SUFFICIENT CONDITION FOR GENERALIZED SAKAGUCHI TYPE SPIRAL-LIKE FUNCTIONS

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In the present paper, the author defines a class of analytic generalized Sakaguchi type spiral-like functions on the open unit disk \( U \) and obtain certain sufficient condition for functions to be in this class. Several corollaries and consequences of the main results are also considered.

1. Introduction and Motivation

Let \( A_n \) denote the class of all functions \( f(z) \) of the form:

\[
f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k
\]

which are analytic in the open unit disk

\[
U := \{ z \in \mathbb{C} : |z| < 1 \}.
\]

In particular, for \( n = 1 \) we write \( A_1 := A \).

A function \( f(z) \in A_n \) is said to be starlike of order \( \alpha \) if it satisfies the inequality

\[
\Re \left[ \frac{zf'(z)}{f(z)} \right] > \alpha \quad (0 \leq \alpha < 1; \ z \in U).
\]
We denote such class by $S_n^*(\alpha)$. For $n = 1$, we denote such class by $S^*(\alpha)$. Further, a function $f \in A_n$ is said to be $\lambda$-spiral-like function of order $\beta$ denoted by $SP_n(\lambda, \beta)$ if and only if the following inequality holds true:

$$\Re \left[ e^{i\lambda} \frac{zf'(z)}{f(z)} \right] > \beta \quad (0 \leq \beta < 1, \quad |\lambda| < \frac{\pi}{2}; \quad z \in \mathbb{U}).$$

(3)

For $\beta = 0$ and $n = 1$, the class $SP_1(\lambda, 0)$ reduces to $S_p(\lambda)$ (see [1]). Špaček [2] proved that members of $S_p(\lambda)$ known as $\lambda$-spiral-like functions that are univalent in the unit disk $\mathbb{U}$.

Recently, Goyal et al. [3] introduced and studied the class $S_n(\beta, t)$ as follows. A function $f(z) \in A_n$ is said to be in the class $S_n(\beta, t)$ if it satisfies

$$\Re \left[ \frac{(1-t)zf'(z)}{f(z) - f(tz)} \right] > \beta \quad (|t| \leq 1, \quad |t| \neq 1)$$

(4)

for some $\beta$ $(0 \leq \beta < 1)$ and for all $z \in \mathbb{U}$.

Motivated by above mentioned work, we define the subclass of $A_n$ as follows:

**Definition 1.1.** A function $f(z) \in A_n$ is said to be in the generalized Sakaguchi type spiral-like class $S_n(\lambda, \beta, s, t)$ if it satisfies

$$\Re \left[ e^{i\lambda} \frac{(s-t)zf'(sz)}{f(sz) - f(tz)} \right] > \beta \cos \lambda \quad (z \in \mathbb{U}),$$

(5)

for some $\beta$ $(0 \leq \beta < 1)$, $s$ and $t$ are real parameters, $s > t$ and $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$.

By specializing the parameters $\lambda, \ n, \ s, \ t$ and $\beta$, we obtain the following subclasses studied by earlier authors. For

- $\lambda = 0, \ s = 1$, the class $S_n(0, \beta, 1, t) = S_n(\beta, t)$ has been studied by Goyal et al. [3];
- $s = n = 1, \lambda = 0$, the class $S_1(0, \beta, 1, t) = S(\beta, t)$ has been studied by Owa et al. [4, 5], Goyal and Goswami [6] and Cho et al. [7];
- $s = 1, \lambda = 0, \ n = 1, \ \beta = 0, \ t = -1$, the class $S_1(0, 0, 1, -1) = S(0, -1)$ has introduced and studied by Sakaguchi [8].

We note that for $\lambda = 0, \ n = 1, \ s = 1, \ t = 0$, the above class reduce to the well-known subclass of $A$ consisting of univalent starlike functions of order $\beta$ [9].

The object of the present paper is to obtain certain sufficient condition for a function $f \in A_n$ to be in the class $S_n(\lambda, \beta, s, t)$.

We need the following lemma for our investigation:

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Lemma 1.2 (see [10]). Let \( \Omega \) be a set in the complex plane \( \mathbb{C} \) and suppose that \( \phi \) is a mapping from \( \mathbb{C}^2 \times U \) to \( \mathbb{C} \) which satisfies \( \phi(ix,y,z) \notin \Omega \) for \( z \in U \), and for all real \( x, y \) such that \( y \leq \frac{n}{2}(1+x^2) \). If the function \( p(z) = 1+c_nz^n+\cdots \) is analytic in \( U \) and \( \phi(p(z),zp'(z);z) \in \Omega \) for all \( z \in U \), then \( \Re(p(z)) > 0 \).

2. Main Results

Unless otherwise stated, we assume throughout our sequel, that \( \lambda \) is real with \( |\lambda| < \frac{\pi}{2} \), \( 0 \leq \beta < 1 \), \( n \in \mathbb{N} \), \( s \) and \( t \) are reals such that \( s > t \).

Theorem 2.1. If \( f(z) \in A_n \) satisfies

\[
\Re \left[ \left( e^{i\lambda} \frac{(s-t)^2zf'(sz)}{f(sz) - f(tz)} \right) \left( \frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tf'(tz)}{f(sz) - f(tz)} + 1 \right) \right] > \frac{Q^2}{4P} + R, \tag{6}
\]

where \( 0 \leq \alpha \leq 1 \) and

\[
P = \alpha(1-\beta) \left\{ \frac{n}{2}(s-t) + s(1-\beta)\cos^2 \lambda \right\} \cos \lambda, \tag{7}
\]

\[
Q = \alpha s(1-\beta)(\beta \cos \lambda - 1)\sin 2\lambda \cos \lambda, \tag{8}
\]

\[
R = \left[ \beta(1-\alpha) - \frac{n\alpha}{2}(1-\beta) \right] (s-t)\cos \lambda + \alpha s\beta^2 \cos^3 \lambda + \alpha s \left( \beta - \frac{1}{2} \right) \sin \lambda \sin 2\lambda, \tag{9}
\]

then \( f(z) \in S_n(\lambda,\beta,s,t) \).

Proof. Define the function \( p(z) \) by

\[
e^{i\lambda} \frac{(s-t)zf'(sz)}{f(sz) - f(tz)} = [(1-\beta)p(z) + \beta] \cos \lambda + isin \lambda. \tag{10}
\]

Then \( p(z) = 1+c_nz^n+\cdots \) is analytic in \( U \) with \( p(0) = 1 \).

Taking logarithmic differentiation on both sides of (10) with respect to \( z \), we get

\[
\frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tf'(tz)}{f(sz) - f(tz)} + 1 = \frac{\alpha szf'(sz)}{f(sz) - f(tz)}
\]

\[
+ \left[ (1-\beta)p(z) + \beta \right] \cos \lambda + isin \lambda + 1 - \alpha. \tag{11}
\]
Therefore, it follows that

\[
\frac{e^{i\lambda} (s - t)^2 z f'(sz)}{f(sz) - f(tz)} \left[ \frac{\alpha sz f''(sz)}{f'(sz)} + \frac{\alpha tz f'(tz)}{f(sz) - f(tz)} + 1 \right] = Lzp'(z) + Mp^2(z) + Np(z) + O = \phi(p(z), zp'(z); z) \text{ (say),}
\]

where

\[
L = (s-t)(1-\beta)cos\lambda,
\]

\[
M = \alpha se^{-i\lambda} (1-\beta)^2 cos^2\lambda,
\]

\[
N = (1-\beta)[(1-\alpha)(s-t)cos\lambda + \alpha se^{-i\lambda}(2\beta cos^2\lambda + i sin2\lambda)],
\]

\[
O = (1-\alpha)(s-t)(\beta cos\lambda + i sin\lambda) + \alpha se^{-i\lambda}(2\beta^2 cos^2\lambda - sin^2\lambda + i\beta sin2\lambda).
\]

Now, for all real \( x \) and \( y \) satisfying \( y \leq -\frac{m}{2}(1 + x^2) \), we have

\[
\phi(ix, y; z) = Ly - Mx^2 + iNx + O
\]

Taking real part on both side of (13), we have

\[
\Re\phi(ix, y; z) \leq -Px^2 + Qx + R
\]

\[
= -\left[ \sqrt{P}x - \frac{Q}{2\sqrt{P}} \right]^2 + \frac{Q^2}{4P} + R
\]

\[
\leq \frac{Q^2}{4P} + R,
\]

where \( P, Q \) and \( R \) are given by (7), (8) and (9) respectively.

Let

\[
\Omega = \{ w : \Re w > \frac{Q^2}{4P} + R \}.
\]

Then

\[
\phi(p(z), zp'(z); z) \in \Omega \quad \text{and} \quad \phi(ix, y; z) \notin \Omega
\]

for all real \( x \) and \( y \) satisfying \( y \leq -\frac{m}{2}(1 + x^2), \ z \in \mathbb{U} \).

Hence by virtue of Lemma 1.2, we obtain the desired result.

If we take \( \lambda = 0 \) in Theorem 2.1, we obtain
Corollary 2.2 (see [11]). If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\Re \left[ \left( \frac{(s-t)^2 zf'(sz)}{f(sz) - f(tz)} \right) \left( \frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tf'(tz)}{f(sz) - f(tz)} + 1 \right) \right]
> \alpha \beta \left\{ s\beta + \frac{n}{2} (s-t) - (s-t) \right\} + \left\{ \beta - \frac{n\alpha}{2} \right\} (s-t)
\]
\[
(0 \leq \alpha \leq 1, \ 0 \leq \beta < 1, \ s > t; \ z \in \mathbb{U}),
\]
then \( f(z) \in \mathcal{S}_n(\beta, s, t) \).

If we take \( s = 1 \) in Corollary 2.2, we obtain

Corollary 2.3 (see [3]). If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\Re \left[ \left( \frac{(1-t)^2 zf'(z)}{f(z) - f(tz)} \right) \left( \frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha tf'(tz)}{f(z) - f(tz)} + 1 \right) \right]
> \alpha \beta \left\{ \beta + \frac{n}{2} (1-t) - (1-t) \right\} + \left\{ \beta - \frac{n\alpha}{2} \right\} (1-t)
\]
\[
(0 \leq \alpha \leq 1, \ 0 \leq \beta < 1, \ |t| \leq 1, \ t \neq 1; \ z \in \mathbb{U}),
\]
then \( f(z) \in \mathcal{S}_n(\beta, t) \).

Taking \( t = -1 \) in Corollary 2.3 gives:

Corollary 2.4. If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\Re \left[ \left( \frac{zf'(z)}{f(z) - f(-z)} \right) \left( \frac{\alpha zf''(z)}{f'(z)} - \frac{\alpha zf'(-z)}{f(z) - f(-z)} + 1 \right) \right]
> \frac{\alpha \beta}{4} \left( \beta + n - 2 \right) + \left( \frac{2\beta - n\alpha}{4} \right)
\]
\[
(0 \leq \alpha \leq 1, \ 0 \leq \beta < 1; \ z \in \mathbb{U}),
\]
then \( f(z) \in \mathcal{S}_n(\beta, -1) \).

By taking \( \beta = 0 \) in Corollary 2.4, we have

Corollary 2.5. If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\Re \left[ \left( \frac{zf'(z)}{f(z) - f(-z)} \right) \left( \frac{\alpha zf''(z)}{f'(z)} - \frac{\alpha zf'(-z)}{f(z) - f(-z)} + 1 \right) \right] > \frac{-n\alpha}{4}
\]
\[
(0 \leq \alpha \leq 1; \ z \in \mathbb{U}),
\]
then \( f(z) \in \mathcal{S}_n(0, -1) \).
Putting $t = 0$ in Corollary 2.3, we obtain the following result.

**Corollary 2.6** (see [12]). If $f(z) \in A_n$ satisfies
\[
\Re \left[ \frac{zf'(z)}{f(z)} \left( \frac{\alpha z f''(z)}{f'(z)} + 1 \right) \right] > \alpha \beta \left( \beta + \frac{n}{2} - 1 \right) + \left( \frac{\beta - n \alpha}{2} \right)
\]
\[(0 \leq \alpha \leq 1, \ 0 \leq \beta \leq 1; \ z \in \mathbb{U}),\]
then $f(z) \in S_n(\beta, 0) = S^*_n(\beta)$.

If we take $n = 1$ and $\beta = 0$ in Corollary 2.6, we obtain

**Corollary 2.7** (see [13]). If $f \in A$ satisfies the inequality
\[
\Re \left[ \frac{zf'(z)}{f(z)} \left( \frac{\alpha z f''(z)}{f'(z)} + 1 \right) \right] > -\frac{\alpha}{2} \quad (z \in \mathbb{U}),
\]
for some $\alpha \ (0 \leq \alpha \leq 1)$, then $f(z) \in S_1(0, 0) = S^*$.

Taking $\lambda = 0, n = 1, \beta = \frac{\alpha}{2}$ and $s = 1$ in Theorem 2.1 yields

**Corollary 2.8** (see [12]). If $f(z) \in A$ satisfies the condition
\[
\Re \left[ \frac{(1-t)^2zf'(z)}{f(z)-f(tz)} \left( \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha t z f'(tz)}{f'(z) + 1} \right) \right] > \frac{\alpha^2}{4} (\alpha - (1-t))
\]
\[(|t| \leq 1, \ t \neq 1, \ 0 \leq \alpha \leq 1; z \in \mathbb{U}),\]
then $f(z) \in S_1(0, \frac{\alpha}{2}, 1, t)$.

Putting $t = 0$ in the Corollary 2.8. we have

**Corollary 2.9.** If $f(z) \in A$ satisfies the condition
\[
\Re \left[ \frac{zf'(z)}{f(z)} \left( \frac{\alpha z f''(z)}{f'(z)} + 1 \right) \right] > -\frac{\alpha^2}{4} (1 - \alpha) \quad (z \in \mathbb{U}),
\]
for some $\alpha \ (\alpha \geq 0)$, then $f(z) \in S_1(0, \frac{\alpha}{2}, 1, 0) = S^*(\frac{\alpha}{2})$.

**Theorem 2.10.** If $f(z) \in A_n$ satisfies the condition
\[
\Re \left[ e^{i\lambda} \frac{f(z)}{z} \left( \frac{\alpha z f'(z)}{f(z)} - \alpha + 1 \right) \right] > -\frac{n \alpha}{2} (1 - \beta) \cos \lambda + \beta \cos \lambda, \quad (15)
\]
then
\[
\Re \left[ e^{i\lambda} \frac{f(z)}{z} \right] > \beta \cos \lambda \quad (16)
\]
Proof. Consider
\[ e^{i\lambda} \frac{f(z)}{z} = [(1 - \beta)p(z) + \beta] \cos \lambda + i \sin \lambda. \] (17)

Taking logarithmic differentiation on both sides of (17) with respect to \( z \) and after simplification, we get
\[ e^{i\lambda} \frac{f(z)}{z} \left( \frac{af'(z)}{f(z)} - \alpha + 1 \right) = \alpha(1 - \beta) \cos \lambda z p'(z) \]
\[ + [(1 - \beta)p(z) + \beta] \cos \lambda + i \sin \lambda = \phi(p(z), zp'(z); z). \] (18)

Therefore, for all real \( x \) and \( y \) satisfying \( y \leq -\frac{n}{2}(1 + x^2) \), we obtain
\[ \phi(ix, y; z) = \alpha(1 - \beta)ycos \lambda + [(1 - \beta)ix + \beta] \cos \lambda + isin \lambda. \] (19)

Taking real part on both sides of (19), we have
\[ \Re \phi(ix, y; z) = \alpha(1 - \beta)ycos \lambda + \beta cos \lambda \]
\[ \leq \alpha(1 - \beta)cos \lambda \left( -\frac{n}{2}(1 + x^2) \right) + \beta cos \lambda \]
\[ = -\frac{n\alpha}{2}(1 - \beta)x^2 \cos \lambda - \frac{n\alpha}{2}(1 - \beta) \cos \lambda + \beta \cos \lambda \]
\[ \leq -\frac{n\alpha}{2}(1 - \beta) \cos \lambda + \beta \cos \lambda. \] (20)

Let \( \Omega = \{ w : \Re w > -\frac{n\alpha}{2}(1 - \beta) \cos \lambda + \beta \cos \lambda \} \).

Then from (15), (18) and (20) we obtain \( \phi(p(z), zp'(z); z) \in \Omega \) and \( \phi(ix, y; z) \notin \Omega \) for all real \( x \) and \( y \) satisfying \( y \leq -\frac{n}{2}(1 + x^2) \). Hence by application of Lemma 1.2, we obtain the desired result. The proof of Theorem 2.10 is thus completed.

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