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# DIFFRACTION OF ELECTROMAGNETIC WAVE ON THE SYSTEM OF METALLIC STRIPS IN THE STRATIFIED MEDIUM

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In this paper, we study diffraction of electromagnetic wave on the system of metallic strips in the stratified medium. The integral equation which represent this problem is solved by Galerkin's method. To this equation the plane diffraction problem for TE-polarizable electromagnetic wave on the system of metallic strips in the stratified medium was reduced.

## 1. Formulation of the problem.

Let planes $z = h_j,\ j = 1..n$ separate the space $(x, y, z)$ into domains $D_0 :\ z < h_1,\ \ D_j :\ h_j < z < h_{j+1},\ \ j = 1..n-1$ and $D_n :\ z > h_n$ filled with dielectric with dielectric indexes $\epsilon_j,\ \ j = 0..n$. Let the ideal conductive infinitely thin metallic strips be placed on the media interfaces parallel to the axis $y$ and segments $[\alpha_{jk}, \beta_{jk}],\ \ k = 1..m_j$ correspond to the strips on the line $z = h_j,\ \ j = 1..n$ in the plane $y = 0$.

Consider plane electromagnetic fields the components of which do

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not depend on the coordinate $y$. Denote by $M_j = \cup_{k=1}^{m_j}(\alpha_{jk}, \beta_{jk})$ and by $N_j$ the complement of $\overline{M_j}$ with respect to the whole real axis.

We need to seek a field arising under diffraction of the plane TE-wave with the potential function $\tilde{u}(x, z)$ which falls down from above on the stratified structure. The potential function $u(x, z)$ of the unknown field should be a solution of the Helmholez equation in the every layer

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + k_j^2 u(x, z) = 0, \quad (x, z) \in D_j \tag{1}$$

and should satisfy the conjugation conditions

$$\begin{gathered}
u(x, h_n + 0) = -\tilde{u}(x, h_n + 0),\ u(x, h_n - 0) = 0,\ x \in M_n; \\
u(x, h_n + 0) - u(x, h_n - 0) = -\tilde{u}(x, h_n + 0), \quad x \in N_n; \\
\frac{\partial u}{\partial z}(x, h_n + 0) - \frac{\partial u}{\partial z}(x, h_n - 0) = -\frac{\partial \tilde{u}}{\partial z}(x, h_n + 0), \quad x \in N_n; \\
u(x, h_j \pm 0) = 0,\ x \in M_j,\ j = 1..n - 1; \\
u(x, h_j + 0) - u(x, h_j - 0) = 0, \quad x \in N_j,\ j = 1..n - 1; \\
\frac{\partial u}{\partial z}(x, h_j + 0) - \frac{\partial u}{\partial z}(x, h_j - 0) = 0, \quad x \in N_j,\ j = 1..n - 1.
\end{gathered} \tag{2}$$

It is convenient to consider solution of the problem (1), (2) as a sum of two functions $u_j(x, z) = u(x, z)$ in $D_j$ completed by zero with respect to the whole plane. To justify the Fourier integral transformation method we will seek $u_j(x, z)$ in the Sobolev spaces of distributions of slow growth at infinity $H_1^{loc}(D_j)$. We can show that the generalized solutions coincide with the classical solutions after the unknown distributions are found.

Consider supplementary conditions providing for uniqueness of solution of the conjugation problem. Let $U_j(\xi, \zeta)$ be the Fourier transform of distribution $u_j(x, z)$. We will seek a solution of the problem (1), (2) in the domains $D_0$ and $D_n$ in class of the outgoing at infinity solutions, i.e., we will assume that the representations

$$u_j(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U_j(\xi, \zeta)\ e^{-i\xi x}\ e^{-i\zeta z}\ d\xi\ d\zeta \tag{3}$$

should contain no elementary harmonics corresponding to those coming from the infinity plane waves under $j = 0$ and $j = n$. Besides we assume that the unknown solution $u(x, z)$ has no addends corresponding to eigne waves of the stratified structure going along the axis $x$ (if such waves exist).

## 2. The jump problem.

Consider the auxiliary jump problem for the Helmholez equation in the stratified medium [3]. We need to seek a solution of the equation (1) in the domain $D_j$ in class of the outgoing at infinity solutions satisfying the conditions

$$
\begin{aligned}
&u(x, h_j + 0) - u(x, h_j - 0) = a_j(x), \\
&\frac{\partial u}{\partial z}(x, h_j + 0) - \frac{\partial u}{\partial z}(x, h_j - 0) = b_j(x)
\end{aligned} \tag{4}
$$

under $j = 1..n$. Assume that conditions (4) are fulfilled on the axis $x$ everywhere except on finite number of points, probably. We will seek functions $u_j(x, z)$ as solutions of the auxiliary Cauchy problems [4] for domains $D_j$ with boundary conditions

$$
\begin{aligned}
u_n(x, h_n + 0) = u_n^+(x), &\quad \frac{\partial u_n}{\partial z}(x, h_n + 0) = v_n^+(x); \\
u_j(x, h_j + 0) = u_j^+(x), &\quad \frac{\partial u_j}{\partial z}(x, h_j + 0) = v_j^+(x), \ \ j = 1..n-1, \\
u_j(x, h_{j+1} - 0) = u_{j+1}^-(x), &\quad \frac{\partial u_j}{\partial z}(x, h_{j+1} - 0) = v_{j+1}^-(x); \\
u_0(x, h_1 - 0) = u_1^-(x), &\quad \frac{\partial u_0}{\partial z}(x, h_1 - 0) = v_1^-(x),
\end{aligned} \tag{5}
$$

where $u_j^{\pm}(x), v_j^{\pm}(x)$ are the auxiliary boundary functions. Note that the Cauchy problems for the Helmholez equation are overexercised. The boundary functions can not be given arbitrary.

We denote

$$\Delta h_j = h_{j+1} - h_j.$$

and

$$\gamma_j^0(\xi) = \left\{|\xi| > k_j \ : \ i\sqrt{\xi^2 - k_j^2}; \quad |\xi| < k_j : -\sqrt{k_j^2 - \xi^2}\right\}.$$

**Theorem 1.** *The solution of the jump problem for the Helmholez equation in the stratified medium exists if and only if when the Fourier transforms* *of the auxiliary boundary functions* $V_j^{\pm}(\xi), U_j^{\pm}(\xi)$ *satisfy the system of equations*

$$V_n^+(\xi) - i\gamma_n^o(\xi)\ U_n^+(\xi) = 0,$$

$$[V_j^+(\xi) - i\gamma_j^o(xi)U_j^+(\xi)] - e^{i\Delta h_j\gamma_j^o(xi)\ [V_{j+1}^-(\xi)-i\gamma_j^o}(\xi)U_{j+1}^-(\xi)] = 0,$$

$$e^{i\Delta h_j\gamma_j^o(\xi)}[V_j^+(\xi)+i\gamma_j^o(\xi)U_j^+(\xi)]-[V_{j+1}^-(\xi)+i\gamma_j^o(\xi)U_{j+1}^-(\xi)]=0, \tag{6}$$

$$V_1^-(\xi) + i\gamma_0^o(\xi)\ U_1^-(\xi) = 0,$$

$$U_j^+(\xi) - U_j^-(\xi) = A_j(\xi), \quad V_j^+(\xi) - V_j^-(\xi) = B_j(\xi), \quad j = 1..n.$$

*Here*

$$\sqrt{2\pi}\ (k_n^2 - \xi^2 - \zeta^2)\ U_n(\xi,\zeta) = e^{ih_n\zeta}\ \left[V_n^+(\xi) - i\zeta U_n^+(\xi)\right],$$

$$\sqrt{2\pi}\ (k_j^2 - \xi^2 - \zeta^2)\ U_j(\xi,\zeta) =$$

$$= e^{ih_j\zeta}\ \left[V_j^+(\xi) - i\zeta U_j^+(\xi)\right] - \tag{7}$$

$$-e^{ih_{j+1}\zeta}\ \left[V_{j+1}^-(\xi) - i\zeta U_{j+1}^-(\xi)\right], \quad j = 1..n-1,$$

$$\sqrt{2\pi}\ (k_0^2 - \xi^2 - \zeta^2)\ U_0(\xi,\zeta) = -e^{ih_1\zeta}\ \left[V_1^-(\xi) - i\zeta U_1^-(\xi)\right].$$

*Proof.* After apply Fourier transformation (3) for (1), (2), (4), and auxiliary boundary conditions (5) we obtain:

$$\sqrt{2\pi}\ (k_n^2 - \xi^2 - \zeta^2)\ U_n(\xi,\zeta) = e^{ih_n\zeta}\ \left[V_n^+(\xi) - i\zeta U_n^+(\xi)\right]; \quad z > h_n$$

and distribution $V_n^{\pm}(\xi), U_n^{\pm}(\xi)$ satisfy the equation

$$V_n^+(\xi) - i\gamma_n^o\quad U_n^+(\xi) = 0,$$

and also at $h_j < z < h_j + 1; \quad j = 1, ..., n-1$:

$$\sqrt{2\pi}\ (k_j^2 - \xi^2 - \zeta^2)\ U_j(\xi,\zeta) = e^{ih_j\zeta}\ \left[V_j^+(\xi) - i\zeta U_j^+(\xi)\right] -$$

$$-e^{ih_{j+1}\zeta}\ \left[V_{j+1}^-(\xi) - i\zeta U_{j+1}^-(\xi)\right],\ j = 1..n-1,$$

and the following are satisfied

$$[V_j^+(\xi) - i\gamma_j^o U_j^+(\xi)] - e^{i\Delta h_j\gamma_j^o}[V_{j+1}^-(\xi) - i\gamma_j^o\ U_{j+1}^-(\xi)] = 0,$$

$$e^{i\Delta h_j\gamma_j^o}[V_j^+(\xi) + i\gamma_j^o U_j^+(\xi)] - [V_{j+1}^-(\xi) + i\gamma_j^o U_{j+1}^-(\xi)] = 0,$$

From the conditions (4) the jump problem it follows that

$$U_j^+(\xi) - U_j^-(\xi) = A_j(\xi), \quad V_j^+(\xi) - V_j^-(\xi) = B_j(\xi); \quad j = 1..n.$$

Similar at $z < h_1$:

$$\sqrt{2\pi}\ (k_0^2 - \xi^2 - \zeta^2)\ U_0(\xi,\zeta) = -e^{ih_1\zeta}\ \left[V_1^-(\xi) - i\zeta U_1^-(\xi)\right],$$

and distribution $U_1^-(\xi), V_1^-(\xi)$ satisfy the equation

$$V_1^-(\xi) + i\gamma_0^o\ U_1^-(\xi) = 0.$$

## 3. The integral equation.

We consider a solution of the diffraction problem in the form

$$u(x,z) = u_d(x,z) + u_m(x,z),$$

where the first addend in the right-hand side is a solution of the problem on the fall of the wave at the media interfaces without metallic strips and the second addend is new unknown function.

The function $u_d(x,z)$ can be found as a solution of the jump problem under the conditions

$$a_n(x) = -\tilde{u}(x, h_n + 0), \quad b_n(x) = -\frac{\partial \tilde{u}}{\partial z}(x, h_n + 0),$$

$$a_j(x) = 0, \quad b_j(x) = 0, \quad j = 1..n.$$

The function $u_m(x,z)$ is also a solution of the jump problem under the conditions

$$a_j(x) = 0, \quad x \in (-\infty, +\infty), \quad j = 0..n;$$

$$b_j(x) = 0, \quad x \in N_j; \quad b_j(x) = \varphi_j(x), \quad x \in M_j, \quad j = 0..n,$$

where $\varphi_j(x)$ are the auxiliary unknown functions which can be found from the boundary conditions on the metallic strips

(8) $u(x, h_n + 0) = -\tilde{u}(x, h_n + 0), \quad u(x, h_j \pm 0) = 0, \quad j = 1..n-1.$

Transform equalities (8) into integral equation with respect to the function $\varphi(x) = \varphi_j(x)$ on $M = \cup_j M_j$ as the following way. Having solved the system of linear algebraic equations (SLAE) (6), express the Fourier transforms of the auxiliary boundary functions $V_j^{\pm}(\xi), U_j^{\pm}(\xi)$ in terms of the function $\varphi(x)$. Having substituted them into equations (7), we seek the Fourier transforms $U_j(\xi,\zeta)$ of the unknown functions $u_j(x,z)$ and by formula (3) we obtain the expressions of these functions. Thus, we have the proof of the following theorem:

**Theorem 2.** *The diffraction problem for TE-wave on the system of the metallic strips in the stratified medium is equivalent to the integral equation of the form*

$$\begin{aligned}
&\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} U_n(\xi,\zeta)\; e^{-i\xi x}e^{-i\zeta h_j}\; d\xi\; d\zeta = -\,\tilde{u}(x,h_n+0),\;\; x\in M_n,\\
(9)\qquad &\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} U_j(\xi,\zeta)\; e^{-i\xi x}e^{-i\zeta h_j}\; d\xi\; d\zeta = 0,\;\; x\in M_j,\;\; j=0...n-1.
\end{aligned}$$

*with respect to the function* $\varphi(x)$.

In the particular case under $n=1$ and $m_1=1$ (there is only one strip $\alpha<x<\beta$ on the boundary $z=h$ of two mediums) the equation (9) has the form

$$\begin{aligned}
(10)\qquad &\int_\alpha^\beta \varphi(t)\left[\frac{-i}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\gamma_0^o(\xi)+\gamma_1^o(\xi)}\; e^{i(t-x)\xi}\; d\xi\right]dt =\\
&= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{U}(\xi)\;\frac{2\gamma_1^o(\xi)}{\gamma_0^o(xi)+\gamma_1^o(\xi)}\; e^{-i\xi x}\; d\xi,\;\; x\in(\alpha,\beta).
\end{aligned}$$

If dielectrics in the upper and lower half-planes are the same, i.e. $k_0=k_1=k$, then $\gamma_0^o(\xi)=\gamma_1^o(\xi)$ also. Having calculated the interior integral by formulas (130) and (134) from [1], we obtain the well known integral equation

$$\frac{i}{4\sqrt{\pi}}\Gamma(1/2)\int_\alpha^\beta \varphi^M(t)H_0^{(1)}(k|t-x|)\; dt = -\tilde{u}(x),\quad x\in(\alpha,\beta),$$

where $H_0^{(1)}(x)$ is the Hankel function.

## 4. The Galerkin method.

Different particular cases of the integral equation of the 1-st kind with logarithmic singularity in the kernel (9) are solved numerically by the Galerkin method with decomposition of the unknown function on every segment $[\alpha_{jk}, \beta_{jk}]$ by Chebyshev polynomials with weight. Note that SLAE with respect to coefficients of decomposition approximating the initial integral equation (9) can be obtained by another method. The Galerkin method can be applied not to the equation (9) with respect to the function $\varphi(x)$ but to the integral equation with respect to its Fourier transform $F(\xi)$ which has been obtained at the preceding stage, e.g., not to the equation (10) under $n = 1$ but to the following equation

$$\int_{-\infty}^{+\infty} F(\xi) \, \frac{1}{\gamma_0^o(\xi) + \gamma_1^o(\xi)} \, e^{-ix\xi} \, d\xi =$$

$$= -i \int_{-\infty}^{+\infty} \tilde{U}(\xi) \, \frac{2\gamma_1^o(\xi)}{\gamma_0^o(\xi) + \gamma_1^o(\xi)} \, e^{-i\xi x} \, d\xi, \quad x \in (\alpha, \beta).$$

Since the Bessel functions are the Fourier transforms of the Chebyshev polynomials together with the weight function, the solution of the last should be decomposed into the sum by these functions. From the Parseval formula it follows that such approach has as a result, just the same SLAE.

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REFERENCES

[1] Ju. A. Brychkov - A. P. Prudnikov, *Integral transformations of the generalized functions*, 1977, Nauka, Moscow.

[2] A. S. Ilyinsky - Ju. G. Smirnov, *Electromagnetic wave diffraction by conducting screens (Pseudo differential operators in diffraction problems)*, 1998 VSP, Zeist, Utrecht, The Netherlands.

[3] A. Maher - N. B. Pleshchinskii, *The jump problem for the Helmholez equation in the stratified medium and its applications*, Izv. vuzov. Matem., 2002 N. 1: pp. 45-56 [in Russian].

[4] I. E. Pleshchinskaya - N. B. Pleshchinskii, *On classification of eigen waves of planar, cylindrical and spherical dielectric waveguide*, In: Mathematical Methods in Electromagnetic Theory. Proc. Int. Conf. $MMET^*$ 98. Kharkov, Ukraine, June (2-5),1998 Vol. 2, pp. 781-783.

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