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SUFFICIENT CONDITION FOR GENERALIZED SAKAGUCHI TYPE SPIRAL-LIKE FUNCTIONS

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In the present paper, the author defines a class of analytic generalized Sakaguchi type spiral-like functions on the open unit disk $\mathbb{U}$ and obtain certain sufficient condition for functions to be in this class. Several corollaries and consequences of the main results are also considered.

1. Introduction and Motivation

Let $\mathcal{A}_n$ denote the class of all functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

In particular, for $n = 1$ we write $\mathcal{A}_1 := \mathcal{A}$.
A function $f(z) \in \mathcal{A}_n$ is said to be starlike of order α if it satisfies the inequality

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > \alpha \quad (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{2}$$

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We denote such class by $\mathcal{S}_n^*(\alpha)$. For $n=1$, we denote such class by $\mathcal{S}^*(\alpha)$. Further, a function $f\in\mathcal{A}_n$ is said to be λ-spiral-like function of order β denoted by $\mathcal{SP}_n(\lambda,\beta)$ if and only if the following inequality holds true:

$$\Re\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\right]>\beta \quad \left(0\leq\beta<1,\ |\lambda|<\frac{\pi}{2};\ z\in\mathbb{U}\right). \tag{3}$$

For $\beta=0$ and $n=1$, the class $\mathcal{SP}_1(\lambda,0)$ reduces to $\mathcal{S}_p(\lambda)$ (see [1]). Špaěck [2] proved that members of $\mathcal{S}_p(\lambda)$ known as λ-spiral-like functions that are univalent in the unit disk $\mathbb{U}$.

Recently, Goyal et al. [3] introduced and studied the class $\mathcal{S}_n(\beta,t)$ as follows. A function $f(z)\in\mathcal{A}_n$ is said to be in the class $\mathcal{S}_n(\beta,t)$ if it satisfies

$$\Re\left[\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right]>\beta \quad (|t|\leq 1,\ |t|\neq 1) \tag{4}$$

for some β $(0\leq\beta<1)$ and for all $z\in\mathbb{U}$.

Motivated by above mentioned work, we define the subclass of $\mathcal{A}_n$ as follows:

Definition 1.1. A function $f(z)\in\mathcal{A}_n$ is said to be in the generalized Sakaguchi type spiral-like class $\mathcal{S}_n(\lambda,\beta,s,t)$ if it satisfies

$$\Re\left[e^{i\lambda}\frac{(s-t)zf'(sz)}{f(sz)-f(tz)}\right]>\beta cos\lambda \quad (z\in\mathbb{U}), \tag{5}$$

for some β $(0\leq\beta<1)$, s and t are real parameters, $s>t$ and λ is real with $|\lambda|<\frac{\pi}{2}$.

By specializing the parameters $\lambda,\ n,\ s,\ t$ and β, we obtain the following subclasses studied by earlier authors. For

- $\lambda=0,\ s=1$, the class $\mathcal{S}_n(0,\beta,1,t)=\mathcal{S}_n(\beta,t)$ has been studied by Goyal et al. [3];
- $s=n=1,\lambda=0$, the class $\mathcal{S}_1(0,\beta,1,t)=\mathcal{S}(\beta,t)$ has been studied by Owa et al. [4, 5], Goyal and Goswami [6] and Cho et al. [7];
- $s=1,\ \lambda=0,\ n=1,\ \beta=0,\ t=-1$, the class $\mathcal{S}_1(0,0,1,-1)=\mathcal{S}(0,-1)$ has introduced and studied by Sakaguchi [8].

We note that for $\lambda=0,\ \ n=1,\ \ s=1,\ \ t=0$, the above class reduce to the well-known subclass of $\mathcal{A}$ consisting of univalent starlike functions of order β [9].

The object of the present paper is to obtain certain sufficient condition for a function $f\in\mathcal{A}_n$ to be in the class $\mathcal{S}_n(\lambda,\beta,s,t)$.

We need the following lemma for our investigation:

Lemma 1.2 (see [10]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that ϕ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\phi(ix,y,z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real x, y such that $y \leq \frac{-n}{2}(1+x^2)$. If the function $p(z) = 1 + c_n z^n + \cdots$ is analytic in $\mathbb{U}$ and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re(p(z)) > 0$.*

2. Main Results

Unless otherwise stated, we assume throughout our sequel, that λ is real with $|\lambda| < \frac{\pi}{2}$, $0 \leq \beta < 1$, $n \in \mathbb{N}$, s and t are reals such that $s > t$.

Theorem 2.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(e^{i\lambda}\frac{(s-t)^2 z f'(sz)}{f(sz)-f(tz)}\right)\left(\frac{\alpha s z f''(sz)}{f'(sz)} + \frac{\alpha t z f'(tz)}{f(sz)-f(tz)} + 1\right)\right] > \frac{Q^2}{4P} + R, \quad (6)$$

where $0 \leq \alpha \leq 1$ and

$$P = \alpha(1-\beta)\left\{\frac{n}{2}(s-t) + s(1-\beta)cos^2\lambda\right\}cos\lambda, \quad (7)$$

$$Q = \alpha s(1-\beta)(\beta cos\lambda - 1)sin2\lambda cos\lambda, \quad (8)$$

$$R = \left[\beta(1-\alpha) - \frac{n\alpha}{2}(1-\beta)\right](s-t)cos\lambda + \alpha s\beta^2 cos^3\lambda + \alpha s\left(\beta - \frac{1}{2}\right)sin\lambda sin2\lambda, \quad (9)$$

then $f(z) \in \mathcal{S}_n(\lambda, \beta, s, t)$.

Proof. Define the function $p(z)$ by

$$e^{i\lambda}\frac{(s-t)zf'(sz)}{f(sz)-f(tz)} = [(1-\beta)p(z)+\beta]cos\lambda + isin\lambda. \quad (10)$$

Then $p(z) = 1 + c_n z^n + \cdots$ is analytic in $\mathbb{U}$ with $p(0) = 1$.
Taking logarithmic differentiation on both sides of (10) with respect to z, we get after simplification

$$\frac{\alpha s z f''(sz)}{f'(sz)} + \frac{\alpha t z f'(tz)}{f(sz)-f(tz)} + 1 = \frac{\alpha s z f'(sz)}{f(sz)-f(tz)} + \frac{\alpha(1-\beta)zp'(z)cos\lambda}{[(1-\beta)p(z)+\beta]cos\lambda + isin\lambda} + 1 - \alpha. \quad (11)$$

Therefore, it follows that

$$e^{i\lambda}\frac{(s-t)^2zf'(sz)}{f(sz)-f(tz)}\left[\frac{\alpha szf''(sz)}{f'(sz)}+\frac{\alpha tzf'(tz)}{f(sz)-f(tz)}+1\right]$$
$$=Lzp'(z)+Mp^2(z)+Np(z)+O$$
$$=\phi(p(z),zp'(z);z)(say), \tag{12}$$

□

where

$$L=\alpha(s-t)(1-\beta)cos\lambda$$
$$M=\alpha se^{-i\lambda}(1-\beta)^2cos^2\lambda$$
$$N=(1-\beta)[(1-\alpha)(s-t)cos\lambda+\alpha se^{-i\lambda}(2\beta cos^2\lambda+isin2\lambda)]$$
$$O=(1-\alpha)(s-t)[\beta cos\lambda+isin\lambda]+\alpha se^{-i\lambda}(\beta^2cos^2\lambda-sin^2\lambda+i\beta sin2\lambda).$$

Now, for all real x and y satisfying $y\le\frac{-n}{2}(1+x^2)$, we have

$$\phi(ix,y;z)=Ly-Mx^2+iNx+O \tag{13}$$

Taking real part on both side of (13), we have

$$\Re\phi(ix,y;z)\le-Px^2+Qx+R$$
$$=-\left[\sqrt{P}x-\frac{Q}{2\sqrt{P}}\right]^2+\frac{Q^2}{4P}+R$$
$$\le\frac{Q^2}{4P}+R, \tag{14}$$

where P, Q and R are given by (7), (8) and (9) respectively.
Let

$$\Omega=\{w:\Re w>\frac{Q^2}{4P}+R\}.$$

Then

$$\phi(p(z),zp'(z);z)\in\Omega\quad\text{and}\quad\phi(ix,y;z)\notin\Omega$$

for all real x and y satisfying $y\le\frac{-n}{2}(1+x^2),\ z\in\mathbb{U}$.
Hence by virtue of Lemma 1.2, we obtain the desired result.
If we take $\lambda=0$ in Theorem 2.1, we obtain

Corollary 2.2 (see [11]). *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(\frac{(s-t)^2 zf'(sz)}{f(sz)-f(tz)}\right)\left(\frac{\alpha szf''(sz)}{f'(sz)}+\frac{\alpha tzf'(tz)}{f(sz)-f(tz)}+1\right)\right] > \alpha\beta\left\{s\beta+\frac{n}{2}(s-t)-(s-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(s-t)$$
$$(0 \le \alpha \le 1,\ 0 \le \beta < 1,\ s > t;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(\beta, s, t)$.

If we take $s = 1$ in Corollary 2.2, we obtain

Corollary 2.3 (see [3]). *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\right)\left(\frac{\alpha zf''(z)}{f'(z)}+\frac{\alpha tzf'(tz)}{f(z)-f(tz)}+1\right)\right] > \alpha\beta\left\{\beta+\frac{n}{2}(1-t)-(1-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(1-t)$$
$$(0 \le \alpha \le 1,\ 0 \le \beta < 1,\ |t| \le 1,\ t \ne 1;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(\beta, t)$.

Taking $t = -1$ in Corollary 2.3 gives:

Corollary 2.4. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(\frac{zf'(z)}{f(z)-f(-z)}\right)\left(\frac{\alpha zf''(z)}{f'(z)}-\frac{\alpha zf'(-z)}{f(z)-f(-z)}+1\right)\right] > \frac{\alpha\beta}{4}(\beta+n-2)+\left(\frac{2\beta-n\alpha}{4}\right)$$
$$(0 \le \alpha \le 1,\ 0 \le \beta < 1;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(\beta, -1)$.

By taking $\beta = 0$ in Corollary 2.4, we have

Corollary 2.5. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(\frac{zf'(z)}{f(z)-f(-z)}\right)\left(\frac{\alpha zf''(z)}{f'(z)}-\frac{\alpha zf'(-z)}{f(z)-f(-z)}+1\right)\right] > \frac{-n\alpha}{4}$$
$$(0 \le \alpha \le 1;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(0, -1)$.

Putting $t=0$ in Corollary 2.3, we obtain the following result.

Corollary 2.6 (see [12]). *If $f(z)\in\mathcal{A}_n$ satisfies*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)}+1\right)\right]>\alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{n\alpha}{2}\right)$$
$$(0\leq\alpha\leq 1,\ \ 0\leq\beta\leq 1;\ \ z\in\mathbb{U}),$$

then $f(z)\in\mathcal{S}_n(\beta,0)=\mathcal{S}_n^*(\beta)$.

If we take $n=1$ and $\beta=0$ in Corollary 2.6, we obtain

Corollary 2.7 (see [13]). *If $f\in\mathcal{A}$ satisfies the inequality*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)}+1\right)\right]>-\frac{\alpha}{2}\quad(z\in\mathbb{U}),$$

for some α $(0\leq\alpha\leq 1)$, then $f(z)\in\mathcal{S}_1(0,0)=\mathcal{S}^*$.

Taking $\lambda=0, n=1, \beta=\frac{\alpha}{2}$ and $s=1$ in Theorem 2.1 yields

Corollary 2.8 (see [12]). *If $f(z)\in\mathcal{A}$ satisfies the condition*

$$\Re\left[\frac{(1-t)^2zf'(z)}{f(z)-f(tz)}\left\{\frac{\alpha zf''(z)}{f'(z)}+\frac{\alpha tzf'(tz)}{f(z)-f(tz)}+1\right\}\right]>\frac{\alpha^2}{4}(\alpha-(1-t))$$
$$(|t|\leq 1,\ \ t\neq 1,\ \ 0\leq\alpha\leq 1; z\in\mathbb{U}),$$

then $f(z)\in\mathcal{S}_1(0,\frac{\alpha}{2},1,t)$.

Putting $t=0$ in the Corollary 2.8. we have

Corollary 2.9. *If $f(z)\in\mathcal{A}$ satisfies the condition*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)}+1\right)\right]>-\frac{\alpha^2}{4}(1-\alpha)\quad(z\in\mathbb{U}),$$

for some α $(\alpha\geq 0)$, then $f(z)\in\mathcal{S}_1(0,\frac{\alpha}{2},1,0)=\mathcal{S}^*(\frac{\alpha}{2})$.

Theorem 2.10. *If $f(z)\in\mathcal{A}_n$ satisfies the condition*

$$\Re\left[e^{i\lambda}\frac{f(z)}{z}\left(\frac{\alpha zf'(z)}{f(z)}-\alpha+1\right)\right]>\frac{-n\alpha}{2}(1-\beta)cos\lambda+\beta cos\lambda,\tag{15}$$

then

$$\Re\left[e^{i\lambda}\frac{f(z)}{z}\right]>\beta cos\lambda\tag{16}$$

Proof. Consider

$$e^{i\lambda}\frac{f(z)}{z} = [(1-\beta)p(z)+\beta]cos\lambda + i\sin\lambda. \tag{17}$$

Taking logarithmic differentiation on both sides of (17) with respect to z and after simplification, we get

$$\begin{aligned} e^{i\lambda}\frac{f(z)}{z}\left(\frac{\alpha z f'(z)}{f(z)}-\alpha+1\right) &= \alpha(1-\beta)cos\lambda\ zp'(z) \\ +[(1-\beta)p(z)+\beta]cos\lambda + i\sin\lambda &= \phi(p(z),zp'(z);z). \end{aligned} \tag{18}$$

Therefore, for all real x and y satisfying $y \leq \frac{-n}{2}(1+x^2)$, we obtain

$$\phi(ix,y;z) = \alpha(1-\beta)ycos\lambda + [(1-\beta)ix+\beta]cos\lambda + isin\lambda. \tag{19}$$

Taking real part on both sides of (19), we have

$$\begin{aligned} \Re\phi(ix,y;z) &= \alpha(1-\beta)ycos\lambda + \beta cos\lambda \\ &\leq \alpha(1-\beta)cos\lambda\left(-\frac{n}{2}(1+x^2)\right)+\beta cos\lambda \\ &= -\frac{n\alpha}{2}(1-\beta)x^2cos\lambda - \frac{n\alpha}{2}(1-\beta)cos\lambda + \beta cos\lambda \\ &\leq \frac{-n\alpha}{2}(1-\beta)cos\lambda + \beta cos\lambda. \end{aligned} \tag{20}$$

Let $\Omega = \left\{w : \Re w > -\frac{n\alpha}{2}(1-\beta)cos\lambda + \beta cos\lambda\right\}$. □

Then from (15), (18) and (20) we obtain $\phi(p(z),zp'(z);z) \in \Omega$ and $\phi(ix,y;z) \notin \Omega$ for all real x and y satisfying $y \leq -\frac{n}{2}(1+x^2)$. Hence by application of Lemma 1.2, we obtain the desired result. The proof of Theorem 2.10 is thus completed.

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