the table

	11	13	17	19	23	29	31
37	41	43	47		53	59	61
67	71	73		79	83	89	
97	101	103	107	109	113		121
127	131		137	139	143	149	151
157		163	167	169	173	179	181
187	191	193	197	199		209	211

Remark 1.3. If

$$\psi_1^{(n)}, \quad \psi_2^{(n)}, \quad \dots, \quad \psi_{T_n}^{(n)}$$

denote the principal terms of the sequence Ψ_{p_n}, we have

$$\psi_1^{(n)} = p_{n+1} \quad \text{and} \quad \psi_{T_n}^{(n)} = k_n + 1 \quad \forall\ n \in \mathbb{N},$$

where T_n and k_n represent respectively the period and the monotony constant of Ψ_{p_n}. Indeed the first equality is a consequence of the remark 1.1 and the second can be proved easily by induction.

Moreover it results

$$p_{n+1}\psi_{T_n}^{(n)} > \psi_{T_n}^{(n)} + (p_{n+1} - 1)k_n \quad \forall\ n \in \mathbb{N}.$$

Therefore the principal terms of the sequence Ψ_{p_n} ($\forall\ n \in \mathbb{N}$) are distributed in the interval $[p_{n+1},\ p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1]$.

Remark 1.4. In the matrix considered in the theorem 1.3 we have $\forall\ n \in \mathbb{N}$

$$\psi_1 + ik - (\psi_T + (i-1)k) = p_n - 1 \quad \text{for} \quad i = 1, 2, ..., p_n - 1,$$

thus the gap $p_n - 1$ between two consecutive terms in the matrsix A_{p_n} appears at least $p_n - 1$ times.

Remark 1.5. The numbers

$$\psi_1, \quad p_n\psi_1, \quad p_n\psi_2, \quad \ldots, \quad p_n\psi_{T-1}$$

considered in the theorem 1.3 are distributed in the matrix A_{p_n} in a such way that in every column is located one and only one of them. In particular $\psi_1 = p_n$ is located in the first column and in the first row; moreover, for $p_n \geq 11$, $p_n\psi_1 = p_n^2$ is located in the first row.

The following definitions will be used in the sequel.

Definition 1.1. The number

$$\rho_n = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n}{1 \cdot 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (p_n - 1)},$$

that is the quotient of the monotony constant by the period of the sequence Ψ_{p_n}, is called mean distance between two consecutive terms of Ψ_{p_n}.

Definition 1.2. Let (a, b) an interval $(a \geq p_n)$. The number

$$\frac{b - a}{\rho_n}$$

is called mean number of the terms of the sequence Ψ_{p_n} that are in the interval (a, b).

Definition 1.3. Let $x \in \mathbb{N}$, $x \geq 3$ and let $\pi(x)$ the number of primes p such that $p \leq x$. The number

$$\frac{x}{\pi(x)}$$

is called mean distance between two consecutive primes less than or equal to x.

The following theorem allows to consider the quantity $2e^{-\gamma}\rho_n$ (γ is the Eulero constant) as an approximation of the mean distance between two consecutive primes less than p_{n+1}^2. Thus ρ_n, the mean distance between two consecutive terms of Ψ_{p_n}, appears as a property which is transferred from the sequence Ψ_{p_n} to the sequence of primes.

Theorem 1.4. The number $\pi(p_{n+1}^2)$ of prime numbers less than p_{n+1}^2 is asymptotic, as n goes to ∞, to

$$\frac{e^\gamma}{2}\frac{p_{n+1}^2}{\rho_n},$$

where γ denotes the Eulero constant and ρ_n is the mean distance between two consecutive terms of Ψ_{p_n}.

Proof. We have

$$\frac{\pi(p_{n+1}^2)}{\frac{p_{n+1}^2}{\rho_n}} = \frac{\pi(p_{n+1}^2)}{\frac{p_{n+1}^2}{\log p_{n+1}^2}} \frac{\frac{p_{n+1}^2}{\log p_{n+1}^2}}{\frac{p_{n+1}^2}{\rho_n}} = \frac{\pi(p_{n+1}^2)}{\frac{p_{n+1}^2}{\log p_{n+1}^2}} \frac{\rho_n}{2\log p_{n+1}} =$$

$$= \frac{\pi(p_{n+1}^2)}{\frac{p_{n+1}^2}{\log p_{n+1}^2}} \frac{\rho_{n+1}}{2\log p_{n+1}} \frac{p_{n+1}-1}{p_{n+1}} \tag{1.5}$$

Now for the prime number theorem (see [7] p. 289) it results

$$\lim_{n\to\infty} \frac{\pi(p_{n+1}^2)}{\frac{p_{n+1}^2}{\log p_{n+1}^2}} = 1$$

and for the Mertens' theorem (see [6] p. 351) it results

$$\lim_{n\to\infty} \frac{\rho_{n+1}}{\log p_{n+1}} = e^\gamma,$$

therefore from 1.5 the thesis follows. □

2. The distribution of twin and cousin terms in the sequence Ψ_{p_n}.

Taking into account the theorem 1.3 we are able to prove the following theorem.

Theorem 2.1. *For* $p_n \geq 5$ *we have*

$$\mu_2(\Psi_{p_n}) = \mu_4(\Psi_{p_n}) = 3 \cdot 5 \cdot 9 \cdot \ldots \cdot (p_n - 2). \tag{2.1}$$

Proof. Let us begin by observing that, for $p_n \geq 7$, in the matrix A_{p_n}, considered in the theorem 1.3, the number of couples of consecutive terms whose difference is 2 or 4 is given respectively by

$$\mu_2(\Psi_{p_{n-1}}) \cdot p_n \quad \text{and} \quad \mu_4(\Psi_{p_{n-1}}) \cdot p_n.$$

On the other hand, by effect of the elimination of the T numbers

$$\psi_1, \quad p_n\psi_1, \quad p_n\psi_2, \quad \ldots, \quad p_n\psi_{T-1}$$

in the matrix A_{p_n}, the number of couples of consecutive terms, whose difference is 2 or 4, that are dropped is given respectively by

$$2\mu_2(\Psi_{p_{n-1}}) \quad \text{and} \quad 2\mu_4(\Psi_{p_{n-1}}).$$

In fact if ψ_i and ψ_{i+1} are principal twin terms of $\Psi_{p_{n-1}}$, the element that must be elimineted in the column i of A_{p_n} is of the form $\psi_i + h_1 k$, where k is the monotony constant of $\Psi_{p_{n-1}}$, $h_1 \in \{0, 1, 2, ..., p_n - 1\}$ and h_1 is such that

$$R\left(\frac{\psi_i + h_1 k}{p_n}\right) = 0.$$

From this it follows

$$R\left(\frac{\psi_{i+1} + h_1 k}{p_n}\right) = R\left(\frac{2 + \psi_i + h_1 k}{p_n}\right) = 2,$$

therefore the element that must be deleted in the column $i + 1$ of A_{p_n} is of the form $\psi_{i+1} + h_2 k$, where $h_2 \in \{0, 1, 2, ..., p_n - 1\}$ and $h_2 \neq h_1$. This implies that the elements $\psi_i + h_1 k$ and $\psi_{i+1} + h_2 k$ are located in two different rows of A_{p_n}. Therefore the number of couples of twin terms that are dropped in the matrix A_{p_n} is $2\mu_2(\Psi_{p_{n-1}})$. Obviousvly the same considerations hold if ψ_i and ψ_{i+1} are cousin terms.

Thus we obtain

$$\mu_2(\Psi_{p_n}) = \mu_2(\Psi_{p_{n-1}}) \cdot p_n - 2\mu_2(\Psi_{p_{n-1}}) = \mu_2(\Psi_{p_{n-1}})(p_n - 2) \tag{2.2}$$

and

$$\mu_4(\Psi_{p_n}) = \mu_4(\Psi_{p_{n-1}}) \cdot p_n - 2\mu_4(\Psi_{p_{n-1}}) = \mu_4(\Psi_{p_{n-1}})(p_n - 2). \tag{2.3}$$

But we also have

$$\mu_2(\Psi_5) = \mu_4(\Psi_5) = 3,$$

therefore from 2.2 and 2.3 the thesis follows easily by induction. □

Remark 2.1. The formula

$$\mu_2(\Psi_{p_n}) = 3 \cdot 5 \cdot 9 \cdot \ldots \cdot (p_n - 2)$$

that holds for $p_n \geq 5$ can be also proved in the following way. Let us begin by observing that the terms of the sequence Σ_{p_n} considered in the theorem 3.1 of [4] represent all the natural numbers p greater than p_n such that p and $p + 2$ are not divisible by p_1, p_2, ..., p_n. Therefore, if p is a principal term of Σ_{p_n} then p and $p + 2$ are principal terms of Ψ_{p_n}. Conversely, if p and $p + 2$ are principal terms of Ψ_{p_n} then p is a principal term of Σ_{p_n}. Taking into account the theorem 3.1 of [4] the asserted formula follows easily.

Finally we observe that the computation of $\mu_2(\Psi_{p_n})$ can be derived from the (more complicated) considerations about twin primes made in [6] on p. 412.

The following definitions are useful to state the next corollary.

Definition 2.1. Let (a, b) an interval $(a \geq p_n)$. The number

$$\frac{b - a}{\dfrac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n}{\mu_2(\Psi_{p_n})}}$$

is called mean number of the pairs of twin terms of the sequence Ψ_{p_n} that are in the interval (a, b).

Definition 2.2. Let (a, b) an interval $(a \geq p_n)$. The number

$$\frac{b - a}{\dfrac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n}{\mu_4(\Psi_{p_n})}}$$

is called mean number of the pairs of cousin terms of the sequence Ψ_{p_n} that are in the interval (a, b).

The following corollary follows immediately from the theorem 2.1.

Corollary 2.1. *The mean number σ_{p_n} of pairs of twin terms (or of pairs of cousin terms) of the sequence Ψ_{p_n} that are in the interval $]p_n^2, p_{n+1}^2[$ is given by*

$$\sigma_{p_n} = \frac{p_{n+1}^2 - p_n^2}{\dfrac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_n}{1 \cdot 3 \cdot 5 \cdot 9 \cdot \ldots \cdot (p_n - 2)}}.$$

Remark 2.2. The theorem 2.1 and the corollary 2.1 explain well the experimental fact that the numbers $\pi_2(x)$ (of the pairs of twin primes less than or equal to $x \in \mathbb{N}$) and $\pi_4(x)$ (of the pairs of cousin primes less than or equal to $x \in \mathbb{N}$) are almost the same (see [8]).

The following theorem improves strongly the theorem 3.4 of [4].

Theorem 2.2. *There exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that*

$$\lim_{k \to \infty} \sigma_{p_{n_k}} = +\infty,$$

where σ_{p_n} denotes the mean number of pairs of twin terms (or of pairs of cousin terms) of Ψ_{p_n} that are in the interval $]p_n^2, p_{n+1}^2[$.

Proof. Let us begin by setting

$$d_n = p_{n+1} - p_n \quad \forall\, n \in \mathbb{N}$$

and let be d_{n_k} such that [(4)]

$$\frac{d_{n_{k+1}}}{d_{n_k}} \geq 2 \quad \forall\, k \in \mathbb{N}.$$

We have

$$\sigma_{p_{n_k}} = \frac{p_{n_k+1}^2 - p_{n_k}^2}{\dfrac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_{n_k}}{1 \cdot 3 \cdot 5 \cdot 9 \cdot \ldots \cdot (p_{n_k} - 2)}} \geq \frac{2d_{n_k} p_{n_k}}{\dfrac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_{n_k}}{1 \cdot 3 \cdot 5 \cdot 9 \cdot \ldots \cdot (p_{n_k} - 2)}}. \tag{2.4}$$

Setting

$$S_k = \frac{2d_{n_k} p_{n_k}}{\dfrac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_{n_k}}{1 \cdot 3 \cdot 5 \cdot 9 \cdot \ldots \cdot (p_{n_k} - 2)}},$$

(4) We can do this in consequence of the theorem 5 of [6].

it results

$$
(2.5)\qquad \frac{S_{k+1}}{S_k} = \frac{d_{n_{k+1}}}{d_{n_k}} \frac{\dfrac{p_{n_{k+1}}}{\dfrac{2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_{n_{k+1}}}{1\cdot 3\cdot 5\cdot 9\cdot \ldots \cdot (p_{n_{k+1}}-2)}}}{\dfrac{p_{n_k}}{\dfrac{2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_{n_k}}{1\cdot 3\cdot 5\cdot 9\cdot \ldots \cdot (p_{n_k}-2)}}} \geq 2,
$$

because the sequence

$$
\left\{ \frac{p_{n_k}}{\dfrac{2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_{n_k}}{1\cdot 3\cdot 5\cdot 9\cdot \ldots \cdot (p_{n_k}-2)}} \right\}_{k\in\mathbb{N}}
$$

is not decreasing for it is a subsequence of the sequence

$$
\left\{ \frac{p_n}{\dfrac{2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_n}{1\cdot 3\cdot 5\cdot 9\cdot \ldots \cdot (p_n-2)}} \right\}_{n\in\mathbb{N}}
$$

which is not decreasing (see theorem 3.4 of [4]).

From 2.5 it follows

$$
(2.6)\qquad \lim_{k\to\infty} S_k = +\infty,
$$

and finally from 2.6 and 2.4 we get the thesis. □

Remark 2.3. The previous theorems and the theorem 3.4 of [4] make furthermore strongly plausible that the set of twin primes (as well as the set of cousin primes) is infinite.

REFERENCES

[1] R. Crandall, Pomerance *Prime Numbers A Computational Perspective*, Springer-Verlag New York (2001).

[2] L. E. Dickson *History of the theory of numbers, vol. I*, Chelsea Publishing Company New York (1952).

[3] G. Fiorito *On Properties of Periodically Monotone Sequences*, Applied Mathematics and Computation, **72** (1995), pp. 259-275

[4] G. Fiorito *On a recursive formula for the sequence of primes and applications to the twin prime problem*, to appear in the Bollettino UMI.

[5] R. K. Guy *Unsolved problems in Number Theory*, Springer-Verlag New York (1994).

[6] G. Hardy, E. Wright *An Introduction to the Theory of Numbers*, Clarendon Press Oxford (1954).

[7] M. B. Nathanson *Elementary Methods in Number Theory*, Springer-Verlag (1999).

[8] M. Wolf *On the twin and cousin primes,* IFTUW 909/96 available from internet at the address WWW.ift.uni.wroc.pl/~mwolf/.

Dipartimento di Matematica e Informatica
Università di Catania,
viale Andrea Doria 6, Italy
e-mail: Fiorito@dmi.unict.it