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COEFFICIENT ESTIMATES FOR A CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS

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In the present investigation, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class $S_\Sigma(\lambda, h)$. The results presented in this paper improve or generalize the recent work of Magesh and Yamini.

1. Introduction and Definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [9] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z\ , \ (z \in U)$$

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and

$$f\left(f^{-1}(w)\right)=w\ ,\ \left(|w|<r_0(f)\ ,\ r_0(f)\geq\frac{1}{4}\right),$$

where

$$f^{-1}(w)=w\ -a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots.$$

A function $f(z)\in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, (see [20]).

Lewin [14] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2|\leq\sqrt{2}$ for $f\in\Sigma$. Netanyahu [18] showed that $max|a_2|=\frac{4}{3}$ if $f(z)\in\Sigma$.

Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^{\star}(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α $(0<\alpha\leq 1)$ respectively (see [18]). Thus, following Brannan and Taha [4], a function $f(z)\in A$ is said to be in the class $S_{\Sigma}^{\star}(\alpha)$ of strongly bi-starlike functions of order α $(0<\alpha\leq 1)$ if each of the following conditions is satisfied:

$$f\in\Sigma,\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\quad(0<\alpha\leq 1,\ z\in U)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}\quad(0<\alpha\leq 1,\ w\in U)$$

and is said to be in the class $K_{\Sigma}(\alpha)$ of strongly bi-convex functions of order α $(0<\alpha<1)$ if each of the following conditions is satisfied:

$$f\in\Sigma,\ \left|\arg\left(1+\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\quad(0<\alpha\leq 1,\ z\in U)$$

and

$$\left|\arg\left(1+\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}\quad(0<\alpha\leq 1,\ w\in U)$$

where g is the extension of f^{-1} to U. The classes $S_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $S^{\star}(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. In fact, the aforecited work of Srivastava et al. [20] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1–3, 7, 8, 10, 13, 15–17, 19, 20, 22, 23]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([6, 11, 12]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \backslash \{1,2\};\ \mathbb{N} = \{1,2,3,\dots\}$) is still an open problem.

In this paper, by using the method [21] different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered by Magesh and Yamini [16] and obtain a better estimate.

2. Coefficient Estimates

Definition 2.1. Let the functions $h, p : U \to \mathbb{C}$ be so constrained that

$$\min\{\Re(h(z)), \Re(p(z))\} > 0$$

and

$$h(0) = p(0) = 1.$$

Definition 2.2. A function $f \in \Sigma$ is said to be in the class $S_{\Sigma}(\lambda, h)$, $0 \leq \lambda \leq 1$, if the following conditions are satisfied:

$$\frac{zf'(z) + \left(2\lambda^2 - \lambda\right) z^2 f''(z)}{4(\lambda - \lambda^2) z + (2\lambda^2 - \lambda) zf'(z) + (2\lambda^2 - 3\lambda + 1) f(z)} \in h(U) \quad (z \in U) \quad (2)$$

and

$$\frac{wg'(w) + \left(2\lambda^2 - \lambda\right) w^2 g''(w)}{4(\lambda - \lambda^2) w + (2\lambda^2 - \lambda) wg'(w) + (2\lambda^2 - 3\lambda + 1) g(w)} \in p(U) \quad (w \in U) \quad (3)$$

where $g(w) = f^{-1}(w)$.

Theorem 2.3. *Let f given by (1) be in the class* $\mathrm{S}_{\Sigma}(\lambda, h)$. *Then*

$$|a_2| \leq \min\left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(1 + 3\lambda - 2\lambda^2)^2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{4(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)}} \right\} \quad (4)$$

and

$$|a_3| \leq \min\left\{ \frac{|h'(0)|^2+|p'(0)|^2}{2(1+3\lambda-2\lambda^2)^2} + \frac{|h''(0)|+|p''(0)|}{8(1+2\lambda^2)}, \right.$$
$$\left. \frac{\left(12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3\right)|h''(0)|+\left|12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1\right||p''(0)|}{8(1+2\lambda^2)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)} \right\}. \quad (5)$$

Proof. Let $f \in S_\Sigma(\lambda, h)$, $0 \leq \lambda \leq 1$. It follows from (2) and (3) that

$$\frac{zf'(z) + \left(2\lambda^2 - \lambda\right) z^2 f''(z)}{4(\lambda - \lambda^2) z + (2\lambda^2 - \lambda) zf'(z) + (2\lambda^2 - 3\lambda + 1) f(z)} = h(z) \qquad (6)$$

and

$$\frac{wg'(w) + \left(2\lambda^2 - \lambda\right) w^2 g''(w)}{4(\lambda - \lambda^2) w + (2\lambda^2 - \lambda) wg'(w) + (2\lambda^2 - 3\lambda + 1) g(w)} = p(w) \qquad (7)$$

where $h(z)$ and $p(w)$ satisfy the conditions of Definiton 2.1. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \cdots,$$

respectively. Since

$$\frac{zf'(z) + \left(2\lambda^2 - \lambda\right) z^2 f''(z)}{4\left(\lambda - \lambda^2\right) z + \left(2\lambda^2 - \lambda\right) zf'(z) + \left(2\lambda^2 - 3\lambda + 1\right) f(z)}$$
$$= 1 + \left(1 + 3\lambda - 2\lambda^2\right) a_2 z + \left[\left(12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1\right) a_2^2 + \left(4\lambda^2 + 2\right) a_3\right] z^2 + \cdots \qquad (8)$$

and

$$\frac{wg'(w) + \left(2\lambda^2 - \lambda\right) w^2 g''(w)}{4\left(\lambda - \lambda^2\right) w + \left(2\lambda^2 - \lambda\right) wg'(w) + \left(2\lambda^2 - 3\lambda + 1\right) g(w)}$$
$$= 1 - \left(1 + 3\lambda - 2\lambda^2\right) a_2 w + \left[\left(12\lambda^4 - 28\lambda^3 + 19\lambda^2 + 2\lambda + 3\right) a_2^2 - \left(4\lambda^2 + 2\right) a_3\right] w^2 + \cdots, \qquad (9)$$

it follows from (6), (7), (8) and (9) that

$$\left(1 + 3\lambda - 2\lambda^2\right) a_2 = h_1, \qquad (10)$$

$$\left(12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1\right) a_2^2 + \left(4\lambda^2 + 2\right) a_3 = h_2, \qquad (11)$$

and

$$-\left(1 + 3\lambda - 2\lambda^2\right) a_2 = p_1, \qquad (12)$$

$$\left(12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3\right)a_2^2-\left(4\lambda^2+2\right)a_3=p_2. \tag{13}$$

From (10) and (12) we obtain

$$h_1=-p_1,$$

and

$$2\left(1+3\lambda-2\lambda^2\right)^2a_2^2=h_1^2+p_1^2. \tag{14}$$

By adding (13) to (11), we find that

$$2\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)a_2^2=h_2+p_2. \tag{15}$$

Therefore, we find from (14) and (15) that

$$|a_2|^2\leq\frac{|h'(0)|^2+|p'(0)|^2}{2\left(1+3\lambda-2\lambda^2\right)^2}.$$

and

$$|a_2|^2\leq\frac{|h''(0)|+|p''(0)|}{4\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)}.$$

So we get the desired estimate on the coefficient $|a_2|$ as claimed in (4).

Next, in order to find the bound on $|a_3|$, by subtracting (13) from (11), we obtain

$$2\left(2+4\lambda^2\right)a_3-2\left(2+4\lambda^2\right)a_2^2=h_2-p_2.$$

Then, in view of (14) and (15), it follows that

$$a_3=\frac{h_1^2+p_1^2}{2\left(1+3\lambda-2\lambda^2\right)^2}+\frac{h_2-p_2}{2\left(2+4\lambda^2\right)}$$

and

$$a_3=\frac{h_2+p_2}{2\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)}+\frac{h_2-p_2}{2\left(2+4\lambda^2\right)}$$

as claimed. This completes the proof of Theorem 2.3. □

Remark 2.4. We note that for $\lambda=\frac{1}{2}$, the class $S_\Sigma(\lambda,h)$ reduces to the class $H_\Sigma(h)$ studied by Srivastava et al. [21].

Corollary 2.5 (see [21])**.** *When $\lambda=\frac{1}{2}$ the results discussed in this article reduce to results in ([21], Corollary 3). If $f\in H_\Sigma(h)$ then*

$$|a_2|\leq\min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{8}},\sqrt{\frac{|h''(0)|+|p''(0)|}{12}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{8}+\frac{|h''(0)|+|p''(0)|}{12},\frac{|h''(0)|}{6}\right\}.$$

Corollary 2.6. *If*

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2 z^2+\ldots \quad (0<\alpha\leq 1),$$

then inequalities (4) and (5) become

$$|a_2|\leq \min\left\{\frac{2\alpha}{1+3\lambda-2\lambda^2},\sqrt{\frac{2}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}}\alpha\right\}$$

and

$$|a_3|\leq \min\left\{\frac{4\alpha^2}{(1+3\lambda-2\lambda^2)^2}+\frac{\alpha^2}{1+2\lambda^2},\frac{2\alpha^2}{1+2\lambda^2}\right\}.$$

Remark 2.7. The estimates on the coefficients $|a_2|$ and $|a_3|$ of Corollary 2.6 are improvement of the estimates obtained in (Theorem 2.1, [16] and Corollary 2.3, [15]).

Remark 2.8. By setting $\lambda=0$ in Corollary 2.6 we get the following consequence.

Corollary 2.9. *Let the functions $f(z)$ given by Taylor-Maclaurin series expansion (1) be in the bi-univalent function class $S_{\Sigma}^{\star}(\alpha)$ $(0<\alpha\leq 1)$. Then*

$$|a_2|\leq\sqrt{2}\alpha$$

and

$$|a_3|\leq 2\alpha^2.$$

Corollary 2.10. *If*

$$\phi(z)=\frac{1+(1-2\alpha)z}{1-z}=1+2(1-\alpha)z+2(1-\alpha)z^2+\cdots \quad (0<\alpha\leq 1),$$

then inequalities (4) and (5) become

$$|a_2|\leq \min\left\{\frac{2(1-\alpha)}{1+3\lambda-2\lambda^2},\sqrt{\frac{2(1-\alpha)}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}}\right\}$$

and

$$|a_3|\leq \min\left\{\frac{4(1-\alpha)^2}{(1+3\lambda-2\lambda^2)^2}+\frac{1-\alpha}{1+2\lambda^2},\frac{(1-\alpha)}{1+2\lambda^2}\right\}.$$

Remark 2.11. The estimates on the coefficients $|a_2|$ and $|a_3|$ of Corollary 2.10 are improvement of the estimates obtained in (Theorem 3.1, [16] and Corollary 3.3, [15]).

Remark 2.12. By setting $\lambda = 0$ in Corollary 2.10 we get the following consequence.

Corollary 2.13 (see [21])**.** *Let the functions $f(z)$ given by Taylor-Maclaurin series expansion (1) be in the bi-univalent function class $S^{\star}_{\Sigma}(\alpha)$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \sqrt{2(1-\alpha)}$$

and

$$|a_3| \leq 1-\alpha.$$

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