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MINIMAL EDGE COLORINGS OF CLASS 2 GRAPHS AND DOUBLE GRAPHS

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A proper edge coloring of a class 2 graph G is minimal if it contains a color class of cardinality equal to the resistance $r(G)$ of G, which is the minimum number of edges that have to be removed from G to obtain a graph which is $\Delta(G)$-edge colorable, where $\Delta(G)$ is the maximum degree of G. In this paper using some properties of minimal edge colorings of a class 2 graph and the notion of reflective edge colorings of the direct product of two graphs, we are able to prove that the double graph of a class 2 graph is of class 1. This result, recently conjectured, is moreover extended to some generalized double graphs.

1. Introduction

Let $G = (V,E)$ be a finite, simple, undirected graph. A *proper edge coloring* of G is a map α from E to a set of colors C such that adjacent edges have different colors [1]. If $\mid C \mid = t$, then α is a proper t-edge coloring of G. For every vertex $v \in V$ the set $C_\alpha(v)$, or simply $C(v)$, is the set of colors assigned to the edges incident to v and $\bar{C}(v) = C \setminus C(v)$ the set of colors missing in $C(v)$.

The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors in a proper edge coloring of G. A famous theorem of *Vizing* [1], states that the

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chromatic index of any graph G satisfies the bounds $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. A graph G is of *class 1* when $\chi'(G) = \Delta(G)$ and is of *class 2* when $\chi'(G) = \Delta(G)+1$.

Let G be a graph of class 2 and maximum degree $\Delta(G) = d$. Suppose that α is a proper $(d+1)$-edge coloring of G and E_i, $1 \leq i \leq d+1$, the color classes of α. Let $m(\alpha) = min\{| E_i || 1 \leq i \leq d+1\}$. The integer $r(G) = min_{\alpha \in C(G)}\{m(\alpha)\}$, where $C(G)$ is the set of all the proper $(d+1)$-edge colorings of G, is defined the *resistance* of G [11].

The same parameter is called color-character in [14] and color-number in [12], where it is investigated in relation to cubic graphs of class 2.

A subgraph F of G is called maximum d-edge colorable if it is d-edge colorable and contains as many edges as possible. We will refer to edges of $E(G) \setminus E(F)$ as *F-uncolored* edges or simply *uncolored*.

In [10] it is proved that every simple graph of class 2 contains a maximum d-edge colorable subgraph such that the uncolored edges form a matching M. Another result of [10] is that if H is a maximum $\Delta(G)$-edge colorable subgraph of a simple graph G, then $\Delta(H) = \Delta(G)$, i.e. H is class 1.

This implies that the resistance of G denotes the minimum number of edges that should be removed from G in order to obtain a graph F of class 1, with $\chi'(F) = \Delta(G)$.

A proper $(d+1)$-edge coloring α of G having a color class L of cardinality $r(G)$ is called a *minimal* $(d+1)$-edge coloring and a color class of a minimal coloring having cardinality $r(G)$ is called *α-minimal* or simply *minimal*. The edges of a minimal color class are still called uncolored.

If u and v are adjacent vertices of G, we write $u \sim v$ and we denote uv the corresponding edge.

Let $e = uv$ be an uncolored edge of a minimal color class of G. Assume that $a \in \bar{C}(u)$, $b \in \bar{C}(v)$. Since F is a maximum d-edge colorable subgraph of G, we have that $a \in C(v)$ and $b \in C(u)$. Let $P(a,b)$ the alternating path starting from the vertex v. By the assumption on F, the path P ends in u; on the contrary we exchange the colors of $P(a,b)$ and obtain a minimal coloring with a missing also in $C(v)$. Thus P is an even path, which together with the edge e forms an odd cycle, denoted $C^e_{a,b}$ and called the (a,b)-*minimal* cycle associated with the edge e. If it is not necessary to mention the colors a,b we may simply call the cycle C^e as a minimal cycle associated with the edge e.
The *direct product* $G \times H$ of two graphs $G = (V,E)$ and $H = (W,F)$ is the graph with vertex set $V(G \times H) = V \times W$ and edge set $E(G \times H) = \{(a,v)(b,w) : ab \in E,\ vw \in F\}$. This product, also called tensor product, Kronecker product, categorical product and conjunction, is commutative and associative (up to isomorphisms).

The direct product of a bipartite graph by any other graph is bipartite.

Recall [8] that the *double graph* of a graph G is defined as $\mathcal{D}[G] = G \times T_2$, where T_2 is the total graph on 2 vertices, i.e. the graph obtained by the complete graph K_2 by adding a loop to each vertex.

Let $V(T_2) = V(K_2) = \{u_1, u_2\}$. Notice that $\mathcal{D}[G] = G \times T_2$ is the union of $G \times K_2$ and the edge disjoint subgraphs $G \times S(u_i u_i)$, for $i = 1, 2$, where $S(u_i u_i)$ denotes the subgraph induced by the loop $u_i u_i$. This is the *canonical decomposition* of $\mathcal{D}[G]$ [9]. In particular, if G is d-regular, then $\mathcal{D}[G]$ is $2d$-regular.

For any $k \in \mathbb{N}$, $k \geq 1$, the *generalized double graph* of a graph G is defined as $\mathcal{D}_k[G] = G \times T_k$, where $T_k = K_k^s$ is the total graph [5]. The generalized double graphs were introduced in [7] as graphs $G[mK_1]$, where $G[H]$ denotes the composition of graphs G and H, also known as the lexicographic product. Recall that the composition of graphs G and H is the graph $G[H]$ with the vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u,v)(u',v') : \textit{either } (u = u' \textit{ and } v \sim v') \textit{ or } u \sim u'\}$. It was also defined as strong tensor product [2].

The main result of this paper is Theorem 3.3 which states that if G is a simple class 2 graph, then $G \times T_2$ is of class 1. This result was conjectured in [9]. In the second section it is introduced the notion of reflective edge colorings of the direct product of graphs. The third section is dedicated to the proof of Theorem 3.3, while last section contains similar results for the generalized double graphs $G \times T_k$, distinguishing the cases of k even or odd.

2. Reflective colorings

In this section we introduce a particular edge coloring of the direct product of a simple graph G by K_2 or T_2.

An edge coloring α of the graph $G \times K_2$ is said to be *reflective* if

$$\alpha((v,u_1)(w,u_2)) = \alpha((v,u_2)(w,u_1)), \ vw \in E(G).$$

Moreover an edge coloring β of the graph $G \times T_2$ is said to be *reflective* if

$$\beta((v,u_1)(w,u_2)) = \beta((v,u_2)(w,u_1)), \ vw \in E(G).$$

and

$$\beta((v,u_1)(w,u_1)) = \beta((v,u_2)(w,u_2)), \ vw \in E(G).$$

These definitions can be immediately extended to the direct product of two more general graphs.

Because $G \times K_2$ is bipartite and $\Delta(G \times K_2) = \Delta(G)$, then $\chi'(G \times K_2) \leq \chi'(G)$.

For every $v_i \in V(G)$, denote $z_i = (v_i, u_1)$ and $z_i' = (v_i, u_2)$.

If $z_iz'_j \in E(G \times K_2)$, then also $z'_iz_j \in E(G \times K_2)$. The vertices z_i, z'_i of $G \times K_2$ and the edges $z_iz'_j$, z'_iz_j are said *corresponding*.

Theorem 2.1. *If G is a graph of class 1, then $G \times T_2$ admits a proper, reflective $2\Delta(G)$-edge coloring.*

Proof. Assume that $G = (V,E)$ is of class 1. Then there exists a proper edge coloring α of G which uses $\Delta(G)$ colors. Let D be the set of colors.

To every edge $e = v_iv_j$, $i < j$, of G we associate the edges $e_1 = (v_i,u_1)(v_j,u_2)$ and $e_2 = (v_i,u_2)(v_j,u_1)$ of $G \times K_2$. We assign to the edges e_1 and e_2 the same color $\alpha(e)$. Note that adjacent edges $(v_i,u_1)(v_j,u_2)$ and $(v_i,u_1)(v_h,u_2)$ are assigned the distinct colors given by α to the adjacent edges v_iv_j and v_iv_h. Thus we obtain an edge coloring α' of $G \times K_2$ which turns out to be proper and reflective.

Now denote $\beta : E \to D'$ a proper edge coloring of G which uses the $\Delta(G)$ colors of a new $\Delta(G)$-set D' disjoint from D. Then we assign to the edges $(v_i,u_1)(v_j,u_1)$ and $(v_i,u_2)(v_j,u_2)$, which belong to the subgraphs $G \times S(u_1u_1)$ and $G \times S(u_2u_2)$ respectively, the same color $\beta(e)$. Thus we have two new edge colorings $\beta_1 : G \times S(u_1u_1) \to D'$ and $\beta_2 : G \times S(u_2u_2) \to D'$ such that $\beta_1((v_i,u_1)(v_j,u_1)) = \beta_2((v_i,u_2)(v_j,u_2)) = \beta(e)$. The union of the colorings α', β_1 and β_2 determines a proper reflective $2\Delta(G)$-edge coloring of $G \times T_2$. □

Denote $M^2(G)$ the multigraph obtained from G by replacing every edge with a pair of two edges in parallel; in other words every edge $e = v_iv_j$ of G is replaced by different edges e_1 and e_2, with the same vertices v_i, v_j.

Proposition 2.2. *There exists a bijection between the set of proper edge colorings of $M^2(G)$ and the set of proper reflective edge colorings of $G \times T_2$.*

Proof. Let α be a proper edge coloring of $M^2(G)$. Denote α' the edge coloring of $G \times T_2$ defined in the following way: for every edge v_iv_j of G we have

$$\alpha'((v_i,u_1)(v_j,u_1)) = \alpha'((v_i,u_2)(v_j,u_2)) = \alpha(e_1),$$

and

$$\alpha'((v_i,u_1)(v_j,u_2)) = \alpha'((v_i,u_2)(v_j,u_1)) = \alpha(e_2).$$

The edge coloring of $G \times T_2$ turns out to be proper and reflective.

On the contrary, let β be a proper, reflective edge coloring of $G \times T_2$. Denote β' the edge coloring of $M^2(G)$ such that

$$\beta'(e_1) = \beta((v_i,u_1)(v_j,u_1)),$$

and

$$\beta'(e_2) = \beta((v_i,u_1)(v_j,u_2)).$$

Because the coloring β is reflective, then the coloring β' is proper and the result follows. □

3. Double graphs

In this section G denotes a simple class 2 graph. In [10] it is proved the following result.

Theorem 3.1. *Let H be any maximum $\Delta(G)$-edge colorable subgraph of a graph G and let $E(G) \setminus E(H) = \{e_i = u_i v_i \mid 1 \le i \le n = r(G)\}$. Assume that H is properly edge colored with colors $1,\dots,\Delta(G)$. Then there is an assignment of colors $\alpha_1 \in \bar{C}(u_1), \beta_1 \in \bar{C}(v_1), \dots, \alpha_n \in \bar{C}(u_n), \beta_n \in \bar{C}(v_n)$ to the uncolored edges such that $E(C^{e_i}_{\alpha_i,\beta_i}) \cap E(C^{e_j}_{\alpha_j,\beta_j}) = \emptyset$, for all $1 \le i, j \le r(G)$.*

In other words there exists a minimal proper edge coloring of G whose minimal cycles are edge-disjoint.

Proposition 3.2. *Let α be a proper minimal edge coloring of a class 2 graph G, $C_1 = C^{e_1}_{a_1,b_1}$ and $C_2 = C^{e_2}_{a_2,b_2}$ minimal edge-disjoint cycles corresponding to the distinct uncolored edges e_1 and e_2, where a_1, a_2, b_1, b_2 are colors of α. If $\{a_1, b_1\} \cap \{a_2, b_2\} \neq \emptyset$, then C_1 and C_2 are vertex-disjoint.*

Proof. Assume that $\{a_1, b_1\} \cap \{a_2, b_2\} \neq \emptyset$; in particular $a_1 = a_2 = a$. Let us assume there is a vertex $v \in V(C_1) \cap V(C_2)$. Since the cycles are edge-disjoint, we have the impossible condition of two distinct edges incident to v, which have the same color. □

In [8] all the graphs of class 2, which are specified, have a double of class 1. Thus we suggested that all double graphs are of class 1. In [9] many other results confirmed the supposition and the related conjecture was stated.

Theorem 3.3. *Let G be a simple class 2 graph; then $G \times T_2$ is of class 1.*

Proof. Let G be a graph of class 2, with maximum degree $\Delta(G) = d$. Denoted $D = \{1, 2, \dots, d\} \cup \{x\}$ a set of $d+1$ colors, let $\alpha : G \to D$ be a proper minimal edge coloring of G, which uses the colors of D. Let A be a color class of cardinality $r(G)$ of α. We assume that the color x is assigned to the edges of A.

Recall that $G \times T_2$ is the union of the subgraphs $G \times K_2$, $G_1 = G \times S(u_1 u_1)$ and $G_2 = G \times S(u_2 u_2)$. Consider the coloring $\alpha_1 : G_1 \to D$ which assigns to the edge $(v, u_1)(w, u_1)$ of G_1, where $vw \in E(G)$, the color $\alpha(vw)$ and the coloring $\alpha_2 : G_2 \to D$ which assigns to the edge $(v, u_2)(w, u_2)$ of G_2 again the color $\alpha(vw)$. In particular $\alpha(vw) = x$, if vw is uncolored, i.e. it is an element of A. Moreover consider the coloring $\alpha_3 : G \times K_2 \to D'$, where $D' = \{1+d, 2+d, \dots, 2d\} \cup \{x'\}$ and $x' \neq x$, the coloring which assigns to the edges $(v, u_1)(w, u_2)$ and $(v, u_2)(w, u_1)$ the same color $\alpha(vw) + d$, if vw is not uncolored, and the color x', if vw is uncolored. Then the coloring α_3 of $G \times K_2$ is reflective and also the coloring α' of $G \times T_2$, union of $\alpha_1, \alpha_2, \alpha_3$, is reflective.

Let $A = \{e_1, e_2, \ldots, e_{r(G)}\}$ and $C_1, C_2, \ldots, C_{r(G)}$ be minimal cycles of G associated with $e_1, e_2, \ldots, e_{r(G)}$ respectively. By Theorem 3.1 we assume that these cycles are edge-disjoint.

Denote $V(C_1) = \{v_1, v_2, \ldots, v_n\}$ the set of vertices of C_1; in particular $C_1 = C^{e_1}_{a,b}$, $e_1 = v_1v_n$ is uncolored and n is odd.

Denoted $z_i = (v_i, u_1)$, $z'_i = (v_i, u_2)$, $1 \le i \le n$, we associate with C_1 the 3 cycles: $H_{1,1} = (z_1, z_2, \ldots, z_n)$, $H_{1,2} = (z'_1, z'_2, \ldots, z'_n)$, and $H_{1,3} = (z_1, z'_2, \ldots, z_n, z'_1, z_2, z'_3, \ldots, z'_n)$. See Fig. 1.

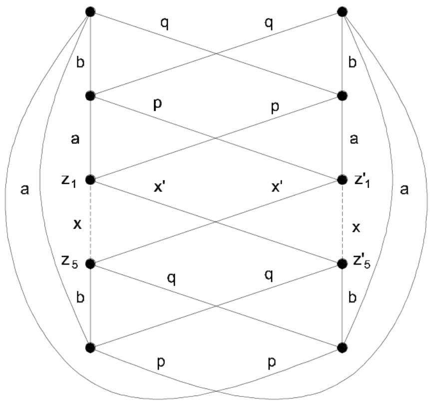

Figure 1: $(n = 5)$

The cycles $H_{1,1}, H_{1,2}$ are colored using the colors a, b and x; in particular the edges z_1z_2 and $z'_1z'_2$ are colored a, the edges $z_{n-1}z_n$ and $z'_{n-1}z'_n$ are colored b while the edges z_1z_n and $z'_1z'_n$ are colored x. The cycle $H_{1,3}$ has a reflective coloring which uses alternatively the colors $p = a + d, q = b + d$ for the edges distinct from $z_1z'_n$ and z'_1z_n, which are colored x'. In particular the edges $z_1z'_2$ and z'_1z_2 are colored p, while the edges $z'_{n-1}z_n$ and $z_{n-1}z'_n$ are colored q.

Now in the cycle $H_{1,2}$ we exchange the colors a and b and in the path $z'_1, z_2, z'_3, \ldots, z'_n$, contained in the cycle $H_{1,3}$, the colors p and q. The new coloring is again proper; moreover the sets of colors of the vertices z_i, z'_i for $2 \le i \le n-1$, z_1 and z_n have not changed, but the sets $C(z'_1)$, in which p is replaced by q and a by b, and $C(z'_n)$, in which the color q is replaced by p and b by a.

In the 4-cycle (z_2, z_3, z'_2, z'_1), which after previous change is alternatively colored b and q with the edge z_2z_3 colored b, we exchange the colors b and q; finally we exchange the colors a, b in the edges of the path z_1, z_2, z'_1.

The new coloring is proper; in particular the sets of colors of the vertices of $G \times T_2$ have not changed, but $C(z_1)$, in which a is replaced by b, $C(z'_1)$, in which

p is replaced by q and $C(z'_n)$, in which q is replaced by p and b by a.

In conclusion we notice that:

1. $C(z_1)$ and $C(z_n)$ are missing of a, which will be assigned to the edge z_1z_n;
2. $C(z'_1)$ and $C(z'_n)$ are missing of b, which will be assigned to the edge $z'_1z'_n$;
3. $C(z_1)$ and $C(z'_n)$ are missing of q, which will be assigned to the edge $z_1z'_n$;
4. $C(z'_1)$ and $C(z_n)$ are missing of p, which will be assigned to the edge z'_1z_n.

We have obtained a proper edge coloring of the subgraph $H_1 = H_{1,1} \cup H_{1,2} \cup H_{1,3}$ where the colors x, assigned to the edges $z_1z_n, z'_1z'_n$, and x', assigned to the edges $z_1z'_n, z'_1z_n$, have been replaced by elements of the set $\{1,2,\dots,2d\}$. See Fig. 2.

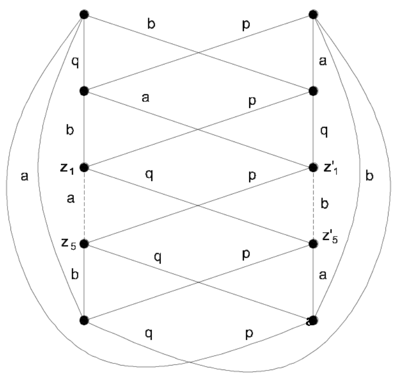

Figure 2: $(n = 5)$

Notice that the edge coloring of $G \times T_2$, in which H_1 has the new coloring, is proper. Moreover the coloring of the subgraph $(G \times T_2) \setminus H_1$ is not changed. Let $C_2 = C^{e_2}_{a',b'}$ another minimal cycle of G, edge-disjoint from C_1 by Theorem 3.1 and H_2 the subgraph formed by the cycles $H_{2,1}, H_{2,2}, H_{2,3}$. By Proposition 3.2, either C_2 is vertex-disjoint from C_1 or $\{a,b\} \cap \{a',b'\} = \emptyset$. In the first case H_2 is vertex-disjoint from H_1; in the second case the new coloring of H_1 has not consequences on the coloring of H_2.

We repeat the same procedure and obtain that also the subgraph $H_2 = H_{2,1} \cup H_{2,2} \cup H_{2,3}$ is colored using colors of $D \cup D'$ distinct from x and x'. By continuing we obtain a proper edge coloring of $G \times T_2$ using the $2d$ colors of $D \cup D'$ without the colors x, x'. Thus $G \times T_2$ is of class 1. $\square$

3.1. Generalized double graphs

Recall that, for any $k \in \mathbb{N}$, $k \geq 1$, the *generalized double graph* of a graph G is defined as $\mathcal{D}_k[G] = G \times T_k$, where $T_k = K_k^s$ is the total graph [5].

Proposition 3.4. *Let $k > 2$ be even and G a simple graph of class 2; then $G \times T_k$ is of class 1.*

Proof. Let $k = 2h$, where $h > 1$. The graph K_{2h} is the union of $2h-1$ matchings $M_1, M_2, \dots, M_{2h-1}$, of cardinality h. Denote M_1^s the subgraph obtained from M_1 by adding a loop to each vertex. For every edge $uv \in M_1$, denote T_{uv} the graph, isomorphic to T_2, formed by the edge uv, by adding a loop to u and v. Thus $M_1^s = \cup_{uv \in M_1} T_{uv}$. Then $G \times T_{2h}$ is decomposed into $G \times M_1^s$ and $G \times M_j$, $2 \leq j \leq 2h-1$. In particular $G \times M_1^s$ is the union of the vertex-disjoint subgraphs $G \times T_{uv}$, for $uv \in M_1$. By Theorem 3.3 $G \times T_{uv}$ is of class 1. Denote $A_1, A_2, \dots, A_{2h}$ disjoint d-sets of colors. We assign to $G \times T_{uv}$, for every edge $uv \in M_1$, a same proper edge coloring which uses $2d$ colors; in particular the colors of the sets A_1 and A_{2h}. Then we assign to $G \times M_j$, $2 \leq j \leq 2h-1$, which is a bipartite graph of maximum degree d, a proper edge coloring which uses the colors of A_j. Thus $G \times T_{2h}$ has a proper edge coloring which uses $2dh$ colors, where $2dh$ is the maximum degree of $G \times T_{2h}$. In other words $G \times T_{2h}$ is of class 1. □

Proposition 3.5. *Let G be a d-regular graph of odd order and class 2. For $k > 1$ odd, $G \times T_k$ is of class 2.*

Proof. The simple graph $G \times T_k$ is kd-regular and has an odd number of vertices. It follows immediately [1] that $G \times T_k$ is of class 2. □

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REFERENCES

[1] S. Fiorini - R. J. Wilson, *Edge-colourings of graphs*, Research Notes in Mathematics, 16 Pitman, London, 1977.

[2] B. L. Garman - R. D. Ringeisen - A. T. White, *On the genus of strong tensor products of graphs*, Canad. J. Math. 28 (1976), 523–532.

[3] M. Kochol, *Three measures of edge-uncolorability*, Discrete Math. 311 (2011), 106–108.

[4] W. Imrich - S. Klavžar, *Product graphs*, Wiley-Interscience, New York, 2000.

[5] M. C. Marino - N. Zagaglia Salvi, *Generalizing double graphs*, Atti della Accademia Peloritana dei Pericolanti, Classe di Sc. Fis., Mat. e Nat. 85 (2007), 1–9.

[6] B. Mohar - T. Pisanski, *Edge-coloring of a family of regular graphs*, Publ. Inst. Math. 33 (47) (1983), 157–162.

[7] B. Mohar - T. Pisanski - J. Shawe-Taylor, *1-Factorization of the composition of regular graphs*, Publ. Inst. Math. 33 (47) (1983), 193–196.

[8] E. Munarini - C. Perelli Cippo - A. Scagliola - N. Zagaglia Salvi, *Double graphs*, Discrete Math. 308 (2008), 242–254.

[9] E. Munarini - C. Perelli Cippo, *On some colorings of a double graph*, Electronic Notes in Discrete Math. 40 (2013), 277–281.

[10] V. V. Mkrtchyan - E. Steffen, *Maximum* Δ- *Edge-Colorable Subgraphs of Class II Graphs*, J. of Graph Theory 70 (4) (2012), 473–482.

[11] V. V. Mkrtchyan - E. Steffen, *Measures of edge-uncolorability*, Discrete Math. 312 (2012), 476–478.

[12] E. Steffen, *Classification and characterizations of snarks*, Discrete Math. 188 (1998), 183–203.

[13] E. Steffen, *Measurements of edge-uncolorability*, Discrete Math. 280 (2004), 191–214.

[14] J. Yan - Q. Huang, *Color-character of uncolorable cubic graphs*, Applied Math. Letters 22 (2009), 1653–1658.

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