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SUBORDINATION PROPERTIES OF CERTAIN SUBCLASSES OF p-VALENT FUNCTIONS DEFINED BY AN INTEGRAL OPERATOR

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In this paper, we investigate inclusion relationships among certain classes of p-valent analytic functions which are defined by means of integral operator.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$ and we write $A(1) = A$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, $z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathrm{U}) \subset g(\mathrm{U}).$$

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For functions f given by (1) and $g \in A(p)$ given by $g(z) = z^p + \sum\limits_{k=p+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

For $p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $l \geq 0$, we define the integral operator $I_p^m(l) : A(p) \to A(p)$ as follows:

$$\begin{aligned}
I_p^0(l) f(z) &= f(z), \\
I_p^1(l) f(z) &= (p+l) z^{-l} \int_0^z t^{l-1} I_p^0(l) f(t) dt \\
&= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l}{k+l} \right) a_k z^k, \\
I_p^2(l) f(z) &= (p+l) z^{-l} \int_0^z t^{l-1} I_p^1(l) f(t) dt \\
&= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l}{k+l} \right)^2 a_k z^k,
\end{aligned}$$

and (in general)

$$\begin{aligned}
I_p^{m+1}(l) f(z) &= (p+l) z^{-l} \int_0^z t^{l-1} I_p^m(l) f(t) dt \\
&= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l}{k+l} \right)^{m+1} a_k z^k \, (p \in \mathbb{N}; m \in \mathbb{N}_0; l \geq 0). \qquad (2)
\end{aligned}$$

From (2), it is easy to verify the identity

$$\begin{aligned}
&z \left(I_p^{m+1}(l) f(z) \right)' \\
&\qquad = (p+l) I_p^m(l) f(z) - l I_p^{m+1}(l) f(z) \quad (p \in \mathbb{N}; m \in \mathbb{N}_0; l \geq 0). \qquad (3)
\end{aligned}$$

By specializing the parameters m, ℓ and p, we obtain the following operators studied by various authors:

(i) $I_1^{\alpha}(1) f(z) = I^{\alpha} f(z)$ (see Jung et al. [6]);

$$= \left\{ f \in A : I^{\alpha} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^{\alpha} a_k z^k ; \alpha > 0 ; z \in \mathrm{U} \right\};$$

(ii) $I_p^{\alpha}(1)f(z) = I_p^{\alpha}f(z)$ (see Shams et al. [14]);

$$= \left\{ f \in A(p) : I_p^{\alpha}f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+1}{k+1} \right)^{\alpha} a_k z^k ; \alpha > 0; z \in \mathrm{U} \right\};$$

(iii) $I_p^m(1)f(z) = D^m f(z)$ (see Patel and Sahoo [11]);

$$= \left\{ f \in A(p) : D^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+1}{k+1} \right)^{m} a_k z^k ; m \in \mathbb{Z}_0 = \{0, \pm 1, \dots\} \right\};$$

(vi) $I_1^m(l)f(z) = I^m f(z)$ (see Flett [5]);

$$= \left\{ f \in A : I^m f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^{m} a_k z^k \ (m \in \mathbb{N}_0;\, z \in \mathrm{U}) \right\};$$

(v) $I_1^m(l)f(z) = I^m f(z)$ (see Salagean [13])

$$= \left\{ f \in A : I^m f(z) = z + \sum_{k=2}^{\infty} k^{-m} a_k z^k \ (m \in \mathbb{N}_0; z \in \mathrm{U}) \right\}.$$

Also we note that:
$I_p^m(0)f(z) = I_p^m f(z)$

$$= \left\{ f \in A(p) : I_p^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{k} \right)^{m} a_k z^k \ (m \in \mathbb{N}_0; z \in \mathrm{U}) \right\}.$$

Definition 1.1. For fixed parameters A and B, with $-1 \leq B < A \leq 1$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, $l \geq 0$ and $p > \eta$, we say that the function $f \in A(p)$ is in the class $S_p^m(l, \eta; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\eta} \left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)} - \eta \right) \prec \frac{1+Az}{1+Bz}. \tag{4}$$

A function $f \in A$ analytic in U is called to be a *convex function of order* α, $\alpha < 1$, if $f'(0) \neq 0$ and

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \ (z \in \mathrm{U}).$$

If $\alpha = 0$, then the function f is called to be *convex*.

It is easy to check that, if $h(z)=\dfrac{1+Az}{1+Bz}$, then $h'(0)\neq 0$ and $\Re\left[1+\dfrac{zh''(z)}{h'(z)}\right]$ $=\Re\left(\dfrac{1-Bz}{1+Bz}\right)>0$, $z\in\mathrm{U}$, whenever $|B|\leq 1$ and $A\neq B$, hence h is convex in U.

If $B\neq -1$, from the fact that $h(\overline{z})=\overline{h(z)}$, $z\in\mathrm{U}$, we deduce that the image $h(\mathrm{U})$ is symmetric with respect to the real axis, and that h maps the unit disc U onto the disc $\left|w-\dfrac{1-AB}{1-B^2}\right|<\dfrac{A-B}{1-B^2}$. If $B=-1$, the function h maps the unit disc U onto the half plane $\Re\{w\}>\dfrac{1-A}{2}$, hence we obtain:

Remark 1.2. The function $f\in A(p)$ is in the class $S_p^m(l,\eta;A,B)$ if and only if

$$\left|\frac{1}{p-\eta}\left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)}-\eta\right)-\frac{1-AB}{1-B^2}\right|<\frac{A-B}{1-B^2}\quad(B\neq -1;z\in\mathrm{U}),\quad(5)$$

and

$$\Re\left\{\frac{1}{p-\eta}\left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)}-\eta\right)\right\}>\frac{1-A}{2}\ (B=-1;z\in\mathrm{U}).\qquad(6)$$

Denoting by $K_p^m(l,\eta;\rho)$ the class of functions $f\in A(p)$ that satisfy the inequality

$$\Re\left\{\frac{1}{p-\eta}\left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)}-\eta\right)\right\}>\rho\ (z\in\mathrm{U}),\qquad(7)$$

where $\rho<1$, from (5) and (6) it follows respectively that

$$S_p^m(l,\eta;A,B)\subset K_p^m\left(l,\eta,\frac{1-A}{1-B}\right),$$

and

$$S_p^m(l,\eta;A,B)=K_p^m\left(l,\eta,\frac{1-A}{2}\right)\Leftrightarrow B=-1.$$

Let us consider the first-order differential subordination

$$H(\varphi(z),z\varphi'(z))\prec h(z).$$

A univalent function q is called its dominant, if $\varphi(z)\prec q(z)$ for all analytic functions φ that satisfy this differential subordination. A dominant $\widetilde{q}$ is called the best dominant, if $\widetilde{q}(z)\prec q(z)$ for all dominants q. For the general theory of the first-order differential subordination and its applications, we refer the reader to [2] and [8].

The object of the present paper is to obtain several inclusion relationships and other interesting properties of functions belonging to $S_p^m(l,\eta;A,B)$ and $K_p^m(l,\eta;\rho)$ by using the method of differential subordination.

2. Preliminaries

To establish our main results, we shall require the following lemmas. The first one deals with the *Briot-Bouquet differential subordinations*.

Lemma 2.1 ([4]). *Let $\beta,\gamma \in \mathbb{C}$, and let h be a convex function with*

$$\Re[\beta h(z)+\gamma] > 0 \quad (z \in \mathrm{U}).$$

If p is analytic in U*, with $p(0)=h(0)$, then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

The class of *starlike (and normalized) functions of order* α in U, $\alpha < 1$, is

$$S^*(\alpha)=\left\{f \in A : \Re\frac{zf'(z)}{f(z)} > \alpha,\, z \in \mathrm{U}\right\}.$$

In particular, the class $S^*(0) \equiv S^*$ is the class of *starlike (normalized) functions*.

For complex numbers a,b and c, the *Gauss hypergeometric function* is defined by

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1+\frac{a\cdot b}{c}\frac{z}{1!}+\frac{a(a+1)\cdot b(b+1)}{c(c+1)}\frac{z^2}{2!}+\dots \\ &= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad a,b \in \mathbb{C},\ c \in \mathbb{C}\setminus\{0,-1,-2,\dots\}, \quad (8) \end{aligned}$$

where $(d)_k = d(d+1)\dots(d+k-1)$ and $(d)_0 = 1$. The series (8) converges absolutely for $z \in \mathrm{U}$, hence it represents an analytic function in U (see [15, chapter 14]).

Lemma 2.2 ([10]). *Let $\beta > 0$, $\beta+\gamma > 0$ and consider the integral operator $J_{\beta,\gamma}$ defined by*

$$J_{\beta,\gamma}(f(z)) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}},$$

where the powers are the principal ones.

If $\sigma \in \left[-\frac{\gamma}{\beta},1\right)$ then the order of starlikeness of the class $J_{\beta,\gamma}(S^*(\sigma))$, i.e. the largest number $\delta(\sigma;\beta,\gamma)$ such that $J_{\beta,\gamma}(S^*(\sigma)) \subset S^*(\delta)$, is given by the number $\delta(\sigma;\beta,\gamma) = \inf\{\Re\{q(z)\} : z \in \mathrm{U}\}$, where

$$q(z)=\frac{1}{\beta Q(z)}-\frac{\gamma}{\beta} \quad \text{and} \quad Q(z)=\int_0^1\left(\frac{1-z}{1-tz}\right)^{2\beta(1-\sigma)} t^{\beta+\gamma-1}dt.$$

Moreover, if $\sigma \in [\sigma_0, 1)$*, where* $\sigma_0 = \max\left\{\frac{\beta-\gamma-1}{2\beta}; -\frac{\gamma}{\beta}\right\}$ *and* $g = J_{\beta,\gamma}(f)$ *with* $f \in S^*(\sigma)$*, then*

$$\Re\left\{\frac{zg'(z)}{g(z)}\right\} > \delta(\sigma;\beta,\gamma)\ (z \in \mathrm{U}),$$

where

$$\delta(\sigma;\beta,\gamma) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{{}_2F_1(1, 2\beta(1-\sigma); \beta+\gamma+1; \frac{1}{2})} - \gamma\right].$$

Lemma 2.3 ([10])**.** *Let* ϕ *be analytic in* U *with* $\phi(0) = 1$ *and* $\phi(z) \neq 0$ *for* $0 < |z| < 1$*, and let* $A, B \in \mathbb{C}$ *with* $A \neq B$*,* $|B| \leq 1$*.*

(i) Let $B \neq 0$ *and* $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ *satisfy either* $\left|\frac{\gamma(A-B)}{B} - 1\right| \leq 1$ *or* $\left|\frac{\gamma(A-B)}{B} + 1\right| \leq 1$*. If* ϕ *satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\phi(z) \prec (1+Bz)^{\frac{\gamma(A-B)}{B}},$$

and this is the best dominant.

(ii) Let $B = 0$ *and* $\gamma \in \mathbb{C}^*$ *be such that* $|\gamma A| < \pi$*, and if* ϕ *satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\phi(z) \prec e^{\gamma Az}$$

and this is the best dominant.

3. Inclusion relationships

Unless otherwise mentioned, we assume throughout this paper that $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, $l \geq 0$, $-1 \leq B < A \leq 1$, $p > \eta \geq 0$ and the power understood as principal values.

Theorem 3.1. *Let*

$$(p-\eta)(1-A) + (l+\eta)(1-B) \geq 0. \tag{9}$$

(i) Supposing that $I_p^m(l)f(z)\neq 0$ for all $z\in\bar{\mathrm{U}}=\mathrm{U}\setminus\{0\}$, then

$$S_p^m(l,\eta;A,B)\subset S_p^{m+1}(l,\eta;A,B).$$

(ii) Moreover, if we suppose in addition that

$$A\leq 1+\frac{1-B}{p-\eta}\min\left\{\frac{2\eta+l-p+1}{2};\eta+l\right\}, \tag{10}$$

then

$$S_p^m(l,\eta;A,B)\subset K_p^{m+1}(l,\eta;\rho(A,B)),$$

where the bound

$$\rho(A,B)=\frac{1}{p-\eta}\left[\frac{p+l}{{}_2F_1(1,2(p-\eta)(A-B)/(1-B);p+l+1;\frac{1}{2})}-(\eta+l)\right], \tag{11}$$

is the best possible.

Proof. Let $f\in S_p^m(l,\eta;A,B)$, and put

$$g(z)=z\left(\frac{I_p^{m+1}(l)f(z)}{z^p}\right)^{1/(p-\eta)}\quad(z\in\mathrm{U}), \tag{12}$$

since $I_p^{m+1}(l)f(z)\neq 0$ for all $z\in\bar{\mathrm{U}}$, the function g is analytic in U, with $g(0)=0$ and $g'(0)=1$. Taking the logarithmic differentiation in (12), we have

$$\phi(z)=\frac{zg'(z)}{g(z)}=\frac{1}{p-\eta}\left(\frac{z(I_p^{m+1}(l)f(z))'}{I_p^{m+1}(l)f(z)}-\eta\right)\quad(z\in\mathrm{U}), \tag{13}$$

then, using the identity (3) in (13), we obtain

$$(p+l)\frac{I_p^m(l)f(z)}{I_p^{m+1}(l)f(z)}=(p-\eta)\phi(z)+\eta+l. \tag{14}$$

Logarithmically differentiating in both sides of (14), and multiplying by z, we have

$$\frac{1}{p-\eta}\left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)}-\eta\right)=\phi(z)+\frac{z\phi'(z)}{(p-\eta)\phi(z)+\eta+l}. \tag{15}$$

Combining (15) together with $f\in S_p^m(l,\eta;A,B)$, we obtain that the function ϕ satisfies the Briot-Bouquet differential subordination

$$\phi(z)+\frac{z\phi'(z)}{(p-\eta)\phi(z)+\eta+l}\prec\frac{1+Az}{1+Bz}.$$

Now we will use Lemma 2.1 for the special case $\beta = p - \eta$ and $\gamma = \eta + l$. Since h is a convex function in U, a simple computation shows that

$$\Re\left\{(p-\eta)\frac{1+Az}{1+Bz}+\eta+l\right\}>0\ (z\in \mathrm{U}),$$

whenever (9) holds, then we have $\phi(z) \prec \frac{1+Az}{1+Bz}$, i.e. $f \in S_p^{m+1}(l,\eta;A,B)$. If, in addition, we suppose that the inequality (10) holds, then all the assumptions of Lemma 2.2 are verified for the above values of β,γ and $\sigma = \dfrac{1-A}{1-B}$. Then it follows the inclusion $S_p^m(l,\eta;A,B) \subset K_p^{m+1}(l,\eta;\rho(A,B))$, where the bound $\rho(A,B)$ given by (11) is the best possible. □

From Theorem 3.1, according to the definition 1.1 and (7), we deduce the next inclusions:

Corollary 3.2. *Let (9) holds.*

(1) Suppose that $I_p^m(l)f(z) \neq 0$ for all $z \in \bar{\mathrm{U}}$, then

$$S_p^m(l,\eta;A,B) \subset S_p^{m+1}(l,\eta;A,B) \subset K_p^{m+1}\left(l,\eta;\frac{1-A}{1-B}\right).$$

(2) If we suppose in addition that (10) holds, then

$$S_p^m(l,\eta;A,B) \subset S_p^{m+1}(l,\eta;A,B) \subset K_p^{m+1}(l,\eta;\rho(A,B)),$$

where $\rho(A,B)$ is given by (11). As a consequence of the last inclusion, we have $\rho(A,B) \geq \dfrac{1-A}{1-B}$.

For the special case $B=-1$, Theorem 3.1 reduces to:

Corollary 3.3. *Let $a > -\dfrac{\eta+l}{p-\eta}$.*

(1) Suppose that $I_p^m(l)f(z) \neq 0$ for all $z \in \bar{\mathrm{U}}$, then

$$K_p^m(l,\eta;a) \subset K_p^{m+1}(l,\eta;a).$$

(2) If we suppose in addition that

$$a \geq \max\left\{-\frac{2\eta+l-p+1}{2(p-\eta)};-\frac{\eta+l}{p-\eta}\right\},$$

then

$$K_p^m(l,\eta;a) \subset K_p^{m+1}(l,\eta;\rho(a)),$$

where the bound

$$\rho(a)=\frac{1}{p-\eta}\left[\frac{p+l}{{}_2F_1(1,2(p-\eta)(1-a);p+l+1;\frac{1}{2})}-(\eta+l)\right],$$

is the best possible.

Theorem 3.4. *If $f\in K_p^{m+1}(l,\eta;\rho)$, where $\rho<1$, then $f\in K_p^m(l,\eta;\rho)$ for $|z|<R$, where*

$$R=\min\{r>0:\theta(r)=0\} \tag{16}$$

and

$$\theta(r)=\frac{2r}{(1-r)(p-\eta)\left|(1-\rho)(1-r)-\left|\rho+\frac{\eta+l}{p-\eta}\right|(1+r)\right|}.$$

Proof. Since $f\in K_p^{m+1}(l,\eta;\rho)$, the function $k(z)$ given by

$$(1-\rho)k(z)+\rho=\frac{1}{p-\eta}\left(\frac{z(I_p^{m+1}(l)f(z))'}{I_p^{m+1}(l)f(z)}-\eta\right), \tag{17}$$

is analytic in U with $k(0)=1$ and $\Re\{k(z)\}>0$. Using the identity (3) in (17) and taking the logarithmic differentiation in the resulting equation, we obtain

$$\frac{1}{p-\eta}\left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)}-\eta\right)-\rho$$

$$=(1-\rho)\left[k(z)+\frac{zk'(z)}{(p-\eta)(1-\rho)k(z)+\rho(p-\eta)+\eta+l}\right], \tag{18}$$

hence

$$\Re\left\{\frac{1}{p-\eta}\left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)}-\eta\right)-\rho\right\}$$

$$\geq(1-\rho)\left[\Re\{k(z)\}-\frac{|zk'(z)|}{(p-\eta)\left|(1-\rho)k(z)-\left|\rho+\frac{\eta+l}{p-\eta}\right|\right|}\right]. \tag{19}$$

By using the well-known results [7]

$$\left|zk'(z)\right|\leq\frac{2r}{1-r^2}\Re\{k(z)\}\ \text{ and }\ \Re\{k(z)\}\geq\frac{1-r}{1+r}\quad(|z|=r<1),$$

together with the inequality (19), we get

$$\Re\left\{\frac{1}{p-\eta}\left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)}-\eta\right)-\rho\right\}\geq(1-\rho)[1-\theta(r)]\Re\{k(z)\}\ (|z|=r). \tag{20}$$

Since the right hand side term of the inequality (20) is nonnegative whenever $|z| \leq R$, where R is given by (16), using the fact that the real part of an analytic function is harmonic, we deduce that $f \in K_p^m(l,\eta;\rho)$ for $|z| < R$. $\square$

For a function $f \in A(p)$, let the integral operator $F_{\delta,p} : A(p) \to A(p)$ defined by (see [3])

$$\begin{aligned} F_{\delta,p}(f(z)) &= \frac{\delta+p}{z^\delta}\int_0^z t^{\delta-1}f(t)dt \\ &= z^p + \sum_{k=p+1}^{\infty}\frac{\delta+p}{\delta+k}a_k z^k = \left(z^p + \sum_{k=p+1}^{\infty}\frac{\delta+p}{\delta+k}z^k\right) * f(z) \\ &= z^p\,{}_2F_1(1,\delta+p;\delta+p+1;z) * f(z)\ (z \in \mathrm{U};\delta > -p), \end{aligned} \tag{21}$$

From (2) and (21), we have

$$z\left(I_p^m(l)\,F_{\delta,p}(f(z))\right)' = (\delta+p)I_p^m(l)\,f(z) - \delta I_p^m(l)\,F_{\delta,p}(f(z))\ \ (z \in \mathrm{U}), \tag{22}$$

and

$$I_p^m(l)\,F_{\delta,p}(f(z)) = F_{\delta,p}(I_p^m(l)\,f(z)),\ f \in A(p).$$

We now prove

Theorem 3.5. *Let $p+\delta > 0$ and*

$$(1-B)(\delta+\eta) + (1-A)(p-\eta) \geq 0. \tag{23}$$

(i) Supposing that $F_{\delta,p}(f(z)) \neq 0$ for all $z \in \bar{\mathrm{U}}$, then

$$F_{\delta,p}(S_p^m(l,\eta;A,B)) \subset S_p^m(l,\eta;A,B).$$

(ii) Moreover, if we suppose in addition that

$$A \leq 1 + \frac{1-B}{p-\eta}\min\left\{\frac{\delta+2\eta-p+1}{2};\delta+\eta\right\}, \tag{24}$$

then

$$F_{\delta,p}(S_p^m(l,\eta;A,B)) \subset K_p^m(l,\eta;r(A,B)),$$

where the bound

$$r(A,B) = \frac{1}{p-\eta}\left[\frac{\delta+p}{{}_2F_1\left(1,\frac{2(p-\eta)(A-B)}{1-B};\delta+p+1;\frac{1}{2}\right)} - (\delta+\eta)\right], \tag{25}$$

is the best possible.

Proof. Let $f \in S_p^m(l,\eta;A,B)$, and suppose that $F_{\delta,p}(f(z)) \neq 0$ for all $z \in \bar{\mathrm{U}}$. Let

$$g(z) = z\left(\frac{I_p^m(l)\,F_{\delta,p}(f(z))}{z^p}\right)^{\frac{1}{p-\eta}}\ (z \in \mathrm{U}), \tag{26}$$

then g is analytic in U, with $g(0) = 0$ and $g'(0) = 1$. Differentiating (26) logarithmically with respect to z, we have

$$\phi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p-\eta}\left(\frac{z\left(I_p^m(l)\,F_{\delta,p}(f(z))\right)'}{I_p^m(l)\,F_{\delta,p}(f(z))} - \eta\right)\ (z \in \mathrm{U}). \tag{27}$$

Now, by using the differential formula (22) in (27), we obtain

$$(p+\delta)\frac{I_p^m(l)\,f(z)}{I_p^m(l)\,F_{\delta,p}(f(z))} = (p-\eta)\phi(z) + (\delta+\eta). \tag{28}$$

Differentiating logarithmically the relation (28) and multiplying by z, we have

$$\frac{1}{p-\eta}\left(\frac{z\left(I_p^m(l)\,f(z)\right)'}{I_p^m(l)\,f(z)} - \eta\right) = \phi(z) + \frac{z\phi'(z)}{(p-\eta)\phi(z)+(\delta+\eta)}. \tag{29}$$

Since $f \in S_p^m(l,\eta;A,B)$, from (29), we obtain that the function ϕ satisfies the Briot-Bouquet differential subordination

$$\phi(z) + \frac{z\phi'(z)}{(p-\eta)\phi(z)+(\delta+\eta)} \prec \frac{1+Az}{1+Bz}.$$

The function $\frac{1+Az}{1+Bz}$ is convex in U, and it is easy to check that

$$\Re\left[(p-\eta)\frac{1+Az}{1+Bz} + (\delta+\eta)\right] > 0\ (z \in \mathrm{U}),$$

whenever (23) holds, then from Lemma 2.1 with $\beta = p-\eta$ and $\gamma = \delta+\eta$, we have $\phi(z) \prec \frac{1+Az}{1+Bz}$, that is, that $F_{\delta,p} \in S_p^m(l,\eta;A,B)$. If we suppose in addition that the inequality (24) holds, then all the assumptions of the Lemma 2.2 are satisfied for β,γ and $\sigma = \dfrac{1-A}{1-B}$, hence it follows the inclusion $F_{\delta,p}(S_p^m(l,\eta;A,B)) \subset K_p^m(l,\eta;r(A,B))$, and the bound $r(A,B)$ given by (25) is the best possible. □

Taking $B = -1$ in Theorem 3.5, we obtain the next corollary:

Corollary 3.6. *Let $p+\delta>0$ and $a\geq -\frac{\delta+\eta}{p-\eta}$.*

(i) Supposing that $F_{\delta,p}f(z)\neq 0$ for all $z\in\bar{\mathrm{U}}$, then

$$F_{\delta,p}(K_p^m(l,\eta;a))\subset K_p^m(l,\eta;a).$$

(ii) If we suppose in addition that

$$a\geq \max\left\{-\frac{\delta+2\eta-p+1}{2(p-\eta)};-\frac{\delta+\eta}{p-\eta}\right\},$$

then

$$F_{\delta,p}(K_p^m(l,\eta;a))\subset K_p^m(l,\eta;r(a)),$$

where the bound

$$r(a)=\frac{1}{p-\eta}\left[\frac{\delta+p}{{}_2F_1\left(1,2(p-\eta)(1-a);\delta+p+1;\frac{1}{2}\right)}-(\delta+\eta)\right],$$

is the best possible.

Theorem 3.7. *Let $\nu\in\mathbb{C}^*$ and let $A,B\in\mathbb{C}$ with $A\neq B$ and $|B|\leq 1$. Suppose that*

$$\begin{aligned}
&\left|\frac{\nu(p+l)(A-B)}{B}-1\right|\leq 1 \text{ or } \left|\frac{\nu(p+l)(A-B)}{B}+1\right|\leq 1, && \text{if } B\neq 0,\\
&|\nu|\leq\frac{\pi}{p+l}, && \text{if } B=0.
\end{aligned}$$

If $f\in A(p)$ with $I_p^{m+1}(l)f(z)\neq 0$ for all $z\in\bar{\mathrm{U}}$, then

$$\frac{I_p^m(l)f(z)}{I_p^{m+1}(l)f(z)}\prec\frac{1+Az}{1+Bz}$$

implies

$$\left(\frac{I_p^{m+1}(l)f(z)}{z^p}\right)^{\nu}\prec q_1(z),$$

where

$$q_1(z)=\begin{cases}(1+Bz)^{\nu(p+l)(A-B)/B}, & \text{if } B\neq 0,\\ e^{\nu(p+l)Az}, & \text{if } B=0,\end{cases}$$

is the best dominant.

Proof. Let us put

$$\phi(z) = \left(\frac{I_p^{m+1}(l)\, f(z)}{z^p} \right)^{\nu} \quad (z \in \mathrm{U}), \tag{30}$$

then ϕ is analytic in U, $\phi(0) = 1$ and $\phi(z) \neq 0$ for all $z \in \mathrm{U}$. Taking the logarithmic derivatives in both sides of (30) and using the identity (3) we have

$$1 + \frac{z\phi'(z)}{\nu(p+l)\phi(z)} = \frac{I_p^m(l)\, f(z)}{I_p^{m+1}(l)\, f(z)} \prec \frac{1+Az}{1+Bz}.$$

Now the assertions of Theorem 3.7 follows by using Lemma 2.3 for $\gamma = \nu(p+l)$. □

Putting $B = -1$ and $A = 1 - 2\rho$, $0 \leq \rho < 1$, in Theorem 3.7, we obtain the following result:

Corollary 3.8. *Assume that $\nu \in \mathbb{C}^*$ satisfies either $|2\nu(p+l)(1-\rho) - 1| \leq 1$ or $|2\nu(p+l)(1-\rho) + 1| \leq 1$. If $f \in A(p)$ with $I_p^{m+1}(l)\, f(z) \neq 0$ for $z \in \bar{\mathrm{U}}$, then*

$$\Re\left\{ \frac{I_p^m(l)\, f(z)}{I_p^{m+1}(l)\, f(z)} \right\} > \rho \quad (z \in \mathrm{U}),$$

implies

$$\left(\frac{I_p^m(l)\, f(z)}{z^p} \right)^{\nu} \prec q_2(z) = (1-z)^{-2\nu(p+l)(1-\rho)},$$

and q_2 is the best dominant.

4. Properties involving the operator $I_p^m(l)$

Theorem 4.1. *If $f \in S_p^m(l, \eta; A, B)$, then, for all $s, t \in \mathbb{C}$ with $|s| \leq 1$, $|t| \leq 1$, and $s \neq t$, the next subordination holds:*

$$\frac{t^p I_p^m(l)\, f(zs)}{s^p I_p^m(l)\, f(zt)} \prec \begin{cases} \left(\dfrac{1+Bzs}{1+Bzt} \right)^{(p-\eta)(A-B)/B}, & (B \neq 0), \\ \exp[(p-\eta)Az(s-t)], & (B = 0). \end{cases} \tag{31}$$

Proof. If $f \in S_p^m(l, \eta; A, B)$, from (4) it follows that

$$\frac{z(I_p^m(l)\, f(z))'}{I_p^m(l)\, f(z)} \prec \frac{p + [pB + (A-B)(p-\eta)]z}{1+Bz} \equiv k(z). \tag{32}$$

Moreover, the function k defined by (32) and the function h given by

$$h(z) \equiv h(z;s,t) = \int_0^z \left(\frac{s}{1-su} - \frac{t}{1-tu} \right) du$$

are convex in U. By combining a general subordination theorem [12, Theorem 4] with (32), we get

$$\left(\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)} - p \right) * h(z) \prec \frac{(p-\eta)(A-B)z}{1+Bz} * h(z). \tag{33}$$

For every analytic function ϕ in U with $\phi(0) = 0$, we have

$$\phi(z) * h(z) = \int_{tz}^{sz} \frac{\phi(u)}{u} du, \tag{34}$$

and thus, from (33) and (34), we deduce

$$\int_{tz}^{sz} \left(\frac{(I_p^m(l)f(u))'}{I_p^m(l)f(u)} - \frac{p}{u} \right) du \prec (p-\eta)(A-B) \int_{tz}^{sz} \frac{du}{1+Bu}.$$

This last subordination implies

$$\exp\left(\int_{tz}^{sz} \left(\frac{(I_p^m(l)f(u))'}{I_p^m(l)f(u)} - \frac{p}{u} \right) du \right) \prec \exp\left((p-\eta)(A-B) \int_{tz}^{sz} \frac{du}{1+Bu} \right),$$

and by simplification, we get the assertion of Theorem 4.1. $\square$

Corollary 4.2. *If $f \in S_p^m(l,\eta;A,B)$, then for $|z| = r < 1$, the next inequalities hold:*

$$\left| I_p^m(l)f(z) \right| \leq \begin{cases} r^p(1+Br)^{(p-\eta)(A-B)/B}, & (B \neq 0), \\ r^p \exp[(p-\eta)Ar], & (B = 0), \end{cases} \tag{35}$$

$$\left| I_p^m(l)f(z) \right| \geq \begin{cases} r^p(1-Br)^{(p-\eta)(A-B)/B}, & (B \neq 0), \\ r^p \exp[-(p-\eta)Ar], & (B = 0), \end{cases} \tag{36}$$

and

$$\left| \arg \frac{I_p^m(l)f(z)}{z^p} \right| \leq \begin{cases} \frac{(p-\eta)(A-B)}{|B|} \sin^{-1}(|B|r), & (B \neq 0), \\ (p-\eta)Ar, & (B = 0). \end{cases} \tag{37}$$

All of the estimates asserted here are sharp.

Proof. Taking $s=1$ and $t=0$ in (4.1), and using the definition of subordination, we obtain

$$\frac{I_p^m(l)f(z)}{z^p}=\begin{cases}(1+Bw(z))^{(p-\eta)(A-B)/B}, & (B\neq 0),\\ \exp[(p-\eta)Aw(z)], & (B=0),\end{cases}\tag{38}$$

where w is analytic function in U with $w(0)=0$ and $|w(z)|\le 1$ for $z\in$ U. According to the well-known *Schwarz's Theorem*, we have $|w(z)|\le|z|$ for all $z\in$ U.

(i) If $B>0$, then we find from (38) that

$$\begin{aligned}\left|\frac{I_p^m(l)f(z)}{z^p}\right|&=\exp\left[\frac{(p-\eta)(A-B)}{B}\log|1+Bw(z)|\right]\\&=|1+Bw(z)|^{\frac{(p-\eta)(A-B)}{B}}\le(1+Br)^{\frac{(p-\eta)(A-B)}{B}}.\end{aligned}$$

(ii) If $B<0$, we can easily obtain

$$\left|\frac{I_p^m(l)f(z)}{z^p}\right|=|1+Bw(z)|^{\frac{(p-\eta)(A-B)}{-B}}\le\left[(1+Br)^{-1}\right]^{\frac{(p-\eta)(A-B)}{-B}}=(1+Br)^{\frac{(p-\eta)(A-B)}{B}}.$$

This proves the inequality (35) for $B\neq 0$. Similarly, we can prove the other inequalities in (35) and (36). Now, for $|z|=r$ and $B\neq 0$, we observe from (38) that

$$\begin{aligned}\left|\arg\frac{I_p^m(l)f(z)}{z^p}\right|&=\frac{(p-\eta)(A-B)}{|B|}|\arg(1+Bw(z))|\\&\le\frac{(p-\eta)(A-B)}{|B|}\sin^{-1}(|B|r),\end{aligned}$$

and, for $B=0$, (37) is a direct consequence of (38).

It is easy to see that all of the estimates in Corollary 4.2 are sharp, being attained by the function f_0 defined by

$$I_p^m(l)f_0(z)=\begin{cases}z^p(1+Bz)^{(p-\eta)(A-B)/B}, & (B\neq 0),\\ z^p\exp[(p-\eta)Az], & (B=0).\end{cases}\tag{39}$$

□

Corollary 4.3. *If $f\in S_p^m(l,\eta;A,B)$, then, for all $|z|=r<1$, the next inequalities hold:*

$$\left|(I_p^m(l)f(z))'\right|\le\begin{cases}r^{p-1}\{p+[\eta B+(p-\eta)A]r\}(1+Br)^{\frac{(p-\eta)(A-B)}{B}-1}, & (B\neq 0),\\ r^{p-1}[p+(p-\eta)Ar]\exp((p-\eta)Ar), & (B=0),\end{cases}\tag{40}$$

$$\left|(I_p^m(l)f(z))'\right| \geq \begin{cases} r^{p-1}\{p-[\eta B+(p-\eta)A]r\}(1-Br)^{\frac{(p-\eta)(A-B)}{B}-1}, & (B\neq 0),\\ r^{p-1}[p-(p-\eta)Ar]\exp(-(p-\eta)Ar), & (B=0), \end{cases} \tag{41}$$

and

$$\left|\arg\frac{(I_p^m(l)f(z))'}{z^{p-1}}\right| \leq \begin{cases} \frac{(p-\eta)(A-B)}{|B|}\sin^{-1}(|B|r)+ \\ +\sin^{-1}\left[\frac{(p-\eta)(A-B)r}{p-[\eta B+(p-\eta)A]Br^2}\right], (B\neq 0), \\ \\ (p-\eta)Ar+\sin^{-1}\left[\frac{A(p-\eta)r}{p}\right], (B=0). \end{cases} \tag{42}$$

All of the estimates asserted here are sharp.

Proof. If we let

$$g(z)=\frac{z(I_p^m(l)f(z))'}{I_p^m(l)f(z)}\ (z\in \mathrm{U}), \tag{43}$$

then g is analytic in U with $g(0)=p$, and

$$g(z)\prec\frac{p+[pB+(A-B)(p-\eta)]z}{1+Bz}.$$

It is known from [1] that the function g satisfies the following sharp inequalities:

$$\frac{p-[\eta B+(p-\eta)A]r}{1-Br}\leq|g(z)|\leq\frac{p+[\eta B+(p-\eta)A]r}{1+Br},\ |z|=r<1, \tag{44}$$

$$\left|g(z)-\frac{p-[\eta B+(p-\eta)A]Br^2}{1-B^2r^2}\right|\leq\frac{(A-B)(p-\eta)r}{1-B^2r^2},\ |z|=r<1, \tag{45}$$

and

$$|\arg g(z)|\leq\sin^{-1}\left[\frac{(A-B)(p-\eta)r}{p-[\eta B+(p-\eta)A]Br^2}\right],\ |z|=r<1. \tag{46}$$

Using (44), (45) and (46), in conjunction with the estimates given by Corollary 4.2, in (43) , we deduce the estimates (40), (41) and (42) of Corollary 4.3. All of the estimates are sharp for the function f_0 defined by (39). □

Remark 4.4. Putting $l=0$ in the above results, we obtain corresponding results for the integral operator I_p^m.

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REFERENCES

[1] M. K. Aouf, *On a subclass of p-valent starlike functions of order* α, Internat. J. Math. Math. Sci. 10 (4) (1987), 733–744.

[2] T. Bulboacă, *Differential Subordinations and Superordinations. Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[3] J. H. Choi - M. Saigo - H. M. Srivastava, *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl. 276 (2002), 432–445.

[4] P. Eenigenburg - S. S. Miller - P. T. Mocanu - M. O. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities 3, I. S. N. M., Vol. 64, Birkhäuser Verlag, Basel 1983, 339–348.

[5] T. M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. 38 (1972), 746–765.

[6] T. B. Jung - Y. C. Kim - H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. 176 (1993), 138–147.

[7] T. H. MacGregor, *Radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. 14 (1963), 514–520.

[8] S. S. Miller - P. T. Mocanu, *Differential Subordination. Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[9] P. T. Mocanu - D. Ripeanu - I. Şerb, *The order of starlikeness of certain integral operators*, Mathematica (Cluj) 23 (46) (1981), 225–230.

[10] M. Obradović - S. Owa, *On certain properties for some classes of starlike functions*, J. Math. Anal. Appl. 145 (1990), 357–364.

[11] J. Patel - P. Sahoo, *Certain subclasses of multivalent analytic functions*, Indian J. Pure Appl. Math. 34 (3) (2003), 487–500.

[12] S. Ruscheweyh - T. Sheil-Small, *Hadamard products of schilicht functions of the Polya-Schoenberg conjecture*, Comment. Math. Helv. 48 (1973), 119–135.

[13] G. S. Salagean, *Subclasses of univalent functions*, Lecture Notes in Math. 1013, Springer-Verlag 1983, 362–372.

[14] S. Shams - S. R. Kulkarni - J. M. Jahangiri, *Subordination properties of p-valent functions defined by integral operators*, Internat. J. Math. Math. Sci. (2006) Art. ID 94572,1–3.

[15] E. T. Whittaker - G. N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principlal Transcendental Functions*, Fourth Edition, Cambridge University Press, Cambridge, 1927.

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