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THE NULL SPACES DIMENSION AND THE EXISTENCE OF THE INTERPOLATING SPLINE-FUNCTION IN BANACH SPACE

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Using the results in papers [2] and [3] in this paper we prove the existence of the interpolating spline-function by the null space dimension of operators A and T.

1. Introduction.

Let X,Y,Z be Banach spaces. Suppose A is a bounded linear operator of X into Z and T is bounded linear operator of X into Y. The null space and the rang of operator A will be denoted by $N(A)$ and $R(A)$, respectively. Let $R(A) = Z$. For a fixed element $z \in Z$ we write by

$$I_z = \{x \in X : A(x) = z\} = A^{-1}(z).$$

Definition 1. *The element $s \in I_z$ for which*

$$||T(s)|| = \inf \{||T(x)|| : x \in I_z\}$$

is called the interpolating spline-function for z in connection with the operators A and T and is denoted by $s = s(z, A, T)$.

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The following theorem has been proved in the case when X,Y and Z are Hilbert spaces (see [1]).

Theorem 1. *Suppose:*

(i) $N(A) + N(T)$ *is closed set in* X

(ii) $N(A) \cap N(T) = \{0\}$,

then for each $z \in Z$ *exists a unique interpolating spline-function* $s = s(z, A, T)$.

If X,Z *are Banach spaces and* Y *is a reflexive Banach space, the following theorem is proved (see [2]).*

Theorem 2. *Suppose:*

(i) $TA^{-1}(z \in Z)$ *is closed and bounded set in* Y

(ii) $N(A) \cap N(T) = \{0\}$,

then for each $z \in Z$ *exists a unique interpolating spline-function* $s = s(z, A, T)$.

2. The main result.

If X is a reflexive Banach space, it is proved the following

Theorem 3. *Suppose:*

(i) $R(T) = Y$

(ii)$N(T)$ *is a finite dimensional subspace in* X

(iii) $N(A) \cap N(T) = \{0\}$,

then for each $z \in Z$ *exists a unique interpolating spline-function* $s = s(z, A, T)$.

Proof. 1. According to Theorem 2 it is enough to show that the set $TA^{-1}(z \in Z)$ is closed set in Y. Since TA^{-1} is a translation of the set $TN(A)$ it is enough to show that the set $TN(A)$ is closed set in Y. Let $y_0 \in clTN(A)$, then there exists a sequence $(y_n) \subset TN(A)$, $||y_n - y_0|| \to 0$. Hence there exists a sequence $(x_n) \subset N(A)$ such that $Tx_n = y_n (n \in \mathbb{N})$. Since $N(T)$ is a finite dimensional subspace in X then exists a closed subspace F in X such that $X = N(T) + F$. The subspace F is Banach space because it is closed subspace in Banach space

X. We denote by T_1 the restriction of operator T in F. Operator T_1 is bijection. According to Theorem of continuity of inverse, the inverse ${T_1}^{-1}$ exists and is bounded. Since $x_n = t_n + f_n(t_n \in N(T), f_n \in F)(n \in \mathbb{N})$, $Tx_n = T_1(f_n) \Rightarrow f_n = {T_1}^{-1}Tx_n(n \to \infty)$. Consequently the sequence (f_n) is bounded in F. Further, we denote by A_1 the restriction of the operator A in $N(T)$. Operator A is a bijection. Let us prove that A_1 is 1-1. Let $x, y \in N(T) \Rightarrow x - y \in N(T)$, then $Ax = Ay$ implies that $x - y \in N(A)$. Since $N(A) \cap N(T) = \{0\}$, then $x = y$. According to Theorem of continuity of inverse, the inverse ${A_1}^{-1}$ exists and is bounded. Since $x_n = t_n + f_n(t_n \in N(T), f_n \in F)(n \in \mathbb{N})$, $A_1 t_n = -Af_n \Rightarrow t_n = -{A_1}^{-1} f_n$. Consequently the sequence (t_n) is bounded in $N(T)$. Hence $(x_n) \subset N(T)$ is a bounded sequence in reflexive Banach space X. Therefore, the sequence (x_n) contains a subsequence $(x_{1,n})$ which converges weakly to $x_0 \in X$. Since $f \cdot T \in X^*(f \in Y^*)$, $(Tx_{1,n})$ converges weakly to $Tx_0 \in Y$ and $Tx_0 = y_0$. Also, the sequence $(Ax_{1,n})$ is weakly convergent to Ax_0. Since $Ax_{1,n} = 0(n \in \mathbb{N})$, $Ax_0 = 0 \Rightarrow x_0 \in N(A) \Rightarrow y_0 = Tx_0 \in TN(A)$.

Proof. 2. Consider the factor space $\tilde{X} = X/N(T)$ and define the operator $\tilde{T} : \tilde{X} \to Y$ by $\tilde{T}\eta x = Tx$, where $\eta : X \to \tilde{X}$ is a natural homomorphism, $\eta x = x + N(T)$. Then $\tilde{T}$ is a continuous bijection. According to Theorem of continuity of inverse, $\tilde{T}M$ is closed in Y for every closed $M \subset \tilde{X}$. By choosing $M = \eta N(A)$, we conclude that $TN(A) = \tilde{T}\eta N(A)$ is closed, and the proof is complete.

Corollary *Suppose:*

(i) $R(T) = Y$

(ii) $N(T)$ *is a finite codimensional subspace in* X

(iii) $N(A) \cap N(T) = \{0\}$,

then for each $z \in Z$ *exists a unique interpolating spline-function* $s = s(z, A, T)$.

Proof. Since $N(T)$ is finite codimensional subspace in X then exists a finite dimensional subspace F in X such that $X = N(T) + F$. The subspace F is Banach space, because it is finite dimensonal subspace in Banach space X.

Analogues results stay for the case of null-space $N(A)$.

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