LE MATEMATICHE
Vol. LXX (2015) – Fasc. I, pp. 21–38
doi: 10.4418/2015.70.1.2

ON A NEW CLASS OF p-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY CONVOLUTION

M. K. AOUF - A. O. MOSTAFA - A. Y. LASHIN - B. M. MUNASSAR

In this paper, we introduce a new class of analytic p-valent functions defined in the open unit disc by using convolution and obtain some results including coefficient inequality, distortion theorems, Hadamard products, radii of starlikeness and convexity and closure theorems of functions in this class.

1. Introduction

Let S_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k;\ p,k \in \mathbb{N} = \{1,2,\ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. A function f belonging to the class S_p is said to be p-valent starlike of order α in U if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < p;\ z \in U). \tag{2}$$

Entrato in redazione: 6 gennaio 2014
AMS 2010 Subject Classification: 30C45.
Keywords: Analytic functions, p-valent functions, Hadamard product, radii of starlikeness and convexity, distortion theorem, coefficient inequality, closure theorem.

Also a function f belonging to the class S_p is said to be p-valent convex of order α in U if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (0\le\alpha<p;\ z\in U). \tag{3}$$

We denote by $S_p^*(\alpha)$ the class of all functions in S_p which are p-valent starlike of order α in U and by $K_p(\alpha)$ the class of all functions in S_p which are p-valent convex of order α in U. We note that $S_p^*(0)=S_p^*$, $S_1^*(\alpha)=S^*(\alpha)$, $K_p(0)=K_p$, $K_1(\alpha)=K(\alpha)$, and

$$f(z)\in K_p(\alpha)\Longleftrightarrow\frac{zf'(z)}{p}\in S_p^*(\alpha). \tag{4}$$

The classes $S_p^*(\alpha)$ and $K_p(\alpha)$ were studied by Patil and Thakare [11] and Owa [10]. For $f\in S_p$ given by (1) and $g\in S_p$ given by

$$g(z)=z^p+\sum_{n=k}^{\infty}b_nz^n \quad (b_n\ge 0), \tag{5}$$

the Hadmard product (or convolution) of f and g is given by

$$(f*g)(z)=z^p+\sum_{n=k}^{\infty}a_nb_nz^n=(g*f)(z). \tag{6}$$

Denote by $S_p^*(f,g,\gamma,\beta,\xi)$ $\left(0<\beta\le 1,\frac{1}{2}\le\xi\le 1,0\le\gamma<\frac{p}{2}\right)$ the subclass of S_p, where f and g are given by (1) and (5), respectively and satisfies

$$\left|\frac{\dfrac{z(f*g)'(z)}{(f*g)(z)}-p}{2\xi\left[\dfrac{z(f*g)'(z)}{(f*g)(z)}-\gamma\right]-\left[\dfrac{z(f*g)'(z)}{(f*g)(z)}-p\right]}\right|<\beta. \tag{7}$$

For $g=\frac{z^p}{(1-z)}$, $k=p+1,p\in\mathbb{N}$ in (7), the class $S_p^*(f,g,\gamma,\beta,\xi)$ reduces to the class $S_p^*\left(f,\frac{z^p}{(1-z)},\gamma,\beta,\xi\right)=S_p^*(\gamma,\beta,\xi)$ (see Kulkarni et al. [9]). LetT_p denote the subclass of S_p consisting of functions of the form

$$f(z)=z^p-\sum_{n=k}^{\infty}a_nz^n,a_n\ge 0\,;z\in U. \tag{8}$$

Further, we define the class $T_p^*(f,g,\gamma,\beta,\xi)$ by

$$T_p^*(f,g,\gamma,\beta,\xi)=S_p^*(f,g,\gamma,\beta,\xi)\cap T_p.$$

We note that:

$$T_p^*\left(f, z^p+\sum_{n=p+1}^{\infty}\left(\frac{p+\ell+\lambda(n-p)}{p+\ell}\right)^m z^n, \gamma, \beta, \xi\right)$$
$$= T_p^m(\lambda, \ell, \gamma, \beta, \xi) \quad (\lambda \geq 0; \ell \geq 0; p \in \mathbb{N};\ m \in \mathbb{N} \cup \{0\})$$

has been defined and studied by Aouf et al. [1], where $I_p^m(\lambda, \ell)$ is the Cãtaş operator (see [3]); also we note that:

1) $T_p^*\left(f, \frac{z^p}{(1-z)}, \gamma, \beta, \xi\right) = T_p^*(\gamma, \beta, \xi)$

$$= \left\{ f \in T_p : \left| \frac{\frac{zf^{'}(z)}{f(z)} - p}{2\xi\left[\frac{zf^{'}(z)}{f(z)} - \gamma\right] - \left[\frac{zf^{'}(z)}{f(z)} - p\right]} \right| < \beta, \right.$$
$$\left. 0 \leq \gamma < \tfrac{p}{2},\ 0 < \beta \leq 1, \tfrac{1}{2} \leq \xi \leq 1, p \in \mathbb{N},\ z \in U \right\},$$

which for $p=1$ reduces to the class $T^*(\gamma, \beta, \xi)$ studied by Kulkarni [8];

2) $T_p^*\left(f, z^p+\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^m z^n, \gamma, \beta, \xi\right)\ (m \in \mathbb{N}_0) = S_p^*(m, \gamma, \beta, \xi)$

$$= \left\{ f \in T_p : \left| \frac{\frac{zD_p^{m+1}f(z)}{D_p^m f(z)} - p}{2\xi\left[\frac{zD_p^{m+1}f(z)}{D_p^m f(z)} - \gamma\right] - \left[\frac{zD_p^{m+1}f(z)}{D_p^m f(z)} - p\right]} \right| < \beta, \right.$$
$$\left. 0 \leq \gamma < \tfrac{p}{2},\ 0 < \beta \leq 1, \tfrac{1}{2} \leq \xi \leq 1,\ p \in \mathbb{N},\ z \in U \right\},$$

where the operator D_p^m has been studied by Kamali and Orhan [6] and Aouf and Mostafa [2];

3) $T_p^*\left(f, z^p+\sum_{n=p+1}^{\infty}\left[\frac{p+\lambda(n-p)}{p}\right]^m z^n, \gamma, \beta, \xi\right)\ (\lambda > 0;\ m \in \mathbb{N}_0)$

$= S_p^*(m, \lambda, \gamma, \beta, \xi)$

$$= \left\{ f \in T_p : \left| \frac{\frac{z\left(D_{p,\lambda}^m f(z)\right)^{'}}{D_{p,\lambda}^m f(z)} - p}{2\xi\left[\frac{z\left(D_{p,\lambda}^m f(z)\right)^{'}}{D_{p,\lambda}^m f(z)} - \gamma\right] - \left[\frac{z\left(D_{p,\lambda}^m f(z)\right)^{'}}{D_{p,\lambda}^m f(z)} - p\right]} \right| < \beta, \right.$$

$$0 \leq \gamma < \tfrac{p}{2},\ 0 < \beta \leq 1, \tfrac{1}{2} \leq \xi \leq 1,\ z \in U \Big\},$$

where the operator $D^m_{p,\lambda}$ has been studied by El-Ashwah and Aouf [5];

4) $$T_p^*\left(f, z^p + \sum_{n=p+1}^{\infty} \binom{n+\lambda-1}{n-p} z^n, \gamma, \beta, \xi\right)(\lambda > -p) = T_p^*(\lambda, \gamma, \beta, \xi)$$

$$= \left\{ f \in T_p : \left| \frac{\frac{z\left(D_p^\lambda f(z)\right)'}{\left(D_p^\lambda f(z)\right)} - p}{2\xi\left[\frac{z\left(D_p^\lambda f(z)\right)'}{\left(D_p^\lambda f(z)\right)} - \gamma\right] - \left[\frac{z\left(D_p^\lambda f(z)\right)'}{\left(D_p^\lambda f(z)\right)} - p\right]} \right| < \beta, \right.$$

$$\left. 0 \leq \gamma < \tfrac{p}{2}, 0 < \beta \leq 1, \tfrac{1}{2} \leq \xi \leq 1, p \in \mathbb{N}, z \in U \right\},$$

which for $p = 1$ reduces to the class $T^*(\lambda, \gamma, \beta, \xi)$ studied by Khairnar and Rajas [7];

5) $$T_p^*\left(f, z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \dots (\alpha_q)_{n-p}}{(\beta_1)_{n-p} \dots (\beta_s)_{n-p}} \cdot \frac{1}{(n-p)!} z^n, \gamma, \beta, \xi\right)$$

$$(\alpha_i > 0, i = 1, \dots, q;\ \beta_j > 0, j = 1, \dots, s; q \leq s+1; q, s \in \mathbb{N}_0)$$

$$= \left\{ f \in T_p : \left| \frac{\frac{z\left(H_{p,q,s}(\alpha_1) f(z)\right)'}{H_{p,q,s}(\alpha_1) f(z)} - p}{2\xi\left[\frac{z\left(H_{p,q,s}(\alpha_1) f(z)\right)'}{H_{p,q,s}(\alpha_1) f(z)} - \gamma\right] - \left[\frac{z\left(H_{p,q,s}(\alpha_1) f(z)\right)'}{H_{p,q,s}(\alpha_1) f(z)} - p\right]} \right| < \beta, \right.$$

$$\left. 0 \leq \gamma < \tfrac{p}{2}, 0 < \beta \leq 1, \tfrac{1}{2} \leq \xi \leq 1, p \in \mathbb{N}, z \in U \right\},$$

where $H_{p,q,s}(\alpha_1)$ is the Dziok-Srivastava operator (see [4]).

2. Coefficient Inequality

Unless otherwise mentioned, we shall assume in the reminder of this paper that $0 < \beta \leq 1,\ \frac{1}{2} \leq \xi \leq 1,\ 0 \leq \gamma < \frac{p}{2},\ n \geq k,\ p < k$, g is given by (5) with $b_n > 0$ and $z \in U$.

Theorem 2.1. *Let the function f be defined by (8). Then f is in the class $T_p^*(f, g, \gamma, \beta, \xi)$ if and only if*

$$\sum_{n=k}^{\infty} [(n-p)(1-\beta) + 2\beta\xi(n-\gamma)] b_n a_n \leq 2\xi\beta(p-\gamma). \tag{9}$$

Proof. Assume that the inequality (9) holds true. We find from (6) that

$$\begin{aligned}
&\left|z(f*g)^{'}(z)-p(f*g)(z)\right|-\beta\left|2\xi\left\{z(f*g)^{'}(z)-\gamma(f*g)(z)\right\}\right. \\
&\left.-\left\{z(f*g)^{'}(z)-p(f*g)(z)\right\}\right| \\
&=\left|\sum_{n=k}^{\infty}(n-p)\,b_na_nz^n\right|-\beta\left|2\xi\left[(p-\gamma)z^p-\sum_{n=k}^{\infty}(n-\gamma)\,b_na_nz^n\right]\right. \\
&\left.+\sum_{n=k}^{\infty}(n-p)\,b_na_nz^n\right| \\
&\leq\sum_{n=k}^{\infty}\left[(n-p)+2\xi\beta(n-\gamma)-\beta(n-p)\right]b_na_n-2\beta\xi(p-\gamma) \\
&=\sum_{n=k}^{\infty}\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]b_na_n-2\beta\xi(p-\gamma)\leq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f\in T_p^*(f,g,\gamma,\beta,\xi)$. Conversely, let $f\in T_p^*(f,g,\gamma,\beta,\xi)$. Then

$$\left|\frac{\dfrac{z(f*g)^{'}(z)}{(f*g)(z)}-p}{2\xi\left[\dfrac{z(f*g)^{'}(z)}{(f*g)(z)}-\gamma\right]-\left[\dfrac{z(f*g)^{'}(z)}{(f*g)(z)}-p\right]}\right|<\beta$$

that is

$$\frac{\left|\sum\limits_{n=k}^{\infty}(n-p)\,b_na_nz^n\right|}{\left|2\xi\left[(p-\gamma)z^p-\sum\limits_{n=k}^{\infty}(n-\gamma)\,b_na_nz^n\right]+\sum\limits_{n=k}^{\infty}(n-p)\,b_na_nz^n\right|}<\beta. \tag{10}$$

Now since $\Re f(z)\leq|f(z)|$ for all z, we have

$$\Re\left\{\frac{\sum\limits_{n=k}^{\infty}(n-p)\,b_na_nz^n}{2\xi\left[(p-\alpha)z^p-\sum\limits_{n=k}^{\infty}(n-\gamma)\,b_na_nz^n\right]+\sum\limits_{n=k}^{\infty}(n-p)\,b_na_nz^n}\right\}<\beta. \tag{11}$$

Choose values of z on the real axis so that $\dfrac{z(f*g)^{'}(z)}{(f*g)(z)}$ is real. Then upon clearing the denominator in (11) and letting $z\to 1^-$ through real values, we have

$$\frac{\sum\limits_{n=k}^{\infty}(n-p)\,b_n a_n}{2\xi\left[(p-\gamma)-\sum\limits_{n=k}^{\infty}(n-\gamma)\,b_n a_n\right]+\sum\limits_{n=k}^{\infty}(n-p)\,b_n a_n}\leq\beta.$$

That is

$$\sum_{n=k}^{\infty}\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]\,b_n a_n\leq 2\xi\beta\,(p-\alpha)\,. \tag{12}$$

This is the required condition, which completes the proof of Theorem 2.1. □

Corollary 2.2. *Let the function f defined by (8) be in the class $T_p^*(f,g,\gamma,\beta,\xi)$. Then we have*

$$a_n\leq\frac{2\xi\beta\,(p-\gamma)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]\,b_n},\quad (n\geq k)\,. \tag{13}$$

The result is sharp for the function f given by

$$f(z)=z^p-\frac{2\xi\beta\,(p-\alpha)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]b_n}z^n,\quad n\geq k. \tag{14}$$

3. Growth and Distortion Theorems

Theorem 3.1. *Let the function f defined by* (8) *be in the class $T_p^*(f,g,\gamma,\beta,\xi)$. Then for $|z|=r<1$, we have*

$$|f(z)|\geq r^p-\frac{2\xi\beta\,(p-\gamma)}{\left[(k-p)(1-\beta)+2\beta\xi(k-\gamma)\right]\,b_k}r^k \tag{15}$$

and

$$|f(z)|\leq r^p+\frac{2\xi\beta\,(p-\gamma)}{\left[(k-p)(1-\beta)+2\beta\xi(k-\gamma)\right]\,b_k}r^k, \tag{16}$$

provided that $b_n\geq b_k\,(n\geq k)$. The equalities in (15) *and* (16) *are attained for the function f given by*

$$f(z)=z^p-\frac{2\xi\beta\,(p-\gamma)}{\left[(k-p)(1-\beta)+2\beta\xi(k-\gamma)\right]\,b_k}z^2, \tag{17}$$

at $z=r$ and $z=re^{i(2n+1)\pi}\,(n\geq k)$.

Proof. Since for $n \geq k$,

$$[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k \sum_{n=k}^{\infty} a_n$$

$$\leq \sum_{n=k}^{\infty} [(n-p)(1-\beta)+2\beta\xi(n-\gamma)] b_k a_n \leq 2\xi\beta(p-\gamma),$$

then

$$\sum_{n=k}^{\infty} a_n \leq \frac{2\xi\beta(p-\gamma)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k}. \tag{18}$$

From (8) and (18), we have

$$|f(z)| \geq r^p - r^k \sum_{n=k}^{\infty} a_n \ \geq r^p - \frac{2\xi\beta(p-\gamma)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k} r^k$$

and

$$|f(z)| \leq r^p + r^k \sum_{n=k}^{\infty} a_n \ \leq r^p + \frac{2\xi\beta(p-\alpha)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k} r^k.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let the function f defined by (8) be in the class $T_p^*(f,g,\alpha,\beta,\xi)$. Then for $|z| = r < 1$,*

$$\left|f'(z)\right| \geq pr^{p-1} - \frac{2k\xi\beta(p-\gamma)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k} r^{k-1} \tag{19}$$

and

$$\left|f'(z)\right| \leq pr^{p-1} - \frac{2k\xi\beta(p-\gamma)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k} r^{k-1}. \tag{20}$$

The equalities in (19) and (20) are attained for the function f given by (17).

Proof. For $n \geq k$, we have

$$\left|f'(z)\right| \leq pr^{p-1} - r^{k-1} \sum_{n=k}^{\infty} n a_n, \tag{21}$$

and by Theorem 2.1, we have

$$\sum_{n=k}^{\infty} n a_n \leq \frac{2k\xi\beta(p-\gamma)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\, b_k}. \tag{22}$$

From (21) and (22), we have

$$\left|f'(z)\right| \geq pr^{p-1} - r^{k-1}\sum_{n=k}^{\infty} na_n$$
$$\geq pr^{p-1} - \frac{2k\xi\beta(p-\gamma)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k} r^{k-1}$$

and

$$\left|f'(z)\right| \leq pr^{p-1} + r^{k-1}\sum_{n=k}^{\infty} na_n$$
$$\geq pr^{p-1} + \frac{2k\xi\beta(p-\gamma)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k} r^{k-1}.$$

This completes the proof of Theorem 3.2. □

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In this section we obtain the radii of p-valent starlikeness, p-valent convexity and p-valent close-to-convexity for functions in the class $T_p^*(f,g,\gamma,\beta,\xi)$.

Theorem 4.1. *Let the function f defined by* (8) *be in the class* $T_p^*(f,g,\gamma,\beta,\xi)$. *Then f is p-valent starlike of order $\delta, 0 \leq \delta < p$ in disc $|z| < R_1$, where*

$$R_1 = \inf_{n \geq k} \left\{ \frac{(p-\delta)[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]b_n}{2\beta\xi(p-\gamma)(n-\delta)} \right\}^{\frac{1}{n-p}}. \tag{23}$$

The result is sharp, with the extremal function f given by (14).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p-\delta \text{ for } |z| < R_1, \tag{24}$$

where R_1 is given by (23). Indeed we find, again from (8) that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{n=k}^{\infty}(n-p)a_n|z|^{n-p}}{1-\sum_{n=k}^{\infty}a_n|z|^{n-p}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p-\delta,$$

if

$$\sum_{n=k}^{\infty} \frac{(n-\delta)}{(p-\delta)} a_n |z|^{n-p} \leq 1. \tag{25}$$

But, by Theorem 2.1, (25) will be true if

$$\frac{(n-\delta)}{(p-\delta)} |z|^{n-p} \leq \frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}{2\beta\xi(p-\gamma)}, \tag{26}$$

that is, if

$$|z| \leq \left\{ \frac{(p-\delta)[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\, b_n}{2\beta\xi(p-\gamma)(n-\delta)} \right\}^{\frac{1}{n-p}}, \quad (n \geq k). \tag{27}$$

Theorem 4.1 follows easily from (27). □

Theorem 4.2. *Let the function f defined by* (8) *be in the class* $T_p^*(f, g, \gamma, \beta, \xi)$. *Then f is convex of order $\delta, (0 \leq \delta < p)$ in the disc $|z| < R_2$ where*

$$R_2 = \inf_{n \geq k} \left\{ \frac{p(p-\delta)[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\, b_n}{2\beta\xi n(p-\gamma)(n-\delta)} \right\}^{\frac{1}{n-1}}. \tag{28}$$

The result is sharp for the function f given by (14).

Proof. We must show that

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \delta, \text{ for } |z| < R_2,$$

where R_2 is given by (28). Indeed we find from (8) that

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq \frac{\sum_{n=k}^{\infty} n(n-p) a_n |z|^{n-p}}{p - \sum_{n=k}^{\infty} n a_n |z|^{n-p}}.$$

Thus

$$\left| \frac{zf''(z)}{f'(z)} + (1-p) \right| \leq 1 - \delta,$$

if

$$\sum_{n=2}^{\infty} \frac{n(n-\delta)}{p(p-\delta)} a_n |z|^{n-p} \leq 1. \tag{29}$$

But, by Theorem 2.1, (29) will be true if

$$\frac{(n-\delta)}{p(p-\delta)}|z|^{n-p} \le \frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}{2\beta\xi n(p-\gamma)},$$

that is, if

$$|z| \le \left\{ \frac{p(p-\delta)[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]b_n}{2\beta\xi n(p-\gamma)(n-\delta)} \right\}^{\frac{1}{n-p}},\ n \ge k. \tag{30}$$

Theorem 4.2 follows easily from (30). □

Corollary 4.3. *Let the function f defined by* (8) *be in the class* $T_p^*(f,g,\gamma,\beta,\xi)$. *Then f is p-valent close-to-convex of order* $\delta, (0 \le \delta < p)$ *in the disc* $|z| < R_3$, *where*

$$R_3 = \inf_{n \ge k} \left\{ \frac{(p-\delta)[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]b_n}{2\beta\xi n(p-\gamma)} \right\}^{\frac{1}{n-p}}. \tag{31}$$

The result is sharp, with the extermal function f given by (14).

5. Closure Theorems

Theorem 5.1. *Let* $\mu_j \ge 0$ *for* $j = 1,2,\dots,m$, *and* $\sum_{j=1}^{m} \mu_j \le 1$. *If the functions* f_j *defined by*

$$f_j(z) = z^p - \sum_{n=k}^{\infty} a_{n,j} z^n \,(a_{n,j} \ge 0; j = 1,2,\dots,m), \tag{32}$$

are in the class $T_p^*(f,g,\gamma,\beta,\xi)$, *for every* $j = 1,2,\dots,m$, *then the function* $h(z)$ *defined by*

$$h(z) = z^p - \sum_{n=k}^{\infty} \left(\sum_{j=1}^{m} \mu_j a_{n,j} \right) z^n$$

is in the class $T_p^*(f,g,\gamma,\beta,\xi)$.

Proof. Since $f_j \in T_p^*(f,g,\gamma,\beta,\xi)$, it follows from Theorem 2.1, that

$$\sum_{n=k}^{\infty} [(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\, b_n a_{n,j} \le 2\xi\beta(p-\alpha),$$

for every $j = 1,2,\dots,m$. Hence

$$\sum_{n=k}^{\infty} [(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\, b_n \left(\sum_{j=1}^{m} \mu_j a_{n,j} \right) =$$

$$= \sum_{j=1}^{m} \mu_j \left(\sum_{n=k-1}^{\infty} \left[(n-p)(1-\beta) + 2\beta\xi (n-\gamma) \right] b_n a_{n,j} \right)$$

$$\leq 2\xi\beta (p-\gamma) \sum_{j=1}^{m} \mu_j \leq 2\beta\xi (p-\gamma).$$

By Theorem 2.1, it follows that $h(z) \in T_p^* (f,g,\gamma,\beta;\xi)$, and so the proof of Theorem 5.1 is completed. □

Theorem 5.2. *Let $f_{k-1}(z) = z^p$ and*

$$f_n(z) = z^p - \frac{2\xi\beta (p-\gamma)}{\left[(n-p)(1-\beta) + 2\beta\xi (n-\gamma) \right] b_n} z^n \quad (n \geq k). \tag{33}$$

Then f is in the class $T_p^* (f,g,\gamma,\beta;\xi)$, if and only if it can by expressed in the form

$$f(z) = \sum_{n=k-1}^{\infty} \mu_n f_n(z), \tag{34}$$

where $\mu_n \geq 0$ $(n \geq k-1)$ and $\sum_{n=k-1}^{\infty} \mu_n = 1$.

Proof. Assume that

$$f(z) = \sum_{n=k-1}^{\infty} \mu_n f_n(z) = z^p - \sum_{n=k}^{\infty} \frac{2\xi\beta (p-\gamma)}{\left[(n-p)(1-\beta) + 2\beta\xi (n-\gamma) \right] b_n} \mu_n z^n.$$

Then it follows that

$$\sum_{n=k}^{\infty} \frac{\left[(n-p)(1-\beta) + 2\beta\xi (n-\gamma) \right] b_n}{2\xi\beta (p-\gamma)} \cdot \frac{2\xi\beta (p-\gamma)}{\left[(n-p)(1-\beta) + 2\beta\xi (n-\gamma) \right] b_n} \mu_n$$

$$= \sum_{n=k}^{\infty} \mu_n = 1 - \mu_{k-1} \leq 1.$$

So, by Theorem 2.1, $f \in T_p^* (f,g,\gamma,\beta,\xi)$. Conversely, assume that the function f defined by (8) belongs to the class $T_p^* (f,g,\gamma,\beta,\xi)$. Then

$$a_n \leq \frac{2\xi\beta (p-\gamma)}{\left[(n-p)(1-\beta) + 2\beta\xi (n-\gamma) \right] b_n} \quad (n \geq k).$$

Setting

$$\mu_n = \frac{\left[(n-p)(1-\beta) + 2\beta\xi (n-\gamma) \right] b_n}{2\xi\beta (p-\gamma)} a_n \quad (n \geq k),$$

and

$$\mu_{k-1} = 1 - \sum_{n=k}^{\infty} \mu_n,$$

we can see that f can by expressed in the form (34). This completes the proof of Theorem 5.2. □

Corollary 5.3. *The extreme points of the class $T_p^*(f,g,\gamma,\beta,\xi)$ are the functions $f_{k-1} = z^p$ and $f_n\,(n \geq k)$ given by* (33).

6. Hadamard Product

For the functions

$$f_j(z) = z^p - \sum_{n=k}^{\infty} a_{n,j} z^n \left(a_{n,j} \geq 0; j = 1,2; p,k \in \mathbb{N}\right), \tag{35}$$

we denote by $(f_1 * f_2)$ the modified Hadamard product (or convolution) of the functions f_1 and f_2, that is,

$$(f_1 * f_2)(z) = z^p - \sum_{n=k}^{\infty} a_{n,1} a_{n,2} z^n. \tag{36}$$

Theorem 6.1. *Let the functions $f_j\,(j = 1,2)$, defined by* (35) *be in the class $T_p^*(f,g,\gamma,\beta,\xi)$. Then $(f_1 * f_2) \in T_p^*(f,g,\mu,\beta,\xi)$,where*

$$\mu = p - \frac{2\beta\xi (p-\alpha)^2 (k-p)\left[(1-\beta)+2\beta\xi\right]}{\left[(k-p)(1-\beta)+2\beta\xi(k-\alpha)\right]^2 b_k - 4\beta^2\xi^2 (p-\gamma)^2}. \tag{37}$$

The result is sharp.

Proof. Employing the technique used ealier by Schild and Silverman [12], we need to find the largest μ such that

$$\sum_{n=k}^{\infty} \frac{\left[(n-p)(1-\beta)+2\beta\xi(n-\mu)\right]\, b_n}{2\xi\beta(p-\mu)} a_{n,1} a_{n,2} \leq 1. \tag{38}$$

Since $f_j \in T_p^*(f,g,\gamma,\beta,\xi)\,(j = 1,2)$, we readily see that

$$\sum_{n=k}^{\infty} \frac{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right] b_n}{2\xi\beta(p-\gamma)} a_{n,1} \leq 1, \tag{39}$$

and

$$\sum_{n=k}^{\infty} \frac{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]\, b_n}{2\xi\beta(p-\gamma)} a_{n,2} \leq 1. \tag{40}$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{n=k}^{\infty} \frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}{2\xi\beta(p-\gamma)} \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{41}$$

Thus it is sufficient to show that

$$\frac{[(n-p)(1-\beta)+2\beta\xi(n-\mu)]\ b_n}{2\xi\beta(p-\mu)} a_{n,1}a_{n,2}$$
$$\leq \frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}{2\xi\beta(p-\gamma)} \sqrt{a_{n,1}a_{n,2}} \quad (n \geq k), \tag{42}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(p-\mu)[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]}{(p-\gamma)[(n-p)(1-\beta)+2\beta\xi(n-\mu)]} \quad (n \geq k). \tag{43}$$

From (41) we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{2\xi\beta(p-\gamma)}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n} \quad (n \geq k). \tag{44}$$

Consequently, we need only to prove that

$$\frac{2\xi\beta(p-\gamma)}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}$$
$$\leq \frac{(p-\mu)[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]}{(p-\gamma)[(n-p)(1-\beta)+2\beta\xi(n-\mu)]}, \tag{45}$$

or, equivalently, that

$$\mu \leq p - \frac{2\beta\xi(p-\gamma)^2(n-p)[(1-\beta)+2\beta\xi]}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]^2 b_n - 4\beta^2\xi^2(p-\gamma)^2} \ (n \geq k). \tag{46}$$

Since

$$\Phi(n) = p - \frac{2\beta\xi(p-\gamma)^2(n-p)[(1-\beta)+2\beta\xi]}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]^2 b_n - 4\beta^2\xi^2(p-\gamma)^2}, \tag{47}$$

is an increasing function of n $(n \geq k)$, letting $n = k$ in (47), we obtain

$$\mu \leq \Phi(k) = p - \frac{2\beta\xi(p-\gamma)^2[(1-\beta)+2\beta\xi]}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]^2 b_k - 4\beta^2\xi^2(p-\gamma)^2}, \tag{48}$$

which proves the main assertion of Theorem 6.1. Finally, by taking the functions $f_j(z)\,(j=1,2)$ given by

$$f_j(z) = z^p - \frac{2\xi\beta(p-\gamma)}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]\ b_k} z^k \ (j=1,2), \tag{49}$$

we can see that the result is sharp. □

Theorem 6.2. *Let the functions* f_j $(j=1,2)$ *defined by* (35) *be in the class* $T_p^*(f,g,\mu_j,\beta,\xi)$. *Then* $(f_1 * f_2) \in T_p^*(f,g,\mu,\beta,\xi)$, *where*

$$\mu = p - \frac{2\xi\beta(p-\mu_1)(p-\mu_2)(k-p)[(1-\beta)+2\beta\xi]}{A_1(\mu_1,p,\beta,\xi,k)\cdot A_2(\mu_2,p,\beta,\xi,k)\ b_k - 4\xi^2\beta^2(p-\mu_1)(p-\mu_2)} \tag{50}$$

and

$$\begin{aligned} A_1(\mu_1,p,\beta,\xi,k) &= [(k-p)(1-\beta)+2\beta\xi(k-\mu_1)] \\ A_2(\mu_2,p,\beta,\xi,k) &= [(k-p)(1-\beta)+2\beta\xi(k-\mu_2)]. \end{aligned} \tag{51}$$

Proof. We need to find the largest μ such that

$$\sum_{n=k}^{\infty} \frac{[(n-p)(1-\beta)+2\beta\xi(n-\mu)]\ b_n}{2\xi\beta(p-\mu)} a_{n,1}a_{n,2} \leq 1.$$

Since

$$(f_1 \in T_p^*(f,g,\mu_1,\beta,\xi) \ and\ f_2 \in T_p^*(f,g,\mu_2,\beta,\xi),$$

then

$$\sum_{n=k}^{\infty} \frac{[(n-p)(1-\beta)+2\beta\xi(n-\mu_1)]\ b_n}{2\xi\beta(p-\mu_1)} a_{n,1} \leq 1$$

and

$$\sum_{n=k}^{\infty} \frac{[(n-p)(1-\beta)+2\beta\xi(n-\mu_2)]\ b_n}{2\xi\beta(p-\mu_2)} a_{n,2} \leq 1.$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=k}^{\infty} \frac{[A_1(\mu_1,p,\beta,\xi,n)]^{\frac{1}{2}}[A_2(\mu_2,p,\beta,\xi,n)]^{\frac{1}{2}}\ b_n}{2\xi\beta\sqrt{(p-\mu_1)(p-\mu_2)}} \sqrt{a_{n,1}a_{n,2}} \leq 1, \tag{52}$$

where

$$\begin{aligned} A_1(\mu_1,p,\beta,\xi,n) &= [(n-p)(1-\beta)+2\beta\xi(n-\mu_1)] \\ A_2(\mu_2,p,\beta,\xi,n) &= [(n-p)(1-\beta)+2\beta\xi(n-\mu_2)]. \end{aligned}$$

Thus we only need to find the largest μ such that

$$\sum_{n=k}^{\infty} \frac{[(n-p)(1-\beta)+2\beta\xi(n-\mu)]\ b_n}{2\xi\beta(p-\mu)} a_{n,1}a_{n,2}$$

$$\leq \sum_{n=k}^{\infty} \frac{[A_1(\mu_1,p,\beta,\xi,n)]^{\frac{1}{2}}[A_2(\mu_2,p,\beta,\xi,n)]^{\frac{1}{2}}\ b_n}{2\xi\beta\sqrt{(p-\mu_1)(p-\mu_2)}} \sqrt{a_{n,1}a_{n,2}}$$

or, equivalently, that

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{p-\mu}{\sqrt{(p-\mu_1)(p-\mu_2)}} \frac{[A_1(\mu_1,p,\beta,\xi,n)]^{\frac{1}{2}}[A_2(\mu_2,p,\beta,\xi,n)]^{\frac{1}{2}}}{[(n-p)(1-\beta)+2\beta\xi(n-\mu)]}\,(n\ge k).$$

Hence, in light of inequality (52), it is sufficient to prove that

$$\frac{2\xi\beta\sqrt{(p-\mu_1)(p-\mu_2)}}{[A_1(\mu_1,p,\beta,\xi,n)]^{\frac{1}{2}}[A_2(\mu_2,p,\beta,\xi,n)]^{\frac{1}{2}}b_n} \le \frac{p-\mu}{\sqrt{(p-\mu_1)(p-\mu_2)}} \frac{[A_1(\mu_1,p,\beta,\xi,n)]^{\frac{1}{2}}[A_2(\mu_2,p,\beta,\xi,n)]^{\frac{1}{2}}}{[(n-p)(1-\beta)+2\beta\xi(n-\mu)]}. \tag{53}$$

It follows from (53) that

$$\mu = p - \frac{2\xi\beta(p-\mu_1)(p-\mu_2)(n-p)[(1-\beta)+2\beta\xi]}{A_1(\mu_1,p,\beta,\xi,n)\cdot A_2(\mu_2,p,\beta,\xi,n)b_n - 4\xi^2\beta^2(p-\mu_1)(p-\mu_2)}.$$

Now, defining the function $\Phi(n)$ by

$$\Phi(n) = p - \frac{2\xi\beta(p-\mu_1)(p-\mu_2)(n-p)[(1-\beta)+2\beta\xi]}{A_1(\mu_1,p,\beta,\xi,n)\cdot A_2(\mu_2,p,\beta,\xi,n)b_n - 4\xi^2\beta^2(p-\mu_1)(p-\mu_2)}.$$

We see that $\Phi(n)$ is an increasing function of n $(n\ge k)$. Therefore, we conclude that

$$\mu = \Phi(k) = p - \frac{2\xi\beta(p-\mu_1)(p-\mu_2)(k-p)[(1-\beta)+2\beta\xi]}{A_1(\mu_1,p,\beta,\xi,k)\cdot A_2(\mu_2,p,\beta,\xi,k)b_k - 4\xi^2\beta^2(p-\mu_1)(p-\mu_2)},$$

where $A_1(\mu_1,p,\beta,\xi,k)$ and $A_2(\mu_2,p,\beta,\xi,k)$ are given by (51), which evidently completes the proof of Theorem 6.2. □

Theorem 6.3. *Let the functions $f_j\,(j=1,2)$ defined by (35) be in the class $T_p^*(f,g,\gamma,\beta,\xi)$. Then the function*

$$h(z) = z^p - \sum_{n=k}^{\infty}\left(a_{n,1}^2 + a_{n,2}^2\right)z^n \tag{54}$$

belongs to the class $T_p^*(f,g,\tau,\beta,\xi)$, where

$$\tau = p - \frac{4\beta\xi(p-\gamma)^2(k-p)[1-\beta+2\beta\xi]}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]^2 b_k - 8\beta^2\xi^2(p-\gamma)^2}. \tag{55}$$

The result is sharp for the functions $f_j\ (j=1,2)$ defined by (49).

Proof. By virtue of Theorem 5.1, we obtain

$$\sum_{n=k}^{\infty}\left[\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}{2\xi\beta(p-\gamma)}\right]^2 a_{n,1}^2$$

$$\leq\left[\sum_{n=k}^{\infty}\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}{2\xi\beta(p-\gamma)}a_{n,1}\right]^2\leq 1 \tag{56}$$

and

$$\sum_{n=k}^{\infty}\left[\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}{2\xi\beta(p-\gamma)}\right]^2 a_{n,2}^2$$

$$\leq\left[\sum_{n=k}^{\infty}\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ b_n}{2\xi\beta(p-\gamma)}a_{n,2}\right]^2\leq 1. \tag{57}$$

It follows from (56) and (57) that

$$\sum_{n=k}^{\infty}\frac{1}{2}\left[\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]b_n}{2\xi\beta(p-\gamma)}\right]^2\left(a_{n,1}^2+a_{n,2}^2\right)\leq 1. \tag{58}$$

Therefore, we need to find the largest τ such that

$$\frac{[(n-p)(1-\beta)+2\beta\xi(n-\tau)]\ b_n}{2\xi\beta(p-\tau)}$$

$$\leq\frac{1}{2}\left[\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]b_n}{2\xi\beta(p-\gamma)}\right]^2\quad(n\geq k), \tag{59}$$

that is, that

$$\tau\leq p-\frac{4\beta\xi(p-\gamma)^2(n-p)[1-\beta+2\beta\xi]}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]^2 b_n-8\beta^2\xi^2(p-\gamma)^2}. \tag{60}$$

Since

$$D(n)=p-\frac{4\beta\xi(p-\gamma)^2(n-p)[1-\beta+2\beta\xi]}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]^2 b_n-8\beta^2\xi^2(p-\gamma)^2}$$

is an increasing function of n $(n\geq k)$, we readily have

$$\tau\leq D(k)=p-\frac{4\beta\xi(p-\gamma)^2(k-p)[1-\beta+2\beta\xi]}{[(k-p)(1-\beta)+2\beta\xi(k-\gamma)]^2 b_k-8\beta^2\xi^2(p-\gamma)^2},$$

and Theorem 6.3 follows at once. □

Remark 6.4. Putting $g(z) = z^p - \sum_{n=k}^{\infty} \left(\frac{p+\ell+\lambda(n-p)}{p+\ell}\right)^m z^n$ $(\lambda \geq 0; \ell \geq 0; p \in \mathbb{N};$ $m \in \mathbb{N} \cup \{0\})$ in all the above results, we obtain the results obtained by Aouf et al. [1].

REFERENCES

[1] M. K. Aouf - A. Y. Lashin - A. O. Mostafa - B. M. Munassar, *On a newclass of p-valent functions with negative coefficients defined by Cãtaş operator*, J. Open Problems Complex Analysis, (To appear).

[2] M. K. Aouf - A. O. Mostafa, *On a subclass of n-p-valent prestarlike functions*, Comput. Math. Appl. 55 (4) (2008), 851-861.

[3] A. Cãtaş, *On certain classes of p-valent functions defined by multiplier transformations*, in: Proc. Book of the International Symposium on Geometric Function Theory and Applications, Istanbul, Turkey, August 2007, 241–250.

[4] J. Dziok - H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. 103 (1999), 1–13.

[5] R. M. El-Ashwah - M. K. Aouf, *Inclusion and neighborhood properties of some analytic p-valent functions*, General. Math. 18 (2) (2010), 183–194.

[6] M. Kamali - H. Orhan, *On a subclass of certain starlike functions with negative coefficients*, Bull. Korean Math. Soc. 41 (1) (2004), 53–71.

[7] S. M. Khairnar - S. M. Rajas, *On a new class of analytic and univalent functions with negative coefficients defined by Ruscheweyh derivative*, Int. J. Contemp. Math. Sci. 4 (34) (2009), 1653–1664.

[8] S. R. Kulkarni, *Some problems connected with univalent functions*, Ph.D. Thesis (1981), Shivaji University, Kolhapur (unpublished).

[9] S. R. Kulkarni - S. B. Joshi - M. K. Aouf, *On p-valent starlike functions*, J. Ramanujan Math. Soc. 9 (1) (1994), 55–68.

[10] S. Owa, *On certain classes of p-valent functions with negative coefficients*, Simon Stevin 59 (4) (1985), 385–402.

[11] D. A. Patil - N. K. Thakare, *On convex hulls and extreme points of p-valent starlike and convex classes with applications*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S) 27 (75) (1983), 145–160.

[12] A. Schild - H. Silverman, *Convolution of univalent functions with negative coeficients*, Ann. Math. Curie-Sklodowska Sect. A 29 (1975), 99–107.

M. K. AOUF
Department of Mathematics
Faculty of Science,
Mansoura University
Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com

A. O. MOSTAFA
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: adelaeg254@yahoo.com

A. Y. LASHIN
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: aylashin@mans.edu.eg

B. M. MUNASSAR
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: bmunassar@yahoo.com