LE MATEMATICHE
Vol. LXXII (2017) – Fasc. I, pp. 3–8
doi: 10.4418/2017.72.1.1

REMARKS ON HYPONORMAL OPERATORS AND ALMOST NORMAL OPERATORS

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In 1984 M. Putinar proved that hyponormal operators are subscalar operators of order two. The proof provided a concrete structure of such operators. We will use this structure to give a sufficient condition for hyponormal operators T with trace-class commutator to admit a direct summand S so that $T \oplus S$ is the sum of a normal operator and a Hilbert-Schmidt operator. We investigate what this sufficient condition amounts to in the case of a weighted shift operator.

1. Introduction.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{C}_1(\mathcal{H})$ and $\mathcal{C}_2(\mathcal{H})$ (or simply $\mathcal{C}_1$ and $\mathcal{C}_2$) the trace class and the Hilbert-Schmidt class, respectively. For arbitrary operators $S, T \in L(\mathcal{H})$, $[S,T]$ will denote the commutator $ST - TS$ and D_S will denote self-commutator of S, that is $[S^*,S]$. An operator $S \in L(\mathcal{H})$ for which $D_S \in \mathcal{C}_1(\mathcal{H})$ ($D_S \geq 0$) is called *almost normal* (*hyponormal*), respectively. The class of operators defined on $\mathcal{H}$ which are almost normal will be denoted by $AN(\mathcal{H})$ and that of hyponormal operators by $H_0^1(\mathcal{H})$.

Entrato in redazione: 20 luglio 2015

AMS 2010 Subject Classification: 47B20

Keywords: Hilbert-Schmidt and trace-class operators, hyponormal operators, almost normal operators

To the memory of my beloved father, Atanasie.

Voiculescu's Conjecture 4 (C_4), (cf. [4] or [3]) states that for $T \in AN(\mathcal{H})$, there exists $S \in AN(\mathcal{H})$ such that $T \oplus S = N + K$, where N is a normal operator and K is a Hilbert-Schmidt operator. This statement is equivalent to the existence of a normal operator $N \in L(\mathcal{K})$, $\mathcal{H} \subset \mathcal{K}$, so that T is unitarily equivalent modulo $\mathcal{C}_2$ to PNP and $[P,N] \in \mathcal{C}_2$, where P is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$, and is an analog of the BDF Theorem concerning the part that each element of $Ext(\cdot)$ has an inverse.

Not much progress has been made towards establishing whether conjecture (C_4) holds or not. The facts remain unknown even for arbitrary almost normal weighted shifts. The most significant result in this direction was obtained by Pasnicu [1] (see Remark (c) at the end of this note).

It is a straightforward exercise (left to the reader) to verify that subnormal operators satisfy (C_4). Since subnormal operators are also hyponormal, it is natural to ask whether the later ones satisfy (C_4). Putinar [2] proved that hyponormal operators are subscalar of order two, that is, they are restrictions to a closed invariant subspace of scalar operators of order two. Putinar's proof was constructive and provided concrete structure information of hyponormal operators $T \in L(\mathcal{H})$, namely they are compressions of normal operators to a semi-invariant subspace, that is there exists an normal operator $N \in L(\mathcal{K})$ with $\mathcal{H} \subset \mathcal{K}$ so the the matrix representation of N is

$$N = \begin{pmatrix} * & * & * \\ 0 & T & * \\ 0 & 0 & * \end{pmatrix}.$$

2. Review of the structure of hyponormal operators.

The purpose of this section is to review Putinar's construction and in section 3 to give a sufficient condition for hyponormal operators to satisfy (C_4).

Let $\mathbb{D}$ be an open disc that includes the spectrum $\sigma(T)$ of a hyponormal operator $T \in L(\mathcal{H})$. Let

$$L^2(\mathbb{D}, \mathcal{H}) = \{ f : \mathbb{D} \to \mathcal{H} |\ ||f||^2_{2,\mathbb{D}} := \int_{\mathbb{D}} ||f(z)||^2 \, d\lambda(z) < +\infty \},$$

where $d\lambda$ is the planar Lebesgue measure. Let $W^2(\mathbb{D}, \mathcal{H})$ consist of those f in $L^2(\mathbb{D}, \mathcal{H})$ so that $\overline{\partial} f$ and $\overline{\partial}^2 f$, in the sense of distributions, belong to $L^2(\mathbb{D}, \mathcal{H})$, where $\overline{\partial}$ is the operator $\partial / \partial \overline{z}$. Endowed with the norm

$$||f||^2_{W^2} := \sum_{k=0}^{2} ||\overline{\partial}^k f||^2_{2,\mathbb{D}},$$

$W^2(\mathbb{D},\mathcal{H})$ becomes a closed subspace of $L^2(\mathbb{D},\mathcal{H})$ in which $C^\infty(\overline{\mathbb{D}},\mathcal{H})$ is a dense subspace. Let $N : L^2(\mathbb{D},\mathcal{H}) \to L^2(\mathbb{D},\mathcal{H})$ be the normal operator defined by $(Nf)(z) = zf(z)$ and let M be the restriction of N to the invariant subspace $W^2(\mathbb{D},\mathcal{H})$. Let $\mathcal{H}_1$ be $\overline{(T-z)W^2(\mathbb{D},\mathcal{H})}$, where

$$T-z : W^2(\mathbb{D},\mathcal{H}) \to W^2(\mathbb{D},\mathcal{H})$$

is defined by

$$((T-z)f)(z) = T(f(z)) - zf(z)$$

and is a bounded operator whose range is invariant for operator M. Let

$$\tilde{M} : W^2(\mathbb{D},\mathcal{H})/\overline{(T-z)W^2(\mathbb{D},\mathcal{H})} \to W^2(\mathbb{D},\mathcal{H})/\overline{(T-z)W^2(\mathbb{D},\mathcal{H})}$$

be defined by $\tilde{M}\tilde{f} = \widetilde{Mf}$, where $\tilde{f} \in W^2(\mathbb{D},\mathcal{H})/\overline{(T-z)W^2(\mathbb{D},\mathcal{H})}$ is the equivalence class of an f in $W^2(\mathbb{D},\mathcal{H})$. Relative to the orthogonal decomposition of $L^2(\mathbb{D},\mathcal{H}) = \mathcal{H}_1 \oplus \mathcal{H}(\mathbb{D}) \oplus \mathcal{H}'$, where $\mathcal{H}(\mathbb{D}) = W^2(\mathbb{D},\mathcal{H})/\overline{(T-z)W^2(\mathbb{D},\mathcal{H})}$, and $\mathcal{H}' = L^2(\mathbb{D},\mathcal{H}) \ominus W^2(\mathbb{D},\mathcal{H})$, the matrix representation of N is

$$N = \begin{pmatrix} A & * & * \\ 0 & \tilde{M} & * \\ 0 & 0 & * \end{pmatrix}.$$

The high point of Putinar's paper is that the initial space $\mathcal{H}$ and the operator T can be recuperated from $\tilde{M}$. More precisely, $\mathcal{H}(\mathbb{D}) = \mathcal{H} \oplus \mathcal{H}''$ and relative to this decomposition, the operator $\tilde{M}$ has representation $\tilde{M} = \begin{pmatrix} T & * \\ 0 & * \end{pmatrix}$. Denoting $\mathcal{H}_2 = \mathcal{H}' \oplus \mathcal{H}''$, then relative to decomposition of $L^2(\mathbb{D},\mathcal{H}) = \mathcal{H}_1 \oplus \mathcal{H} \oplus \mathcal{H}_2$, the matrix representation of N is

$$N = \begin{pmatrix} A & B & C \\ 0 & T & D \\ 0 & 0 & E \end{pmatrix}. \tag{1}$$

With the notation used above, we conclude this section with the following.

Theorem 2.1. *Let T be an operator in $AN(\mathcal{H}) \cap H_0^1(\mathcal{H})$. If the operator A (in equation (1)) belongs to $AN(\mathcal{H}_1)$, then T satisfies (C_4).*

Proof. Since operator N is normal, $[A^*,A] = BB^* + CC^*$, $[T^*,T] = DD^*$, and $[E^*,E] = -(C^*C + D^*D)$. Since T is almost normal, $DD^* \in \mathcal{C}_1$, or equivalently $D \in \mathcal{C}_2$. The hypothesis about A implies $BB^* + CC^* \in \mathcal{C}_1$, and thus BB^* and CC^* are both in $\mathcal{C}_1$ (since they are both nonnegative), which implies that B and C are both Hilbert-Schmidt operators. Furthermore, the operator E is almost normal since $[E^*,E] = -(C^*C + D^*D)$. Thus $N - A \oplus T \oplus E$ is a Hilbert-Schmidt operator. □

3. Application.

In this section we find the matrix representation of the operator A when the operator T is a weighted shift. Let $\{e_n\}_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ and so that $Te_n = w_{n+1}e_{n+1}$, $n \geq 0$. A weighted shift operator is hyponormal if and only if the sequence $\{|w_n|\}_{n\geq 1}$ is nondecreasing. We can further assume that $w_n \geq 0$ since such weighted shifts are unitarily equivalent, and that $w_n \uparrow w$. We further assume that $\mathbb{D}$ is a disc centered at the origin. Let $E_{ijk}(z,\bar{z}) = z^i\bar{z}^j e_k$, $i,j,k \geq 0$, and let $F_{i,j,k} = (T-z)E_{ijk} = w_{k+1}E_{i,j,k+1} - E_{i+1,j,k}$. With the disc $\mathbb{D}$ centered at the origin, we have $E_{ijk} \perp E_{rst}$ when $(i,j,k) \neq (r,s,t)$, (recall that the scalar product in $L^2(\mathbb{D},\mathcal{H})$ is defined by $\langle f,g\rangle_{L^2(\mathbb{D},\mathcal{H})} = \int_{\mathbb{D}} \langle f(z),g(z)\rangle_{\mathcal{H}} d\lambda(z)$). Let $G_n = \{F_{i,j,k} |\ i+j+k = n\}$ listed in the following order

$$F_{n,0,0}, F_{n-1,1,0}, \ldots, F_{0,n,0}; F_{n-1,0,1}, F_{n-2,1,1}, F_{n-3,2,1}, \ldots, F_{0,n-1,1}; \ldots; F_{0,0,n}.$$

Lemma 3.1. *Any vector of G_m is orthogonal on any vector of G_n when $m \neq n$.*

Proof. Assume that $m > n$. Let $F_{i,j,k} \in G_m$ and $F_{r,s,t} \in G_n$. If $|k-t| \geq 2$, then regardless of m,n, we have $F_{i,j,k} \perp F_{r,s,t}$. If $k = t$, then $F_{i,j,k} = w_{k+1}z^i\bar{z}^j e_{k+1} - z^{i+1}\bar{z}^j e_k$ and $F_{r,s,k} = w_{k+1}z^r\bar{z}^s e_{k+1} - z^{r+1}\bar{z}^s e_k$ and since $i+j \neq r+s$, one obtains $F_{i,j,k} \perp F_{r,s,k}$. If $k = t+1$, then $F_{r,s,k-1} = w_k z^r\bar{z}^s e_k - z^{r+1}\bar{z}^s e_{k-1}$ and $r+s = n-k+1 < m-k+1 = i+j+1$ and thus $F_{i,j,k} \perp F_{r,s,k-1}$. Similarly, if $k = t-1$, then $F_{r,s,k+1} = w_{k+2}z^r\bar{z}^s e_{k+2} - z^{r+1}\bar{z}^s e_{k+1}$ and $r+s+1 = n-k < m-k = i+j$ and thus $F_{i,j,k} \perp F_{r,s,k+1}$. □

According to the above lemma, the space $\mathcal{H}_1$ can be decomposed as

$$\bigoplus_{n\geq 0} span(G_n),$$

where $span(G_n)$ denotes the linear span of all vectors in G_n.

Recall that operator A in matrix (1) is a compression of the normal operator N which is the operator of multiplication by variable z. Therefore

$$AF_{i,j,k} = zF_{i,j,k} = z(w_{k+1}E_{i,j,k+1} - E_{i+1,j,k}) = F_{i+1,j,k},$$

and consequently $A(span(G_n)) \subseteq span(G_{n+1})$. Relative to decomposition (2) of $\mathcal{H}_1$, the operator A can be written

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ A_{10} & 0 & 0 & \ldots \\ 0 & A_{21} & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

with each $A_{n+1,n} : span(G_n) \to span(G_{n+1})$.

After orthonormalization of each subspace $span(G_n)$ and redenoting its new vectors by

$$G_{n,0,0}, G_{n-1,1,0}, \dots, G_{0,n,0}; G_{n-1,0,1}, G_{n-2,1,1}, G_{n-3,2,1}, \dots, G_{0,n-1,1}; \dots; G_{0,0,n},$$

each operator $A_{n+1,n}$ has a matrix representation $\tilde{A}_{n+1,n}$.

Theorem 3.2. *The operator A is almost normal if and only if*

$$\operatorname{tr}(\tilde{A}^*_{n+1,n}\tilde{A}_{n+1,n}) \le m < +\infty,\ n \ge 0.$$

Proof. The matrix representation of $[A^*, A]$ relative to the orthonormalized basis of $\oplus_{n\ge 0} span(G_n)$ is

$$\begin{pmatrix} \tilde{A}^*_{10}\tilde{A}_{10} & 0 & 0 & \dots \\ 0 & \tilde{A}^*_{21}\tilde{A}_{21} - \tilde{A}_{10}\tilde{A}^*_{10} & 0 & \dots \\ 0 & 0 & \tilde{A}^*_{32}\tilde{A}_{32} - \tilde{A}_{21}\tilde{A}^*_{21} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Furthermore, operator A is subnormal, thus it is hyponormal and therefore

$$\tilde{A}^*_{n+1,n}\tilde{A}_{n+1,n} - \tilde{A}_{n,n-1}\tilde{A}^*_{n,n-1} \ge 0,\ n \ge 1.$$

This, if A is almost normal, $\operatorname{tr}(P_n[A^*, A]P_n) \uparrow \operatorname{tr}[A^*, A]$, where P_n is the orthogonal projection onto $\oplus_{k=0}^{n+1} span(G_k)$. On other hand,

$$\operatorname{tr}(P_n[A^*, A]P_n) = \operatorname{tr}(\tilde{A}^*_{n+1,n}\tilde{A}_{n+1,n}),$$

and thus

$$\operatorname{tr}(\tilde{A}^*_{n+1,n}\tilde{A}_{n+1,n}) \le \operatorname{tr}[A^*, A].$$

The converse results in a similar way. □

Remark 3.3. (a) Relative to conjecture (C_4), one can assume that the operator T has norm less than 1 since multiplication by a constant preserves both hyponormality and (C_4), and thus one can choose the disc $\mathbb{D}$ to have radius less than 1.

(b) A calculation of matrix $\tilde{A}_{n+1,n}$ is useful in order to determine (sufficient and/or necessary) conditions to satisfy the hypothesis of Theorem 3.2.

Let the orthonormalized vectors of G_n be split in subgroups $L_0^n, L_1^n, \dots, L_n^n$, with each L_k consisting of

$$G_{n-k,0,k}, G_{n-k-1,1,k}, \dots, G_{0,n-k,k}.$$

Since all initial vectors $F_{n-k,k,0}$ are orthogonal on each other,

$$G_{n-k,k,0} = \frac{F_{n-k,k,0}}{||F_{n-k,k,0}||_2},\ k = 0,1,\ldots,n.$$

Thus, for $k = 0,1,\ldots,n$,

$$AG_{n-k,k,0} = z\frac{F_{n-k,k,0}}{||F_{n-k,k,0}||_2} = \frac{F_{n-k+1,k,0}}{||F_{n-k,k,0}||_2} = \frac{||F_{n-k+1,k,0}||_2}{||F_{n-k,k,0}||_2} G_{n-k+1,k,0},$$

that is the k^{th} vector of subgroup L_0^n is mapped into k^{th} vector of subgroup L_0^{n+1}. Finally, the last vector of each group $L_0^{n+1}, L_1^{n+1}, \ldots, L_{n+1}^{n+1}$, is not in the range of $A_{n+1,n}$ and thus is not in the range of A.

(c) Pasnicu [1] proved that a weighted shift satisfies conjecture (C_4) if $w_n \uparrow w,\ w > 0$ and there exists $p > 0$ so that the series $\sum_n (w - w_n)^p$ is convergent.

We conclude the note with the following.

Problem 3.4. Can theorem 3 be used to prove that (C_4) holds for arbitrary hyponormal weighted shifts, or at least for some weight-sequence $w_n \uparrow w,\ w > 0$ for which the series $\sum_n (w - w_n)^p$ is divergent of any $p > 0$?

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