REMARKS ON HYPONORMAL OPERATORS AND ALMOST NORMAL OPERATORS

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In 1984 M. Putinar proved that hyponormal operators are subscalar operators of order two. The proof provided a concrete structure of such operators. We will use this structure to give a sufficient condition for hyponormal operators $T$ with trace-class commutator to admit a direct summand $S$ so that $T \oplus S$ is the sum of a normal operator and a Hilbert-Schmidt operator. We investigate what this sufficient condition amounts to in the case of a weighted shift operator.

1. Introduction.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $C_1(\mathcal{H})$ and $C_2(\mathcal{H})$ (or simply $C_1$ and $C_2$) the trace class and the Hilbert-Schmidt class, respectively. For arbitrary operators $S, T \in L(\mathcal{H})$, $[S, T]$ will denote the commutator $ST - TS$ and $D_S$ will denote self-commutator of $S$, that is $[S^*, S]$. An operator $S \in L(\mathcal{H})$ for which $D_S \in C_1(\mathcal{H})$ ($D_S \geq 0$) is called almost normal (hyponormal), respectively. The class of operators defined on $\mathcal{H}$ which are almost normal will be denoted by $AN(\mathcal{H})$ and that of hyponormal operators by $H^1_0(\mathcal{H})$.
Voiculescu’s Conjecture 4 (C₄), (cf. [4] or [3]) states that for \( T \in AN(\mathcal{H}) \), there exists \( S \in AN(\mathcal{H}) \) such that \( T \oplus S = N + K \), where \( N \) is a normal operator and \( K \) is a Hilbert-Schmidt operator. This statement is equivalent to the existence of a normal operator \( N \in L(\mathcal{K}) \), \( \mathcal{H} \subset \mathcal{K} \), so that \( T \) is unitarily equivalent modulo \( C_2 \) to \( PNP \) and \( [P,N] \in C_2 \), where \( P \) is the orthogonal projection from \( \mathcal{K} \) onto \( \mathcal{H} \), and is an analog of the BDF Theorem concerning the part that each element of \( Ext(\cdot) \) has an inverse.

Not much progress has been made towards establishing whether conjecture (\( C_4 \)) holds or not. The facts remain unknown even for arbitrary almost normal weighted shifts. The most significant result in this direction was obtained by Pasnicu [1] (see Remark (c) at the end of this note).

It is a straightforward exercise (left to the reader) to verify that subnormal operators satisfy (\( C_4 \)). Since subnormal operators are also hyponormal, it is natural to ask whether the later ones satisfy (\( C_4 \)). Putinar [2] proved that hyponormal operators are subscalar of order two, that is, they are restrictions to a closed invariant subspace of scalar operators of order two. Putinar’s proof was constructive and provided concrete structure information of hyponormal operators \( T \in L(\mathcal{H}) \), namely they are compressions of normal operators to a semi-invariant subspace, that is there exists an normal operator \( N \in L(\mathcal{K}) \) with \( \mathcal{H} \subset \mathcal{K} \) so the the matrix representation of \( N \) is

\[
N = \begin{pmatrix}
* & * & * \\
0 & T & * \\
0 & 0 & *
\end{pmatrix}.
\]

2. Review of the structure of hyponormal operators.

The purpose of this section is to review Putinar’s construction and in section 3 to give a sufficient condition for hyponormal operators to satisfy (\( C_4 \)).

Let \( \mathbb{D} \) be an open disc that includes the spectrum \( \sigma(T) \) of a hyponormal operator \( T \in L(\mathcal{H}) \). Let

\[
L^2(\mathbb{D}, \mathcal{H}) = \{ f : \mathbb{D} \to \mathcal{H} | \|f\|_{L^2(\mathbb{D})}^2 := \int_{\mathbb{D}} \|f(z)\|^2 d\lambda(z) < +\infty \},
\]

where \( d\lambda \) is the planar Lebesgue measure. Let \( W^2(\mathbb{D}, \mathcal{H}) \) consist of those \( f \) in \( L^2(\mathbb{D}, \mathcal{H}) \) so that \( \overline{\partial}f \) and \( \overline{\partial}^2 f \), in the sense of distributions, belong to \( L^2(\mathbb{D}, \mathcal{H}) \), where \( \overline{\partial} \) is the operator \( \partial/\partial \bar{z} \). Endowed with the norm

\[
\|f\|_{W^2}^2 := \sum_{k=0}^2 \|\overline{\partial}^k f\|_{L^2(\mathbb{D})}^2,
\]
W^2(\mathbb{D}, \mathcal{H}) becomes a closed subspace of L^2(\mathbb{D}, \mathcal{H}) in which C^\infty(\overline{\mathbb{D}}, \mathcal{H}) is a dense subspace. Let N : L^2(\mathbb{D}, \mathcal{H}) \to L^2(\mathbb{D}, \mathcal{H}) be the normal operator defined by (Nf)(z) = zf(z) and let M be the restriction of N to the invariant subspace W^2(\mathbb{D}, \mathcal{H}). Let \mathcal{H}_1 be \((T - z)W^2(\mathbb{D}, \mathcal{H}),\) where

\[ T - z : W^2(\mathbb{D}, \mathcal{H}) \to W^2(\mathbb{D}, \mathcal{H}) \]

is defined by

\[ ((T - z)f)(z) = T(f(z)) - zf(z) \]

and is a bounded operator whose range is invariant for operator M. Let

\[ \tilde{M} : W^2(\mathbb{D}, \mathcal{H})/(T - z)W^2(\mathbb{D}, \mathcal{H}) \to W^2(\mathbb{D}, \mathcal{H})/(T - z)W^2(\mathbb{D}, \mathcal{H}) \]

be defined by \( \tilde{M}f = \overline{Mf}, \) where \( f \in W^2(\mathbb{D}, \mathcal{H})/(T - z)W^2(\mathbb{D}, \mathcal{H}) \) is the equivalence class of an \( f \) in \( W^2(\mathbb{D}, \mathcal{H}). \) Relative to the orthogonal decomposition of \( L^2(\mathbb{D}, \mathcal{H}) = \mathcal{H}_1 \oplus \mathcal{H}(\mathbb{D}) \oplus \mathcal{H}', \) where \( \mathcal{H}(\mathbb{D}) = W^2(\mathbb{D}, \mathcal{H})/(T - z)W^2(\mathbb{D}, \mathcal{H}), \) and \( \mathcal{H}' = L^2(\mathbb{D}, \mathcal{H}) \oplus W^2(\mathbb{D}, \mathcal{H}), \) the matrix representation of \( N \) is

\[ N = \begin{pmatrix} A & * & * \\ 0 & \tilde{M} & * \\ 0 & 0 & * \end{pmatrix}. \]

The high point of Putinar’s paper is that the initial space \( \mathcal{H} \) and the operator \( T \) can be recuperated from \( \tilde{M}. \) More precisely, \( \mathcal{H}(\mathbb{D}) = \mathcal{H} \oplus \mathcal{H}'' \) and relative to this decomposition, the operator \( \tilde{M} \) has representation \( \tilde{M} = \begin{pmatrix} T & * \\ 0 & \ast \end{pmatrix}. \) Denoting \( \mathcal{H}_2 = \mathcal{H}' \oplus \mathcal{H}'', \) then relative to decomposition of \( L^2(\mathbb{D}, \mathcal{H}) = \mathcal{H}_1 \oplus \mathcal{H} \oplus \mathcal{H}_2, \) the matrix representation of \( N \) is

\[ N = \begin{pmatrix} A & B & C \\ 0 & T & D \\ 0 & 0 & E \end{pmatrix}. \]  \hspace{1cm} (1)

With the notation used above, we conclude this section with the following.

**Theorem 2.1.** Let \( T \) be an operator in \( AN(\mathcal{H}) \cap H^1_0(\mathcal{H}). \) If the operator \( A \) (in equation (1)) belongs to \( AN(\mathcal{H}_1), \) then \( T \) satisfies (C4).

**Proof.** Since operator \( N \) is normal, \( [A^*, A] = BB^* + CC^*, \) \( [T^*, T] = DD^*, \) and \( [E^*, E] = -(C^*C + D^*D). \) Since \( T \) is almost normal, \( DD^* \in C_1, \) or equivalently \( D \in C_2. \) The hypothesis about \( A \) implies \( BB^* + CC^* \in C_1, \) and thus \( BB^* \) and \( CC^* \) are both in \( C_1 \) (since they are both nonnegative), which implies that \( B \) and \( C \) are both Hilbert-Schmidt operators. Furthermore, the operator \( E \) is almost normal since \( [E^*, E] = -(C^*C + D^*D). \) Thus \( N - A \oplus T \oplus E \) is a Hilbert-Schmidt operator. \( \square \)
3. Application.

In this section we find the matrix representation of the operator $A$ when the operator $T$ is a weighted shift. Let $\{e_n\}_{n \geq 0}$ be an orthonormal basis of $\mathcal{H}$ and so that $Te_n = w_{n+1}e_{n+1}$, $n \geq 0$. A weighted shift operator is hyponormal if and only if the sequence $\{|w_n|\}_{n \geq 1}$ is nondecreasing. We can further assume that $w_n \geq 0$ since such weighted shifts are unitarily equivalent, and that $w_n \uparrow w$. We further assume that $\mathbb{D}$ is a disc centered at the origin. Let $E_{ijk}(z, \bar{z}) = z^i\bar{z}^j/e_k$, $i, j, k \geq 0$, and let $F_{i,j,k} = (T-z)E_{ijk} = w_{k+1}E_{i,j,k+1} - E_{i+1,j,k}$. With the disc $\mathbb{D}$ centered at the origin, we have $E_{ijk} \perp E_{rs}$ when $(i,j,k) \neq (r,s,t)$, (recall that the scalar product in $L^2(\mathbb{D}, \mathcal{H})$ is defined by $\langle f, g \rangle_{L^2(\mathbb{D}, \mathcal{H})} = \int_\mathbb{D} \langle f(z), g(z) \rangle_{\mathcal{H}} d\lambda(z)$). Let $G_n = \{F_{i,j,k} \mid i + j + k = n\}$ listed in the following order

$$F_{n,0,0}, F_{n-1,1,0}, \ldots, F_{0,n,0}; F_{n-1,0,1}, F_{n-2,1,1}, F_{n-3,2,1}, \ldots, F_{0,n-1,1}; \ldots; F_{0,0,n}.$$

**Lemma 3.1.** Any vector of $G_m$ is orthogonal to any vector of $G_n$ when $m \neq n$.

**Proof.** Assume that $m > n$. Let $F_{i,j,k} \in G_m$ and $F_{r,s,t} \in G_n$. If $|k-t| \geq 2$, then regardless of $m, n$, we have $F_{i,j,k} \perp F_{r,s,t}$. If $k = t$, then $F_{i,j,k} = w_{k+1}z^i\bar{z}^j e_{k+1} - z^{i+1}\bar{z}^{j+1} e_k$ and $F_{r,s,k} = w_{k+1}z^r\bar{z}^s e_{k+1} - z^{r+1}\bar{z}^{s+1} e_k$ and since $i + j \neq r + s$, one obtains $F_{i,j,k} \perp F_{r,s,k}$. If $k = t + 1$, then $F_{r,s,k-1} = w_{k+1}z^r\bar{z}^s e_k - z^{r+1}\bar{z}^{s+1} e_{k-1}$ and $r + s = n - k + 1 < m - k + 1 = i + j + 1$ and thus $F_{i,j,k} \perp F_{r,s,k-1}$. Similarly, if $k = t - 1$, then $F_{r,s,k+1} = w_{k+1}z^r\bar{z}^s e_{k+2} - z^{r+1}\bar{z}^{s+1} e_{k+1}$ and $r + s + 1 = n - k < m - k = i + j$ and thus $F_{i,j,k} \perp F_{r,s,k+1}$.

According to the above lemma, the space $\mathcal{H}_1$ can be decomposed as

$$\bigoplus_{n \geq 0} \text{span}(G_n),$$

where $\text{span}(G_n)$ denotes the linear span of all vectors in $G_n$.

Recall that operator $A$ in matrix (1) is a compression of the normal operator $N$ which is the operator of multiplication by variable $z$. Therefore

$$AF_{i,j,k} = zF_{i,j,k} = z(w_{k+1}E_{i,j,k+1} - E_{i+1,j,k}) = F_{i+1,j,k},$$

and consequently $A(\text{span}(G_n)) \subseteq \text{span}(G_{n+1})$. Relative to decomposition (2) of $\mathcal{H}_1$, the operator $A$ can be written

$$A = \begin{pmatrix}
0 & 0 & 0 & \cdots \\
A_{10} & 0 & 0 & \cdots \\
0 & A_{21} & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$
with each $A_{n+1,n} : \text{span}(G_n) \to \text{span}(G_{n+1})$.

After orthonormalization of each subspace $\text{span}(G_n)$ and redenoting its new vectors by $G_{n,0,0}, G_{n-1,1,0}, \ldots, G_{0,n,0}; G_{n-1,0,1}, G_{n-2,1,1}, G_{n-3,2,1}, \ldots, G_{0,n,1}; \ldots; G_{0,0,n}$, each operator $A_{n+1,n}$ has a matrix representation $\tilde{A}_{n+1,n}$.

**Theorem 3.2.** The operator $A$ is almost normal if and only if

$$\text{tr}(\tilde{A}_{n+1,n}^{*} \tilde{A}_{n+1,n}) \leq m < +\infty, \ n \geq 0.$$ 

**Proof.** The matrix representation of $[A^{*}, A]$ relative to the orthonormalized basis of $\bigoplus_{n \geq 0} \text{span}(G_n)$ is

$$
\begin{pmatrix}
\tilde{A}_{10}^{*} & 0 & 0 & \cdots \\
0 & \tilde{A}_{21}^{*} - \tilde{A}_{10}^{*} & 0 & \cdots \\
0 & 0 & \tilde{A}_{32}^{*} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Furthermore, operator $A$ is subnormal, thus it is hyponormal and therefore

$$\tilde{A}_{n+1,n}^{*} \tilde{A}_{n+1,n} - \tilde{A}_{n,n-1}^{*} \tilde{A}_{n,n-1} \geq 0, \ n \geq 1.$$ 

This, if $A$ is almost normal, $\text{tr}(P_n[A^{*}, A]P_n) \uparrow \text{tr}[A^{*}, A]$, where $P_n$ is the orthogonal projection onto $\bigoplus_{k=0}^{n+1} \text{span}(G_k)$. On other hand,

$$\text{tr}(P_n[A^{*}, A]P_n) = \text{tr}(\tilde{A}_{n+1,n}^{*} \tilde{A}_{n+1,n}),$$

and thus

$$\text{tr}(\tilde{A}_{n+1,n}^{*} \tilde{A}_{n+1,n}) \leq \text{tr}[A^{*}, A].$$

The converse results in a similar way. \qed

**Remark 3.3.**

(a) Relative to conjecture (C₄), one can assume that the operator $T$ has norm less than 1 since multiplication by a constant preserves both hyponormality and (C₄), and thus one can choose the disc $\mathbb{D}$ to have radius less than 1.

(b) A calculation of matrix $\tilde{A}_{n+1,n}$ is useful in order to determine (sufficient and/or necessary) conditions to satisfy the hypothesis of Theorem 3.2.

Let the orthonormalized vectors of $G_n$ be split in subgroups $L^n_0, L^n_1, \ldots, L^n_n$, with each $L_k$ consisting of $G_{n-k,0,k}, G_{n-k-1,1,k}, \ldots, G_{0,n-k,k}$. 

Since all initial vectors $F_{n-k,k,0}$ are orthogonal on each other,

$$G_{n-k,k,0} = \frac{F_{n-k,k,0}}{||F_{n-k,k,0}||^2}, \quad k = 0, 1, \ldots, n.$$  

Thus, for $k = 0, 1, \ldots, n$,

$$AG_{n-k,k,0} = z \frac{F_{n-k,k,0}}{||F_{n-k,k,0}||^2} = \frac{F_{n-k+1,k,0}}{||F_{n-k,k,0}||^2} = \frac{||F_{n-k+1,k,0}||^2}{||F_{n-k,k,0}||^2} G_{n-k+1,k,0},$$

that is the $k^{th}$ vector of subgroup $L_0^n$ is mapped into $k^{th}$ vector of subgroup $L_{n+1}^0$. Finally, the last vector of each group $L_{n+1}^0, L_{n+1}^1, \ldots, L_{n+1}^n$, is not in the range of $A_{n+1,n}$ and thus is not in the range of $A$.

(c) Pasnicu [1] proved that a weighted shift satisfies conjecture (C4) if $w_n \uparrow w$, $w > 0$ and there exists $p > 0$ so that the series $\sum_n (w - w_n)^p$ is convergent.

We conclude the note with the following.

**Problem 3.4.** Can theorem 3 be used to prove that (C4) holds for arbitrary hyponormal weighted shifts, or at least for some weight-sequence $w_n \uparrow w$, $w > 0$ for which the series $\sum_n (w - w_n)^p$ is divergent of any $p > 0$?

**REFERENCES**


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