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A DETERMINISTIC INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS WITH RAMP-TYPE DEMAND RATE AND SHORTAGES UNDER PERMISSIBLE DELAY IN PAYMENTS

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In this paper, we have proposed an inventory model for non-instantaneous deteriorating items, having Ramp-type demand rate with a time dependent holding cost. In addition, the shortage is allowed, which is partially backlogged. In the genuine business sector, for getting more profit one of the best tools is the trade credit or delay in payments. Furthermore, in our model we have considered as the credit-period is offered by the suppliers to retailers for settling the account. Presented model serves in minimizing the total inventory cost by finding an optimal solution. Some useful lemmas and algorithms have been discussed to illustrate the optimal solution. Several numerical examples are given to test and verify the theoretical results. Finally, the conclusion of the proposed model is discussed.

1. Introduction

The best known inventory model is the classical square-root Economic Order Quantity (EOQ) model developed by F. Harris [9] in 1915. In 1977 Donaldson [5] was the first scientist, included a linear demand in the EOQ model rather

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Table 1: Major Characteristic of Inventory Models on selected researchers

Authors & year	Deterioration	Demand Rate	Permissible Delay in Payment	Shortages allowed	Holding Cost
Goyal [8] (1985)	not considered	constant	not considered	Yes	constant
Aggarwal & Jaggi [1] (1995)	instantaneous	constant	considered	No	constant
Hwang & Shinn [11] (1997)	instantaneous	constant price	considered	No	constant
Jamal, Sarker & Wang [13] (1997)	instantaneous	constant	considered	Yes	constant
Jamal, Sarker & Wang [12](2000)	instantaneous	constant	considered	No	constant
Chang & Dye [3] (2001)	instantaneous	constant	considered	Yes(Partial)	constant
Ouyang, Wu, & Yang [15] (2006)	non-instantaneous	constant	considered	No	constant
Wu, Ouyang, & Yang [25] (2006)	non-instantaneous	stock dependent	considered	No	constant
Geetha & Uthayakumar [2] (2010)	non-instantaneous	constant	considered	(Partial)	constant
Skouri, Konstantaras, Papachristos, & Teng [18] (2011)	instantaneous	ramp-type	considered	Partial	constant
Shah, Soni, & Patel [17] (2013)	non-instantaneous	advertisement and selling price	considered	Partial	time dependent
Soni [19] (2013)	non-instantaneous	price and stock sensitive	no	no	constant
Wu, Skouri, Teng, & Ouyang [26] (2014)	non-instantaneous	price and stock sensitive	no	no	constant
Vandana & Sharma [22] (2016)	non-instantaneous	quadratic	considered	Partial	constant
Present Model	non-instantaneous	ramp-type	considered	Partial	time dependent

than steady request. In 1995, R. M. Hill [10] proposed a time dependent demand known as "Ramp-Type Demand". This sort of demand typically shows up when some new brand of consumer good goes into the business sector. In any case, it is inspected that the demand rate of another brand of buyer trading, when goes to the business sector, increments toward the start of the season for a specific time (say, μ) and after that remaining parts consistent for the rest time period.

In the inventory models two types of shortages are allowed. First, in which customers are willing to wait known as complete backlogging, firstly considered by Deb and Chudhary[4]. Second, in which customers are not willing to wait during shortage periods, is known as partial backlog.

Trade credit is an open record with a seller, who gives a chance for retailers/buyers' that purchase now and pay later. Trade credit is a course of action, between organizations to buy merchandise or administrations without making quick money installment. An EOQ model with constant demand rate under the conditions of permissible delay in payments has been developed by Goyal [8] in 1985. The Goyal's model was extended by Dave[6] in 1985, with considering the fact that, the selling price is necessarily higher than its purchase cost. Next, Jamal et al. [12] generalized their model for shortages. Later on, Teng et al. [20] considered that the selling price not equal to the purchase price and modified the Goyal's [8] model. After that, many researcher works on this aspect, such as [2], [14], [22], [23], and etc.

In 2006, Ouyang et al.[15] developed an inventory model for non - instantaneous deterioration items over a constant demand rate to replace the instantaneous deteriorating. Afterwards, a researcher who worked on this aspect are Wu et al.[25], Maihami and Kamal Abadi [14], Valliathal and Uthayakumar [21], and etc. Many researchers worked on this aspect, such as Geetha and Uthayakumar [2], Wu et al. [25], they considered a constant demand rate with the constant holding cost. But, in reality some non-instantaneous deteriorating items like as fruits, vegetables, etc. having a ramp-type demand rate. The comparative parameters of such inventory models designed by the researchers are given in Table (1).

In the inventory models, numerous of authors are considered holding costs as a constant and known. Yet, as a general rule, holding expense is not constant. A few researchers are considered varying holding cost for details [7, 16, 24]. In this paper, we therefore propose an inventory model for non-instantaneous deteriorating items having a ramp-type demand rate with delay in payment and time varying holding cost.

The rest of the paper is described as below: In Sections 2 and 3, we discussed the notation and assumptions of the proposed model used throughout the paper. In Section 4, the mathematical formulation, to minimize the total annual inventory cost is established. Section 5, presents useful theorem to characterize the optimal solution. In Section 6, we develop the algorithm to solve our numerical examples. Several numerical examples are provided in Section 7. Finally, the managerial implications and conclusion of the proposed model is presented in Section 8.

2. Notation

The useful notation is given as below:

- p The purchasing cost per unit
- h The holding cost per unit per unit time excluding the capital cost
- s The shortage cost for backlogged items per unit per year
- O The cost of lost sales per unit
- p_1 The selling price per unit
- t_d The length of time in which the product exhibits no deterioration
- μ The parameter of the ramp type demand function
- t_1 The length of time in which there is no inventory shortage $(t_1 > t_d)$
- T The duration of the replenishment cycle $(T > t_1)$
- Q The order quantity

S	The inventory system at the begging of each cycle
t_1^*	The optimal length of time in which there is no inventory shortage
I_{max}	The maximum inventory level
I_e	The interest earned per dollar per unit
I_p	The interest charged per dollar per unit
M	The trade credit-period
$TC(t_1)$	The total minimum relevant cost for the inventory system
TC^*	The optimal total minimum cost

3. Assumption

1. The inventory framework included a single type of commodity.

2. Replenishment rate is infinite, replenishment size is constant and lead time is zero.

3. The decay rate θ is non-instantaneous and constant.

4. Let us assume $\beta(t)$ be the fraction, where t is the waiting time up to the next replenishment.

   We consider $\beta(t) = \frac{1}{1+\delta t}$, where δ known as the backlogging parameter is a positive constant.

5. The demand rate $D(t)$ is assumed to be a Ramp-Type function of time,

   $$D(t) = D_0[t - (t-\mu)H(t-\mu)], \;\; D_0 > 0$$

   Where $H(t-\mu)$ is the well known Heaviside's function defined as follows:
   $H(t-\mu) = \begin{cases} 1; & t \geq \mu \\ 0; & t < \mu \end{cases}$

   Here we consider $\mu < t_1$ case only.

6. For the sake of simplicity, assumed t_d is constant, $\mu < t_1$, and $t_d < t_1$.

7. Holding cost $h(t)$ per unit is assumed as time dependent, i.e. $h(t) = h + at$, where $a, h > 0$.

8. In the exchange credit-period M, the record is not settled, created deals income is saved in an enthusiasm bearing record. Toward the end of the period, the retailer pays off all units purchased and starts to pay off the capital opportunity cost.

4. Mathematical formulation of Model

4.1. Model 1. When $t_d < \mu$ -

For this situation, inventory level declines due to demand rate in time period $[0,t_d]$. After that, in time interval $[t_d,\mu]$ deterioration of the item begins, and inventory level rapidly decreases due to demand and deterioration rate both. Now, inventory level ultimately reaches to zero inventory level at the end of the time interval $[\mu,t_1]$. Finally, the shortage occurs (partial backlogging) in the time interval $[t_1,T]$. The mathematical formulation of this situation is given as below:

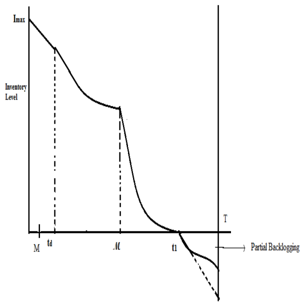

Figure 1: Inventory model for $0 < M \leq t_d$

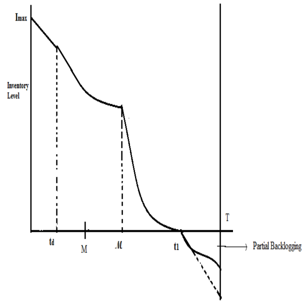

Figure 2: Inventory model for $t_d < M \leq \mu$

In the time period $[0,t_d]$ (there is no deterioration) the differential equation representing the inventory status is given as

$$\frac{dI_{11}(t)}{dt} = -D_0 t; \;\; 0 \leq t \leq t_d, \tag{1}$$

with boundary conditions, $I_{11}(0) = I_{max}$. Then, in $[t_d,\mu]$, the differential equation showing the inventory status as

$$\frac{dI_{12}(t)}{dt} + \theta I_{12}(t) = -D_0 t; \;\; t_d < t \leq \mu. \tag{2}$$

In $[\mu,t_1]$ the market demand will be constant, thus the differential equation is as

$$\frac{dI_{13}(t)}{dt} + \theta I_{13}(t) = -D_0 \mu; \;\; \mu < t \leq t_1, \tag{3}$$

with boundary conditions $I_{12}(\mu) = I_{13}(\mu)$, and $I_{13}(t_1) = 0$. During the time interval $[t_1,T]$ the differential equation representing the inventory level with partial backlogging, i.e.,

$$\frac{dI_{14}(t)}{dt} = \frac{-D_0\mu}{1+\delta(T-t)}; \;\; with \;\; t_1 < t \leq T, \tag{4}$$

with boundary conditions $I_{14}(t_1) = 0$. Now, solving all above Equations with boundary conditions, we have

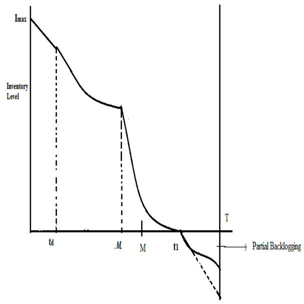

Figure 3: Inventory system for $\mu < M \leq t_1$

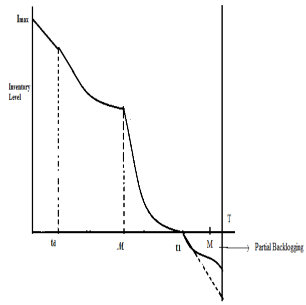

Figure 4: Inventory system for $t_1 < M \leq T$

$$I_{11}(t) = \frac{(-D_0 t^2)}{2} + I_{max} \tag{5}$$

$$I_{12}(t) = \frac{D_0}{\theta^2}(e^{(-\theta t)}(\mu\theta e^{\theta t_1} - \theta e^{\theta\mu}) - \theta t + 1) \tag{6}$$

$$I_{13}(t) = \frac{D_0\mu}{\theta}(e^{(\theta(t_1-t))} - 1), \tag{7}$$

and

$$I_{14}(t) = \frac{D_o\mu}{\delta}\log\left(\frac{1+\delta(T-t)}{1+\delta(T-t_1)}\right). \tag{8}$$

By continuity of t_d, we have $I_{11}(t_d) = I_{12}(t_d)$ and get the value of I_{max}, is as

$$I_{max} = \frac{D_0 t_d^2}{2} + \frac{D_0}{\theta^2}(e^{(-\theta t)}(\mu\theta e^{\theta t_1} - \theta e^{\theta\mu}) - \theta t_d + 1), \tag{9}$$

letting $t = T$ in equation (8), we can obtain the maximum amount of demand backlogged per cycle as

$$S = -I_{14}(T) = \frac{D_0}{\delta}\log(1+\delta(T-t_1)). \tag{10}$$

Hence, the order quantity per cycle is given by

$$\begin{aligned} Q &= I_{max} + S \\ &= D_0(\tfrac{1}{2}t_d^2 + \tfrac{1}{\theta^2}(e^{(\theta(t_1-t_d))}(\theta t_1 - 1) - t_d\theta + 1)) + \tfrac{D_0}{\delta}\log(1+\delta(T-t_1)) \end{aligned} \tag{11}$$

Now, obtained the total inventory cost per cycle, which consists the following costs:

i. A - The ordering cost.

ii. HC - The inventory holding cost

$$\begin{aligned} HC &= h\Big[\int_0^{t_d} I_{11}(t)dt + \int_{t_d}^{\mu} I_{12}(t)dt + \int_{\mu}^{t_1} I_{13}(t)dt\Big] \\ &\quad + a\Big[\int_0^{t_d} tI_{11}(t)dt + \int_{t_d}^{\mu} tI_{12}(t)dt + \int_{\mu}^{t_1} tI_{13}(t)dt\Big] \\ &= \tfrac{D_0}{(48\theta^2)}\Bigg(a\Bigg(-24\mu^2(-1+e^{(\mu\theta)}\theta) - 2{t_d}^3\theta(-1+e^{(\mu\theta)}\theta)(-4+3t_d\theta) + 4\mu^3\theta \\ &\quad (2+4e^{(\mu\theta)}\theta + 3{t_1}^2(-1+\theta)\theta) - 2\mu^4\theta^2(8t_1(-1+\theta) + 3e^{(\mu\theta)}\theta + 4{t_1}^2\theta^2) \\ &\quad +3\mu^5\theta^2(-2+2\theta+2t_1\theta^2+{t_1}^2\theta^3) + \mu\theta^2\big(8{t_1}^3 + 2{t_1}^4 + 2{t_d}^3(-4+3t_d\theta) \\ &\quad +2t_1{t_d}^3\theta(-4+3t_d\theta) + {t_1}^2{t_d}^3\theta^2(-4+3t_d\theta)\big)\Bigg) + 4h\Bigg(-2{t_d}^2\theta \\ &\quad (-1+e^{(\mu\theta)}\theta)(-3+2t_d\theta) + 6\mu^2\theta(1+e^{(\mu\theta)}\theta + {t_1}^2(-1+\theta)\theta) - \mu^3\theta^2 \\ &\quad (6t_1(-1+\theta) + 2e^{(\mu\theta)}\theta + 3{t_1}^2\theta^2) + \mu^4\theta^2(-2+2\theta+2t_1\theta^2+{t_1}^2\theta^3) \\ &\quad +\mu\Bigg(12 - 12e^{(\mu\theta)}\theta + 2{t_1}^3\theta^2 - 6{t_d}^2\theta^2 + 4{t_d}^3\theta^3 + 2t_1{t_d}^2\theta^3(-3+2t_d\theta) \\ &\quad +{t_1}^2\theta^2(6 - 3{t_d}^2\theta^2 + 2{t_d}^3\theta^3)\Bigg)\Bigg)\Bigg). \end{aligned} \tag{12}$$

iii. SC - The shortage cost due to backlog

$$\begin{aligned} SC &= s\Big(\int_{t_1}^{T} -I_4(t)dt\Big) \\ &= \tfrac{sD_0\mu}{\delta^2}(\log(\tfrac{1}{1+\delta(T-t_1)}) + \delta(T-t_1)) \end{aligned} \tag{13}$$

iv. OC - The opportunity cost due to lost sales

$$\begin{aligned} OC &= oD_0\mu\Big(\int_{t_1}^{T}(1 - \tfrac{1}{1+\delta(T-t)})dt\Big) \\ &= \tfrac{oD_0\mu}{\delta}(-\log(1+\delta(T-t_1)) + \delta(T-t_1)). \end{aligned} \tag{14}$$

v. DC - The deterioration cost

$$
\begin{aligned}
DC &= p\theta(\int_{t_d}^{\mu} I_{12}dt + \int_{\mu}^{t_1} I_{13}dt) \\
&= \tfrac{D_0 p\theta}{6}\Bigg(3\mu^3 - \mu^4 - 6\mu^2 t_1 + 3\mu^3 t_1 + 3\mu t_1{}^2 - 3\mu^2 t_1{}^2 + \mu t_1{}^3 \\
&\quad +6\Bigg(\mu - t_d - \tfrac{\mu^2\theta}{2} + \tfrac{t_d{}^2\theta}{2} + \tfrac{(\mu-t_d)\theta}{12}(6 - 3(\mu+t_d)\theta + (\mu^2 \\
&\quad +\mu t_d + t_d{}^2)\theta^2)(-2e^(\mu\theta) + \mu(2 + t_1\theta(2+t_1\theta)))\Bigg)/\theta^2\Bigg). \qquad (15)
\end{aligned}
$$

vi. The interest payable - For each cycle, we need to consider the cases where the length of the credit-period is longer or shorter than the length of time in which the product exhibits no deterioration (t_d) and the length of period with positive inventory of the item (t_1). Thus, we have four cases, given as below

Case 1. $0 < M \leq t_d$ - In this case, payment for items is settled and the retailer starts paying the capital opportunity cost for the items, see Figure (11). Thus, we have

$$
\begin{aligned}
IP_1 &= pI_p\left[\int_M^{t_d} I_{11}(t)dt + \int_{t_d}^{\mu} I_{12}(t)dt + \int_{\mu}^{t_1} I_{13}(t)dt\right] \\
&= \tfrac{D_0 I_p p}{6}\Bigg(3\mu^3 - \mu^4 + M^3 - 6\mu^2 t_1 + 3\mu^3 t_1 + 3\mu t_1{}^2 - 3\mu^2 t_1{}^2 + \mu t_1{}^3 \\
&\quad -3Mt_d{}^2 + 2t_d{}^3 + \tfrac{1}{\theta^2}\Bigg\{6\Bigg[\mu - t_d - \tfrac{\mu^2\theta}{2} + \tfrac{t_d{}^2\theta}{2} + \tfrac{\mu-t_d}{12}\theta(6 - 3(\mu+t_d)\theta \\
&\quad +(\mu^2 + \mu t_d + t_d{}^2)\theta^2)\Bigg(-2e^{(\mu\theta)} + \mu(2 + t_1\theta(2+t_1\theta))\Bigg)\Bigg]\Bigg\} \\
&\quad +\tfrac{1}{\theta^2}\Bigg\{6(-M+t_d)\big[1 - t_d\theta + \tfrac{\theta}{4}(2 + t_d\theta(-2+t_d\theta))\big(-2e^{(\mu\theta)} \\
&\quad +\mu(2 + t_1\theta(2+t_1\theta))\big)\big]\Bigg\}\Bigg) \qquad (16)
\end{aligned}
$$

Case 2. $t_d < M \leq \mu$ -

$$
\begin{aligned}
IP_2 &= pI_p\left(\int_M^{\mu} I_{12}(t)dt + \int_{\mu}^{t_1} I_{13}(t)dt\right) \\
&= \tfrac{D_0 I_p p}{6}\Big\{3\mu M^2 - \mu M^3 - 6\mu M t_1 + 3\mu M^2 t_1 + 3\mu t_1{}^2 - 3\mu M t_1{}^2 + \mu t_1{}^3
\end{aligned}
$$

$$+\frac{1}{\theta^2}\Bigg[6\Bigg(\mu - M - \frac{\mu^2\theta}{2} + \frac{M^2\theta}{2} + \frac{(\mu-M)\theta}{12}(6 - 3(\mu+M)\theta + (\mu^2+\mu M$$
$$+M^2)\theta^2)\big(-2e^{(\mu\theta)} + \mu(2+t_1\theta(2+t_1\theta))\big)\Bigg)\Bigg]\Bigg\} \tag{17}$$

Case 3. If $\mu < M \leq t_1$ - In this case, the interest payable is

$$IP_3 = \frac{-pI_pD_0\mu}{\theta^2}(1 + \theta t_1 - e^{(\theta(t_1-M))} - M\theta) \tag{18}$$

Case 4. $t_1 < M \leq T$ - In this case interest payable is 0, i.e.,

$$IP_4 = 0. \tag{19}$$

vii. Interest earned - For simplicity, we use Geetha and Uthayakumar [2] approach throughout this paper. That is, we assume that during the time when the account is not settled, the retailer sells the goods and continues to accumulate sales revenue and earns the interest rate I_e. Therefore, the interest earned per year (denote by IE) is given below for the four different Cases

Case 1. $0 < M \leq t_d$ - In this case, the interest earned is

$$\begin{aligned} IE_{11} &= p_1I_e\int_0^M D_0t^2dt \\ &= \tfrac{1}{3}pI_eD_0M^3 \end{aligned} \tag{20}$$

Case 2. $t_d < M \leq \mu$ - In this case, the interest earned is

$$\begin{aligned} IE_{12} &= p_1I_e\int_0^M D_0t^2dt \\ &= \tfrac{1}{3}pI_eD_0M^3 \end{aligned} \tag{21}$$

Case 3. $\mu < M \leq t_1$ - In this case, the interest earned is

$$\begin{aligned} IE_{13} &= p_1I_e\left(\int_0^\mu D_0t^2dt + \int_\mu^M D_0\mu t dt\right) \\ &= p_1I_eD_0\mu\left[\frac{\mu^2}{3} + \frac{M^2-\mu^2}{2}\right] \end{aligned} \tag{22}$$

Case 4. $t_1 < M \leq T$ - For this case, the interest earned is

$$\begin{aligned} IE_{14} &= p_1 I_e(\int_0^{t_1}(D_0 t^2)dt + \int_0^{t_1}(D_0 t\mu)dt \\ &\quad +(M-t_1)(\int_0^{t_1}(D_0 t)dt + \int_0^{t_1}(D_0\mu)dt)) \\ &= p_1 I_e(\tfrac{1}{3}D_0 t_1^3 + \tfrac{1}{2}D_0\mu t_1^2 + (M-t_1)(\tfrac{1}{2}D_0 t_1^2 + D_0\mu t_1)). \qquad (23) \end{aligned}$$

Therefore, the total minimum relevant cost per unit time is denoted by $TC(t_1)$ is given by

$$TC(t_1) = \begin{cases} TC_{11}(t_1) = \frac{A+HC+Sc+OC+DC+IP_{11}-IE_{11}}{T}; \; 0 < M \leq t_d \\ TC_{12}(t_1) = \frac{A+HC+SC+OC+DC+IP_{12}-IE_{12}}{T}; \; t_d < M \leq \mu \\ TC_{13}(t_1) = \frac{A+HC+Sc+OC+DC+IP_{13}-IE_{13}}{T}; \; \mu < M \leq t_1 \\ TC_{14}(t_1) = \frac{A+HC+SC+OC+DC+IP_{14}-IE_{14}}{T}; \; t_1 < M \leq T \end{cases}$$

4.2. Model 2. When $\mu < t_d$ -

For this situation, inventory level declines due to only demand rate in time period $[0,\mu]$ and $[\mu,t_d]$. After that, in time interval $[t_d,t_1]$ deterioration of item begins, and inventory level rapidly decreases due to demand and deterioration rate both reaches to zero inventory level. Finally, the shortage occurs (partial backlogging) in the time interval $[t_1,T]$. The mathematical formulation of this situation is given as below:

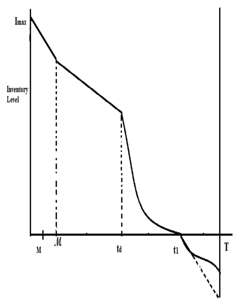

Figure 5: Inventory system for $0 < M \leq \mu$

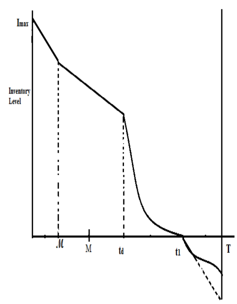

Figure 6: Inventory system for $\mu < M \leq t_d$

During the time interval $[0,\mu]$ the differential equation representing the inventory status is as below

$$\frac{dI_{21}(t)}{dt} = -D_0 t;\ 0 \le t \le \mu, \tag{24}$$

with boundary conditions as below: $I_{21}(0) = I_{max}$. In $[t_d, \mu]$, the differential equation representing the inventory status is given by,

$$\frac{dI_{22}(t)}{dt} = -D_0\mu;\ \mu \le t \le t_d, \tag{25}$$

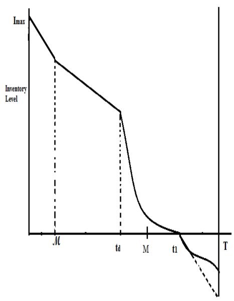

Figure 7: Inventory system for $t_d < M \le t_1$

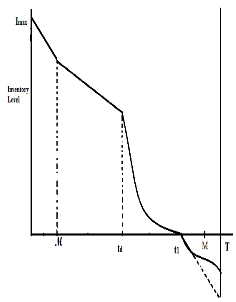

Figure 8: Inventory system for $t_1 < M \le T$

During the time interval $[t_d, t_1]$ market demand will be constant. So in this case, the differential equation representing the inventory status is given by;

$$\frac{dI_{23}(t)}{dt} + \theta I_{23}(t) = -D_0\mu; \quad t_d < t \le t_1, \tag{26}$$

with boundary conditions $I_{22}(t_d) = I_{23}(t_d)$ (by continuity of $t = t_d$) and $I_{23}(t_1) = 0$. In the time interval $[t_1, T]$ the differential equation representing the inventory level with partial backlog given as

$$\frac{dI_{24}(t)}{dt} = \frac{-D_0\mu}{1+\delta(T-t)};\ \ t_1 < t \le T, \tag{27}$$

with boundary conditions $I_{24}(t_1) = 0$. Now, solving all Equations with boundary conditions, we get

$$I_{21}(t) = \tfrac{(-D_0 t^2)}{2} + Imax \tag{28}$$

$$I_{22}(t) = -D_0\mu(t_d - t) + \tfrac{D_0\mu}{\theta}(e^{\theta(t_1 - t_d)} - 1) \tag{29}$$

$$I_{23}(t) = \frac{D_0\mu}{\theta}(e^{(\theta(t_1-t))} - 1), \tag{30}$$

and

$$I_{24}(t) = \frac{D_o\mu}{\delta}\log\left(\frac{1+\delta(T-t)}{1+\delta(T-t_1)}\right). \tag{31}$$

To find the value of I_max, we equate $I_{22}(t_d)$ = $I_{23}(t_d)$ (from continuity at $t = \mu$) and get

$$I_{max} = D_0\mu(t_d - \frac{\mu}{2}) + \frac{D_0\mu}{\theta}(e^{\theta(t_1-t_d)} - 1), \tag{32}$$

letting $t = T$, in Equation (31) and obtained the maximum amount of demand backlogged per cycle as

$$S = -I_{24}(T) = \frac{D_0}{\delta}\log(1+\delta(T-t_1)). \tag{33}$$

Hence, the order quantity per cycle is given by

$$\begin{aligned} Q &= I_{max} + S \\ &= D_0\mu(t_d + \frac{1}{\theta}(e^{(\theta(t_1-t_d))} - 1)) + \frac{D_0}{\delta}\log(1+\delta(T-t_1)). \end{aligned} \tag{34}$$

Now, we can obtained the total inventory cost per cycle, which consists the following costs -

i. A - The ordering cost.

ii. HC - The inventory holding cost

$$\begin{aligned} HC &= h\left[\int_0^{\mu} I_{21}(t)dt + \int_{\mu}^{t_d} I_{22}(t)dt + \int_{t_d}^{t_1} I_{23}(t)dt\right] \\ &\quad + a\left[\int_0^{\mu} tI_{21}(t)dt + \int_{\mu}^{t_d} tI_{22}(t)dt + \int_{t_d}^{t_1} tI_{23}(t)dt\right] \\ &= \frac{D_0\mu}{24}\Big((t_1-t_d)^2\big(4h(3+t_1-t_d) + a(t_1{}^2 + (8-3t_d)t_d + 2t_1(2+t_d))\big) \\ &\quad + 2(\mu-t_d)(6h(\mu-2t_1+t_d) + 2a(2\mu^2 - 3\mu t_1 + 2\mu t_d - 3t_1t_d + 2t_d{}^2) \\ &\quad - 3(t_1-t_d)^2(2h + a(\mu+t_d))\theta) \\ &\quad + \mu\big(-4h\mu - 3a\mu^2 - 6h(\mu-2t_d)(t_1-t_d)\theta(2+t_1\theta-t_d\theta) \\ &\quad - 3a\mu(\mu-2t_d)(t_1-t_d)\theta(2+t_1\theta-t_d\theta)\big)\Big). \end{aligned} \tag{35}$$

iii. SC - The shortage cost due to backlog

$$\begin{aligned} SC &= s\int_{t_1}^{T} -I_{24}(t)dt \\ &= \tfrac{sD_0\mu}{\delta^2}\log\left(\tfrac{1}{(1+\delta(T-t_1))}+\delta(T-t_1)\right). \end{aligned} \tag{36}$$

iv. OC - The opportunity cost due to lost sales

$$\begin{aligned} OC &= oD_0\mu\int_{t1}^{T}\left(1-\tfrac{1}{1+\delta(T-t)}\right)dt \\ &= \tfrac{oD_0\mu}{\delta}(-\log(1+\delta(T-t_1))+\delta(T-t_1)). \end{aligned} \tag{37}$$

v. DC - The deterioration cost is

$$\begin{aligned} DC &= p(S-[\int_{t_d}^{t_1} D_0\mu dt]) \\ &= \tfrac{D_0\mu p\theta}{6}(t_1-t_d)^2(3+t_1-t_d) \end{aligned} \tag{38}$$

vi. The interest payable - For this case, there are four cases arises given as

Case 1. When $0 < M \leq \mu$ - In this case, payment for items is settled and the retailer starts paying the capital opportunity cost for the items given as below

$$\begin{aligned} IP_{21} &= pI_p\left[\int_{M}^{\mu} I_{21}(t)dt+\int_{\mu}^{t_d} I_{22}(t)dt+\int_{t_d}^{t_1} I_{23}(t)dt\right] \\ &= \tfrac{D_0I_pp}{12}\Big(4\mu^3+2M^3-12\mu^2t_1+6\mu t_1{}^2+2\mu t_1{}^3-6\mu t_1{}^2t_d+6\mu t_1t_d{}^2-2\mu t_d{}^3 \\ &\quad -6\mu^2t_1{}^2\theta+12\mu^2t_1t_d\theta+6\mu t_1{}^2t_d\theta-6\mu^2t_d{}^2\theta-12\mu t_1t_d{}^2\theta+6\mu t_d{}^3\theta \\ &\quad -3\mu(\mu-M)(\mu-2t_d)(t_1-t_d)\theta(2+t_1\theta-t_d\theta)\Big). \end{aligned} \tag{39}$$

Case 2. When $\mu < M \leq t_d$ - In this case, the interest payable is

$$\begin{aligned} IP_{22} &= pI_p\left[\int_{M}^{t_d} I_{22}(t)dt+\int_{t_d}^{t_1} I_{23}(t)dt\right] \\ &= \tfrac{D_0I_p\mu p}{6}\Big(3M^2+t_1{}^3+3t_1{}^2(1+t_d(-1+\theta))+3t_1t_d{}^2(1-2\theta)+t_d{}^3(-1+3\theta) \\ &\quad -3M(t_1{}^2\theta+t_d{}^2\theta+t_1(2-2t_d\theta))\Big) \end{aligned} \tag{40}$$

Case 3. $t_d < M \leq t_1$ - For this case the interest payable is written as:

$$\begin{aligned} IP_{23} &= pI_p \int_M^{t_1} I_{23}(t)dt \\ &= -\frac{pI_pD_0\mu}{\theta^2}(1+t_1\theta - e^{(\theta(t_1-M))} - M\theta) \end{aligned} \tag{41}$$

Case 4. When $t_1 < M \leq T$ - In this case there is no opportunity cost, Therefore interest payable is 0, i.e.,

$$IP_4 = 0$$

vii. Interest earned - We assume that during the time when the account is not settled, the retailer sells the goods and continues to accumulate sales revenue and earns the interest rate I_e. Therefore, the interest earned per year (denote by IE) is given below for the four different cases,

Case 1. $0 < M \leq \mu$ - In this case, the interest earned is

$$\begin{aligned} IE_{21} &= p_1 I_e \int_0^M D_0 t^2 dt \\ &= \tfrac{1}{3} p I_e D_0 M^3 \end{aligned} \tag{42}$$

Case 2. $\mu < M \leq t_d$ - In this case, the interest earned is

$$\begin{aligned} IE_{22} &= p_1 I_e \left[\int_0^\mu D_0 t^2 dt + \int_\mu^M D_0 \mu t dt \right] \\ &= p_1 I_e D_0 \mu \left[\frac{\mu^2}{3} + \frac{M^2-\mu^2}{2} \right] \end{aligned} \tag{43}$$

Case 3. $t_1 < M \leq t_1$ - In this case, the interest earned is

$$\begin{aligned} IE_{23} &= p_1 I_e \left[\int_0^\mu D_0 t^2 dt + \int_\mu^M D_0 \mu t dt \right] \\ &= p_1 I_e D_0 \mu \left[\frac{\mu^2}{3} + \frac{M^2-\mu^2}{2} \right] \end{aligned} \tag{44}$$

Case 4. $t_1 < M \leq T$ - In this case, the interest earned is

$$\begin{aligned} IE_{24} &= p_1 I_e \Big(\int_0^{t_1} (D_0 t^2) dt + \int_0^{t_1} (D_0 t \mu) dt \\ &\quad + (M - t_1) \Big(\int_0^{t_1} (D_0 t) dt + \int_0^{t_1} (D_0 \mu) dt \Big) \Big) \\ &= p_1 I_e \left(\tfrac{1}{3} D_0 t_1^3 + \tfrac{1}{2} D_0 \mu t_1^2 + (M - t_1)\left(\tfrac{1}{2} D_0 t_1^2 + D_0 \mu t_1 \right) \right) \end{aligned} \tag{45}$$

Therefore, the total minimum relevant cost per unit time is denoted by $TC(t_1)$ is given by

$$TC(t_1) = \begin{cases} TC_{21}(t_1) = \frac{A+HC+Sc+OC+DC+IP_{21}-IE_{21}}{T}; 0 < M \leq \mu \\ TC_{22}(t_1) = \frac{A+HC+SC+OC+DC+IP_{22}-IE_{22}}{T}; \mu < M \leq t_d \\ TC_{23}(t_1) = \frac{A+HC+Sc+OC+DC+IP_{23}-IE_{23}}{T}; t_d < M \leq t_1 \\ TC_{24}(t_1) = \frac{A+HC+SC+OC+DC+IP_{24}-IE_{24}}{T}; t_1 < M \leq T \end{cases}$$

5. Theoretical Results

5.1. Theoretical Result for Model 1 -

Our theoretical proof is inspired by the proof of Geetha and Uthayakumar [2]. For simplicity, we use Geetha and Uthayakumar [2] approach in our proposed model.

Case 1. $0 < M \leq t_d$ - To obtain the first order necessary condition for $TC_{11}(t_1)$ to be minimized, we differentiate $TC_{11}(t_1)$ with respect to t_1 and set the result equal to zero, i.e.,

$$\frac{dTC_{11}(t_1)}{dt_1} = 0.$$

$$\begin{aligned} \frac{dTC_{11}(t_1)}{dt_1} = {} & \frac{D_0\mu}{6T}\Bigg\{-\frac{6Lo(T-t_1)}{1+L(T-t_1)} - \frac{6s(T-t_1)}{1+L(T-t_1)} + I_p p\Bigg(6t_1 + 3{t_1}^2 + 6\mu t_1(-1+\theta) - 3{t_d}^2\theta \\ & -3t_1{t_d}^2\theta^2 + 2{t_d}^3\theta^2 + 2t_1{t_d}^3\theta^3 + \mu^3\theta^2(1+t_1\theta) - 3\mu^2(-1+\theta+t_1\theta^2) \\ & -3M(1+t_1\theta)(2-2t_d\theta+{t_d}^2\theta^2)\Bigg) + h\Bigg(3{t_1}^2 + 6\mu t_1(-1+\theta) + \mu^3\theta^2(1+t_1\theta) \\ & +{t_d}^2\theta(-3+2t_d\theta) - 3\mu^2(-1+\theta+t_1\theta^2) + t_1(6-3{t_d}^2\theta^2+2{t_d}^3\theta^3)\Bigg) \\ & +p\theta\Bigg(-6\mu + 3\mu^2 + 6t_1 - 6\mu t_1 + 3{t_1}^2 + (\mu - t_d)(1+t_1\theta)(6-3t_d\theta+\mu^2\theta^2 \\ & +{t_d}^2\theta^2 + \mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \qquad (46) \end{aligned}$$

Now, put $t_1 = t_d$ in Equation (46), and equate as

$$
\begin{aligned}
\Delta_{11} &= \tfrac{D_0\mu}{6T}(-\tfrac{6\delta o(T-t_d)}{1+\delta(T-t_d)} - \tfrac{6s(T-t_d)}{1+\delta(T-t_d)} + p\theta(6\mu t_d(-1+\theta)+\mu^3\theta^2(1+t_d\theta) \\
&\quad -3\mu^2(-1+\theta+t_d\theta^2)+{t_d}^2(3-3\theta+2t_d\theta^2-{t_d}^2\theta^3)) \\
&\quad +h(6\mu t_d(-1+\theta)+\mu^3\theta^2(1+t_d\theta)-3\mu^2(-1+\theta \\
&\quad +t_d\theta^2)+t_d(6-3t_d(-1+\theta)-{t_d}^2\theta^2+2{t_d}^3\theta^3)) \\
&\quad +I_p p(6\mu t_d(-1+\theta)+\mu^3\theta^2(1+t_d\theta)-3\mu^2(-1+\theta+t_d\theta^2) \\
&\quad -3M(2-{t_d}^2\theta^2+{t_d}^3\theta^3)+t_d(6+3t_d-3t_d\theta-{t_d}^2\theta^2+2{t_d}^3\theta^3))). \quad (47)
\end{aligned}
$$

Lemma 5.1. **(a)** *If $\Delta_{11} \leq 0$ then, the optimal solution of t_1 say t_1^* not only exists but is unique.*

(b) *If $\Delta_{11} > 0$, then the optimal solution of t_1 is $t_1^* = t_d$.*

Proof. Put $t_1 = \chi$, where $\chi \in [t_d, t_1]$ in Equation (46), then Equation (46) becomes

$$
\begin{aligned}
\psi_{11}(\chi) &= \tfrac{D_0\mu}{6T}\Bigg\{-\tfrac{6Lo(T-\chi_1)}{1+L(T-\chi_1)} - \tfrac{6s(T-\chi_1)}{1+L(T-\chi_1)} + I_p p\Bigg(6\chi_1+3{\chi_1}^2 \\
&\quad +6\mu\chi_1(-1+\theta)-3{t_d}^2\theta-3\chi_1{t_d}^2\theta^2+2{t_d}^3\theta^2+2\chi_1{t_d}^3\theta^3 \\
&\quad +\mu^3\theta^2(1+\chi_1\theta)-3\mu^2(-1+\theta+\chi_1\theta^2)-3M(1+\chi_1\theta)(2-2t_d\theta+{t_d}^2\theta^2)\Bigg) \\
&\quad +h\Bigg(3{\chi_1}^2+6\mu\chi_1(-1+\theta)+\mu^3\theta^2(1+\chi_1\theta)+{t_d}^2\theta(-3+2t_d\theta) \\
&\quad -3\mu^2(-1+\theta+\chi_1\theta^2)+\chi_1(6-3{t_d}^2\theta^2+2{t_d}^3\theta^3)\Bigg) \\
&\quad +p\theta\Bigg(-6\mu+3\mu^2+6\chi_1-6\mu\chi_1+3{\chi_1}^2+(\mu-t_d) \\
&\quad (1+\chi_1\theta)(6-3t_d\theta+\mu^2\theta^2+{t_d}^2\theta^2+\mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \quad (48)
\end{aligned}
$$

Now, we differentiate Equation (48) with respect to χ_1, we get

$$
\begin{aligned}
\frac{d\psi_{11}}{d\chi} &= \tfrac{D_0\mu}{6T}\Bigg(\tfrac{6\delta o}{(1+L(T-\chi_1))^2} + \tfrac{6s}{(1+L(T-\chi_1))^2} + h(6+6\chi_1+6\mu(-1+\theta)-3\mu^2\theta^2 \\
&\quad -3t_d^2\theta^2+\mu^3\theta^3+2t_d^3\theta^3) \\
&\quad +I_p p\Bigg(6+6\chi_1+6\mu(-1+\theta)-6M\theta-3\mu^2\theta^2+6Mt_d\theta^2-3t_d^2\theta^2 \\
&\quad +\mu^3\theta^3-3Mt_d^2\theta^3+2t_d^3\theta^3\Bigg)+p\theta(6-6\mu+6\chi_1+(\mu-t_d)\theta(6-3(\mu+t_d)\theta \\
&\quad +(\mu^2+\mu t_d+t_d^2)\theta^2))\Bigg). \quad (49)
\end{aligned}
$$

By, our assumption of $D_0, \mu, \theta and \delta > 0$, it's clear that $\frac{d\psi_{11}}{d\chi} > 0$. Which easily establishes the convexity of our lemma. Hence the solution of above equations as $t_1 = t_1^*$ is not only exists, but unique also. □

Case 2. $t_d < M \leq \mu$ - To obtain the first order necessary condition for $TC_{12}(t_1)$ to be minimized, we differentiate $TC_{12}(t_1)$ with respect to t_1 and set the result equal to zero, i.e.,

$$\frac{dTC_{12}(t_1)}{dt_1} = 0.$$

$$\begin{aligned}
\frac{dTC_{12}(t_1)}{dt_1} &= \frac{D_0\mu}{6T}\Bigg\{-\frac{6Lo(T-t_1)}{1+L(T-t_1)} - \frac{6s(T-t_1)}{1+L(T-t_1)} + I_p p\Bigg(3t_1(2+t_1)+6\mu(1+t_1\theta) \\
&\quad -3\mu^2\theta(1+t_1\theta)+\mu^3\theta^2(1+t_1\theta)-M^3\theta^2(1+t_1\theta)-6M(2+t_1+t_1\theta) \\
&\quad +3M^2(1+\theta+t_1\theta^2)\Bigg)+h\Bigg(3{t_1}^2+6\mu t_1(-1+\theta)+\mu^3\theta^2(1+t_1\theta)+{t_d}^2\theta \\
&\quad (-3+2t_d\theta)-3\mu^2(-1+\theta+t_1\theta^2)+t_1(6-3{t_d}^2\theta^2+2{t_d}^3\theta^3)\Bigg) \\
&\quad +p\theta\Bigg(-6\mu+3\mu^2+6t_1-6\mu t_1+3{t_1}^2+(\mu-t_d)(1+t_1\theta)(6-3t_d\theta+\mu^2\theta^2 \\
&\quad +{t_d}^2\theta^2+\mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \qquad (50)
\end{aligned}$$

Put $t_1 = \mu$ in (50) and equate is

$$\begin{aligned}
\Delta_{12}(\mu) &= \frac{D_0\mu}{6T}\Bigg\{-\frac{6Lo(T-\mu)}{1+L(T-\mu)} - \frac{6s(T-\mu)}{1+L(T-\mu)} \\
&\quad +I_p p\Bigg(3\mu(2+\mu)+6\mu(1+\mu\theta)-3\mu^2\theta(1+\mu\theta)+\mu^3\theta^2(1+\mu\theta) \\
&\quad -M^3\theta^2(1+\mu\theta)-6M(2+\mu+\mu\theta)+3M^2(1+\theta+\mu\theta^2)\Bigg)+h\Bigg(3\mu^2 \\
&\quad +6\mu^2(-1+\theta)+\mu^3\theta^2(1+\mu\theta)+{t_d}^2\theta(-3+2t_d\theta)-3\mu^2(-1+\theta+\mu\theta^2) \\
&\quad +\mu(6-3{t_d}^2\theta^2+2{t_d}^3\theta^3)\Bigg)+p\theta\Bigg((\mu-t_d)(1+\mu\theta)(6-3t_d\theta+\mu^2\theta^2 \\
&\quad +{t_d}^2\theta^2+\mu\theta(-3+t_d\theta))\Bigg)\Bigg\} \qquad (51)
\end{aligned}$$

Lemma 5.2. **(a)** *If $\Delta_{12} \leq 0$ then, the optimal solution of t_1 say t_1^* not only exists but is unique.*

(b) *If $\Delta_{12} > 0$, then the optimal solution of t_1 is $t_1^* = \mu$.*

Proof. The proof is same as Lemma (5.1). □

Case 3. $\mu < M \leq t_1$ - To obtain the first order necessary condition for $TC_{13}(t_1)$ to be minimized, we differentiate $TC_{13}(t_1)$ with respect to t_1 and set the result equal to zero, i.e.,

$$\frac{dTC_{13}}{dt_1} = 0.$$

$$\begin{aligned}\frac{dTC_{13}}{dt_1} = {} & \tfrac{D_0\mu}{6T}\Bigg\{-\tfrac{6Lo(T-t_1)}{1+L(T-t_1)} - \tfrac{6s(T-t_1)}{1+L(T-t_1)} + 3I_p p(M^2 - 2M(1+t_1) + t_1(2+t_1)) \\ & + h\Bigg(3{t_1}^2 + 6\mu t_1(-1+\theta) + \mu^3\theta^2(1+t_1\theta) + {t_d}^2\theta(-3+2t_d\theta) \\ & -3\mu^2(-1+\theta+t_1\theta^2) + t_1(6 - 3{t_d}^2\theta^2 + 2{t_d}^3\theta^3)\Bigg) + p\theta\Bigg(-6\mu + 3\mu^2 \\ & +6t_1 - 6\mu t_1 + 3{t_1}^2 + (\mu - t_d)(1+t_1\theta)(6 - 3t_d\theta + \mu^2\theta^2 + {t_d}^2\theta^2 \\ & +\mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \qquad (52)\end{aligned}$$

Then, we put $t_1 = M$ in (52) and equate is as

$$\begin{aligned}\Delta_{13}(M) = {} & \tfrac{D_0\mu}{6T}\Bigg\{-\tfrac{6Lo(T-M)}{1+L(T-M)} - \tfrac{6s(T-M)}{1+L(T-M)} + h\Bigg(3M^2 + 6\mu M(-1+\theta) + \mu^3\theta^2(1+M\theta) \\ & +{t_d}^2\theta(-3+2t_d\theta) - 3\mu^2(-1+\theta+M\theta^2) + M(6 - 3{t_d}^2\theta^2 + 2{t_d}^3\theta^3)\Bigg) \\ & +p\theta\Bigg(-6\mu + 3\mu^2 + 6M - 6\mu M + 3M^2 + (\mu - t_d)(1+M\theta)(6 - 3t_d\theta \\ & +\mu^2\theta^2 + {t_d}^2\theta^2 + \mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \qquad (53)\end{aligned}$$

Lemma 5.3. **(a)** *If $\Delta_{13} \leq 0$ then, the optimal solution of t_1 say t_1^* not only exists but is unique.*

(b) *If $\Delta_{13} > 0$, then the optimal solution of t_1 is $t_1^* = M$.*

Proof. The proof is same as Lemma (5.1). □

Case 4. $t_1 < M \leq T$ - To obtain, the first order necessary condition for $TC_{14}(t_1)$ to be minimized, we differentiate $TC_{14}(t_1)$ with respect to t_1 and set the result equal to zero, i.e.,

$$\frac{dTC_{14}}{dt_1} = 0.$$

$$\begin{aligned}\frac{dTC_{14}}{dt_1} &= \frac{D_0}{6T}\Bigg\{-\frac{6Lo\mu(T-t_1)}{1+L(T-t_1)}-\frac{6s\mu(T-t_1)}{1+L(T-t_1)}+3I_ep_1(-2\mu M+2\mu t_1-2Mt_1+t_1{}^2)\\&+h\mu\Bigg(3t_1{}^2+6\mu t_1(-1+\theta)+\mu^3\theta^2(1+t_1\theta)+t_d{}^2\theta(-3+2t_d\theta)-3\mu^2(-1+\theta\\&+t_1\theta^2)+t_1(6-3t_d{}^2\theta^2+2t_d{}^3\theta^3)\Bigg)+\mu p\theta\Bigg(-6\mu+3\mu^2+6t_1-6\mu t_1+3t_1{}^2\\&+(\mu-t_d)(1+t_1\theta)(6-3t_d\theta+\mu^2\theta^2+t_d{}^2\theta^2+\mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \end{aligned} \tag{54}$$

Then, we put $t_1 = \mu$ in (54) and equate as, get,

$$\begin{aligned}\Delta_{141}(\mu) &= \frac{D_0}{6T}\Bigg\{-\frac{6Lo\mu(T-\mu)}{1+L(T-\mu)}-\frac{6s\mu(T-\mu)}{1+L(T-\mu)}+3I_ep_1(-4\mu M+3\mu^2)\\&+h\mu\Bigg(3\mu^2+6\mu^2(-1+\theta)+\mu^3\theta^2(1+\mu\theta)+t_d{}^2\theta(-3+2t_d\theta)-3\mu^2(-1+\theta\\&+\mu\theta^2)+\mu(6-3t_d{}^2\theta^2+2t_d{}^3\theta^3)\Bigg)+\mu p\theta\Bigg((\mu-t_d)(1+\mu\theta)(6-3t_d\theta\\&+(\mu^2+t_d^2)\theta^2+\mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \end{aligned} \tag{55}$$

Next, put $t_1 = M$ in (54), and equate as,

$$\begin{aligned}\Delta_{142}(M) &= \frac{D_0}{6T}\Bigg\{-\frac{6Lo\mu(T-M)}{1+L(T-M)}-\frac{6s\mu(T-M)}{1+L(T-M)}-3I_ep_1M^2+h\mu\Bigg(3M^2+6\mu M(-1+\theta)\\&+\mu^3\theta^2(1+M\theta)+t_d{}^2\theta(-3+2t_d\theta)-3\mu^2(-1+\theta+M\theta^2)+M(6-3t_d{}^2\theta^2\\&+2t_d{}^3\theta^3)\Bigg)+\mu p\theta\Bigg(-6\mu+3\mu^2+6M-6\mu M+3M^2+(\mu-t_d)(1+M\theta)\\&(6-3t_d\theta+(\mu^2+t_d^2)\theta^2+\mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \end{aligned} \tag{56}$$

Lemma 5.4. **(a)** *If $\Delta_{141} \le 0 \le \Delta_{142}$ then, the optimal solution of t_1 say t_1^* not only exist but unique and $t_1^* \in [\mu, M]$.*

(b) *If $\Delta_{141} > 0$, then the optimal solution of t_1 is $t_1^* = \mu$.*

(c) *If $\Delta_{142} \le 0$, then the optimal solution of t_1 is $t_1^* = M$.*

Proof. **Proof of** *a* - If $\Delta_{141} \le 0 \le \Delta_{142}$ then, we put $t_1 = \chi$ in (54) and we get,

$$\begin{aligned}\Psi_{14}(\chi) \quad = \quad & \tfrac{D_0}{6T}\Bigg\{-\tfrac{6Lo\mu(T-\chi)}{1+L(T-\chi)}-\tfrac{6s\mu(T-\chi)}{1+L(T-\chi)}+3I_e p_1(-2\mu M+2\mu\chi-2M\chi+\chi^2)\\ & +h\mu\Bigg(3\chi^2+6\mu\chi(-1+\theta)+\mu^3\theta^2(1+\chi\theta)+{t_d}^2\theta(-3+2t_d\theta)\\ & -3\mu^2(-1+\theta+\chi\theta^2)+\chi(6-3{t_d}^2\theta^2+2{t_d}^3\theta^3)\Bigg)+\mu p\theta\Bigg(-6\mu\\ & +3\mu^2+6\chi-6\mu\chi+3\chi^2+(\mu-t_d)(1+\chi\theta)(6-3t_d\theta+\mu^2\theta^2\\ & +{t_d}^2\theta^2+\mu\theta(-3+t_d\theta))\Bigg)\Bigg\}. \qquad (57)\end{aligned}$$

Since, the first-order derivative of $\Psi_{14}(\chi)$ with respect to $\chi \in [\mu, M]$ is

$$\tfrac{d\Psi_{14}(\chi)}{d\chi} > 0.$$

$\Psi_{14}(\chi)$ is a strictly increasing function of χ in the interval $\chi \in [\mu, M]$. Moreover, by assumption $\Delta_{141}(\mu) \leq 0$ and $\Delta_{142}(M) \geq 0$. That is, $\Delta_{141}(\mu) \leq 0 \leq \Delta_{142}(M)$. Thus, we can find a unique value $\chi \in [\mu, M]$ such that which implies that the solution of $\Psi_{14}(\chi) = 0$ not only exists and unique.

Proof of b - On the other hand, if $\Delta_{141} > 0$, then $\Psi_{14}(\mu) > 0$. then we get $\Psi_{14}(\chi) > 0$. So, TC_{14} is a strictly increasing function of T in the interval $[\mu, M]$. Thus, TC_{14} has a minimum value at $t_1^* = \mu$.

Proof of c - On the other hand, if $\Delta_{142} < 0$, then $\Psi_{14}(M) < 0$. Since, $\Psi_{14}(\chi)$ is a strictly increasing function of χ in the interval $[\mu, M]$. Thus, we get $\Psi_{14}(\chi) < 0$ for all $\chi \in [\mu, M]$. This implies that $\Psi_{14}(\chi) < 0$, for all $t_1 \in [\mu, M]$.So, TC_{14} is a strictly decreasing function of T in the interval $[\mu, M]$. Thus, TC_{14} has a minimum value at $t_1^* = M$. □

5.2. Theoretical Result For Model 2 -

Case 1. $0 < M \leq \mu$ - To obtain, the first order necessary condition for $TC_{21}(t_1)$ to be minimized, we differentiate $TC_{21}(t_1)$ with respect to t_1 and set the result equal to zero i.e.

$$\frac{dTC_{21}}{dt_1} = 0.$$

$$\begin{aligned}\frac{dTC_{21}}{dt_1} &= \tfrac{D_0\mu}{2T}\Bigg\{-\tfrac{(2\delta o(T-t_1))}{(1+\delta(T-t_1))}-\tfrac{2s(T-t_1)}{1+\delta(T-t_1)}+p({t_1}^2-2t_1(-1+t_d)+(-2+t_d)t_d)\theta\\ &-h\Bigg(-{t_1}^2-2t_1(1+t_d(-1+\theta))+{t_d}^2(-1+2\theta)+\mu^2\theta(1+t_1\theta-t_d\theta)\\ &-2\mu(1+t_1\theta-t_d\theta)(-1+t_d\theta)\Bigg)-I_pp\Bigg(-{t_1}^2+\mu^2\theta(1+t_1\theta-t_d\theta)\\ &-\mu(1+t_1\theta-t_d\theta)(-2+M\theta+2t_d\theta)+2t_1(-1+t_d-t_d\theta+Mt_d\theta^2)\\ &-t_d(t_d-2M\theta-2t_d\theta+2Mt_d\theta^2)\Bigg)\Bigg\}. \qquad (58)\end{aligned}$$

Now, put $t_1=\mu$ in Equation (58), and equate as

$$\begin{aligned}\Delta_{21} &= \tfrac{D_0\mu}{2T}\Bigg\{-\tfrac{(2\delta o(T-\mu))}{(1+\delta(T-\mu))}-\tfrac{2s(T-\mu)}{1+\delta(T-\mu)}+p(\mu^2-2\mu(-1+t_d)+(-2+t_d)t_d)\theta\\ &-h\Bigg(-\mu^2-2\mu(1+t_d(-1+\theta))+{t_d}^2(-1+2\theta)+\mu^2\theta(1+(\mu-t_d)\theta)\\ &-2\mu(1+\mu\theta-t_d\theta)(-1+t_d\theta)\Bigg)-I_pp\Bigg(-\mu^2+\mu^2\theta(1+(\mu-t_d)\theta)\\ &-\mu(1+(\mu-t_d)\theta)(-2+(M+2t_d)\theta)+2\mu(-1+t_d-t_d\theta+Mt_d\theta^2)\\ &-t_d(t_d-2(M+t_d)\theta+2Mt_d\theta^2)\Bigg)\Bigg\}. \qquad (59)\end{aligned}$$

Lemma 5.5. **(a)** *If $\Delta_{21}\le 0$ then, the optimal solution of t_1 say t_1^* not only exist but unique.*

(b) *If $\Delta_{21}>0$, then the optimal solution of t_1 is $t_1^*=\mu$.*

Proof. The proof is same as Lemma (5.1). □

Case 2. $\mu<M\le t_d$ - To obtain the first order necessary condition for $TC_{22}(t_1)$ to be minimized, the total cost per unit time $TC_{22}(t_1)$, differentiate $TC_{22}(t_1)$ with respect to t_1 and set the result equal to zero, i.e..

$$\frac{dTC_{22}}{dt_1}=0.$$

$$\begin{aligned}\frac{dTC_{22}}{dt_1} &= \tfrac{D_0\mu}{2T}\Bigg\{-\tfrac{(2\delta o(T-t_1))}{(1+\delta(T-t_1))}-\tfrac{2s(T-t_1)}{1+\delta(T-t_1)}+p({t_1}^2-2t_1(-1+t_d)+(-2+t_d)t_d)\theta\\ &+I_pp({t_1}^2+2t_1(1+t_d(-1+\theta))+{t_d}^2(1-2\theta)-2M(1+t_1\theta-t_d\theta))\\ &-h(-{t_1}^2-2t_1(1+t_d(-1+\theta))+{t_d}^2(-1+2\theta)+\mu^2\theta(1+t_1\theta-t_d\theta)\\ &-2\mu(1+t_1\theta-t_d\theta)(-1+t_d\theta))\Bigg\}. \qquad (60)\end{aligned}$$

Now, we put $t_1 = t_d$ in Equation (60), and equate

$$\begin{aligned}\Delta_{22} \quad = \quad & \tfrac{D_0\mu}{2T}\Big\{-\tfrac{(2\delta o(T-t_d))}{(1+\delta(T-t_d))}-\tfrac{2s(T-t_d)}{1+\delta(T-t_d)}+I_p p({t_d}^2+2t_d(1+t_d(-1+\theta))\\ & +{t_d}^2(1-2\theta)-2M)-h\Big(-{t_d}^2-2t_d(1+t_d(-1+\theta))+{t_d}^2(-1+2\theta)+\mu^2\theta\\ & (1+t_d\theta-t_d\theta)-2\mu(1+t_d\theta-t_d\theta)(-1+t_d\theta)\Big)\Big\}. \qquad (61)\end{aligned}$$

Lemma 5.6. **(a)** *If $\Delta_{22} \le 0$ then, the optimal solution of t_1 say t_1^* not only exist but unique.*

(b) *If $\Delta_{22} > 0$, then the optimal solution of t_1 is $t_1^* = t_d$.*

Proof. The proof is same as Lemma (5.1). □

Case 3. $t_d < M \le t_1$ - To obtain, the first order necessary condition for $TC_{23}(t_1)$ to be minimized, we differentiate $TC_{23}(t_1)$ with respect to t_1 and set the result equal to zero i.e.

$$\frac{dTC_{23}}{dt_1} = 0.$$

$$\begin{aligned}\frac{dTC_{23}}{dt_1} \quad = \quad & \tfrac{D_0\mu}{2T}\Big\{-\tfrac{(2\delta o(T-t_1))}{(1+\delta(T-t_1))}-\tfrac{2s(T-t_1)}{1+\delta(T-t_1)}+I_p p(M^2-2M(1+t_1)+t_1(2+t_1))\\ & +p({t_1}^2-2t_1(-1+t_d)+(-2+t_d)t_d)\theta\\ & -h\Big(-{t_1}^2-2t_1(1+t_d(-1+\theta))+{t_d}^2(-1+2\theta)\\ & +\mu^2\theta(1+(t_1-t_d)\theta)-2\mu(1+(t_1-t_d)\theta)(-1+t_d\theta)\Big)\Big\}. \qquad (62)\end{aligned}$$

Then, we put $t_1 = M$ in Equation (62) and equate as

$$\begin{aligned}\Delta_{23} \quad = \quad & \tfrac{D_0\mu}{2T}\Big\{-\tfrac{(2\delta o(T-M))}{(1+\delta(T-M))}-\tfrac{2s(T-M)}{1+\delta(T-M)}+p(M^2-2M(-1+t_d)+(-2+t_d)t_d)\theta\\ & -h\Big(-M^2-2M(1+t_d(-1+\theta))+{t_d}^2(-1+2\theta)+\mu^2\theta(1+(M-t_d)\theta)\\ & -2\mu(1+(M-t_d)\theta)(-1+t_d\theta)\Big)\Big\}. \qquad (63)\end{aligned}$$

Lemma 5.7. **(a)** *If $\Delta_{23} \leq 0$ then, the optimal solution of t_1 say t_1^* not only exist but unique.*

(b) *If $\Delta_{23} > 0$, then the optimal solution of t_1 is $t_1^* = M$.*

Proof. The proof is same as Lemma (5.1). □

Case 4. $t_1 < M \leq T$ - To obtain, the first order necessary condition for $TC_{24}(t_1)$ to be minimized, we differentiate $TC_{24}(t_1)$ with respect to t_1 and set the result equal to zero, i.e.,

$$\frac{dTC_{24}}{dt_1} = 0.$$

$$\begin{aligned}\frac{dTC_{24}}{dt_1} &= \frac{D_0\mu}{2T}\Bigg\{-\frac{(2\delta o(T-t_1))}{(1+\delta(T-t_1))} - \frac{2s(T-t_1)}{1+\delta(T-t_1)} + Iep1(-2\mu M + 2\mu t_1 - 2Mt_1 + {t_1}^2) \\ &\quad + \mu p({t_1}^2 - 2t_1(-1+t_d) + (-2+t_d)t_d)\theta - h\mu\Bigg(-{t_1}^2 - 2t_1(1+t_d(-1 \\ &\quad + \theta)) + {t_d}^2(-1+2\theta) + \mu^2\theta(1+t_1\theta - t_d\theta) \\ &\quad -2\mu(1+t_1\theta - t_d\theta)(-1+t_d\theta)\Bigg)\Bigg\}. \qquad (64)\end{aligned}$$

Then, put $t_1 = t_d$ in Equation (64), and equate as

$$\begin{aligned}\Delta_{241}(t_d) &= \frac{D_0\mu}{2T}\Bigg\{-\frac{(2\delta o(T-t_d))}{(1+\delta(T-t_d))} - \frac{2s(T-t_d)}{1+\delta(T-t_d)} + I_e p_1(-2\mu M + 2\mu t_d - 2Mt_d + {t_d}^2) \\ &\quad + \mu p({t_d}^2 - 2t_d(-1+t_d) + (-2+t_d)t_d)\theta - h\mu \\ &\quad \Bigg(-{t_d}^2 - 2t_d(1+t_d(-1+\theta)) + {t_d}^2(-1+2\theta) + \mu^2\theta(1+t_d\theta - t_d\theta) \\ &\quad -2\mu(1+t_d\theta - t_d\theta)(-1+t_d\theta)\Bigg)\Bigg\}. \qquad (65)\end{aligned}$$

Now, we again put $t_1 = M$ in Equation (64), and set as

$$\begin{aligned}\Delta_{242}(M) &= \frac{D_0\mu}{2T}\Bigg\{-\frac{(2\delta o(T-M))}{(1+\delta(T-M))} - \frac{2s(T-M)}{1+\delta(T-M)} - I_e p_1 M^2) + \mu p(M^2 - 2M(-1+t_d) \\ &\quad + (-2+t_d)t_d)\theta - h\mu\Bigg(-M^2 - 2M(1+t_d(-1+\theta)) + {t_d}^2(-1+2\theta) \\ &\quad + \mu^2\theta(1+M\theta - t_d\theta) - 2\mu(1+M\theta - t_d\theta)(-1+t_d\theta)\Bigg)\Bigg\}. \qquad (66)\end{aligned}$$

Lemma 5.8. **(a)** *If, $\Delta_{241} \le 0 \le \Delta_{242}$ then, the optimal solution of t_1 say t_1^* not only exist but unique and $t_1^* \in [\mu, M]$.*

(b) *If, $\Delta_{241} > 0$, then the optimal solution of t_1 is $t_1^* = t_d$.*

(c) *If, $\Delta_{242} \le 0$, then the optimal solution of t_1 is $t_1^* = M$.*

Proof. **proof of** (a) - If $\Delta_{241} \le 0 \le \Delta_{242}$ then, we put $t_1 = \chi$ in Equation (64) and we get,

$$\begin{aligned}\Psi_{24}(\chi) \quad = \quad & \tfrac{D_0\mu}{2T}\Bigg\{-\tfrac{(2\delta o(T-\chi))}{(1+\delta(T-\chi))} - \tfrac{2s(T-\chi)}{1+\delta(T-\chi)} + Iep1(-2\mu M \\ & +2\mu\chi - 2M\chi + \chi^2) + \mu p(\chi^2 - 2\chi(-1+t_d) + (-2+t_d)t_d)\theta \\ & -h\mu\Bigg(-\chi^2 - 2\chi(1+t_d(-1+\theta)) + {t_d}^2(-1+2\theta) + \mu^2\theta(1+\chi\theta \\ & -t_d\theta) - 2\mu(1+\chi\theta - t_d\theta)(-1+t_d\theta)\Bigg)\Bigg\}. \qquad (67)\end{aligned}$$

Since, the first-order derivative of $\Psi_{24}(\chi)$ with respect to $\chi \in [t_d, M]$ is

$$\tfrac{d\Psi_{24}(\chi)}{d\chi} > 0.$$

$\Psi_{24}(\chi)$ is a strictly increasing function of χ in the interval $\chi \in [\mu, M]$. Moreover, by assumption $\Delta_{241}(t_d) \le 0$ and $\Delta_{242}(M) \ge 0$. That is, $\Delta_{241}(t_d) \le 0 \le \Delta_{242}(M)$. Thus, we can find a unique value $\chi \in [t_d, M]$ such that which implies that the solution of $\Psi_{24}(\chi) = 0$ not only exists but also is unique.

proof of (b) - On the other hand, if $\Delta_{241} > 0$. Then, $\Psi_{24}(t_d) > 0$ and we get $\Psi_{24}(\chi) > 0$. So, $TC_{24}(t_1)$ is a strictly increasing function of T in the interval $[t_d, M]$. So $TC_{24}(t_1)$ has a minimum value at $t_1^* = t_d$.
proof of (c) - In the other way, if $\Delta_{242} < 0$, and $\Psi_{24}(M) < 0$. Since, $\Psi_{24}(\chi)$ is a strictly increasing function of x in the interval $[t_d, M]$, we can get $\Psi_{24}(\chi) < 0$ for all $\chi \in [t_d, M]$. This implies that $\Psi_{24}(\chi) < 0$, for all $t_1 \in [\mu, M]$.So, TC_{24} is a strictly decreasing function of T in the interval $[\mu, M]$. Hence, $TC_{24}(t_1)$ has a minimum value at $t_1^* = M$.

□

6. Computational Algorithm

6.1. For Model 1 -

The procedure to find the optimal solution of $TC(t_1)$ is given as below:

Step(1) Find the minimum of TC_{11} say t_1^* as follows:

(a) If $\Delta_{11} \le 0$ then, the optimal solution of t_1 say t_1^* exist and unique.

(b) If $\Delta_{11} > 0$, then the optimal solution of t_1 is $t_1^* = t_d$.

Step(2) Find the minimum of TC_{12} say t_1^* as follows:

(a) If $\Delta_{12} \leq 0$ then, the optimal solution of t_1 say t_1^* exist and unique.

(b) If $\Delta_{12} > 0$, then the optimal solution of t_1 is $t_1^* = \mu$.

Step(3) To find the minimum of TC_{13} say t_1^* as follows:

(a) If $\Delta_{13} \leq 0$ then, the optimal solution of t_1 say t_1^* exist and unique.

(b) If $\Delta_{13} > 0$, then the optimal solution of t_1 is $t_1^* = M$.

Step(4) To find the minimum of TC_{14} say t_1^* as follows:

(a) If $\Delta_{141}(\mu) \leq 0 \leq \Delta_{142}$, then the total annual inventory cost $TC_{14}(t_1)$ has a minimum value at the point $t_1^* \in [\mu, M]$.

(b) If $\Delta_{142} \leq 0$, then the total annual inventory cost $TC_{14}(t_1)$ has a minimum value at the point $t_1^* = M$.

(c) If $\Delta_{141} \geq 0$, then the total annual inventory cost $TC_{14}(t_1)$ has a minimum value at the point $t_1^* = \mu$.

Step(5) To find the minimum $TC(t_1)$, we find a $\min TC(t_1) = \min\{TC_{11}(t_1), TC_{12}(t_1), TC_{13}(t_1), TC_{14}(t_1)\}$ and accordingly select the optimal value of $t_1 = t_1^*$ and total relevant cost $TC(t_1)$.

6.2. For Model 2 -

The procedure to find the optimal solution of t_1^* is given as below:

Step(1) Find the minimum of TC_{21} say t_1^* as follows:

(a) If $\Delta_{21} \leq 0$ then, the optimal solution of t_1 say t_1^* exist and unique.

(b) If $\Delta_{21} > 0$, then the optimal solution of t_1 is $t_1^* = \mu$.

Step(2) Find the minimum of TC_{22} say t_1^* as follows:

(a) If $\Delta_{22} \leq 0$ then, the optimal solution of t_1 say t_1^* exist and unique.

(b) If $\Delta_{22} > 0$, then the optimal solution of t_1 is $t_1^* = t_d$.

Step(3) To find the minimum of TC_{23} say t_1^* as follows:

(a) If $\Delta_{23} \leq 0$ then, the optimal solution of t_1 say t_1^* exist and unique.

(b) If $\Delta_{23} > 0$, then the optimal solution of t_1 is $t_1^* = M$.

Step(4) To find the minimum of TC_{24} say t_1^* as follows:

(a) If $\Delta_{241}(t_d) \leq 0 \leq \Delta_{242}(M)$, then the total annual inventory cost $TC_{14}(t_1)$ has a minimum value at the point $t_1^* \in [\mu, M]$.

(b) If $\Delta_{242} \leq 0$, then the total annual inventory cost $TC_{14}(t_1)$ has a minimum value at the point $t_1^* = M$.

(c) If $\Delta_{241} \geq 0$, then the total annual inventory cost $TC_{14}(t_1)$ has a minimum value at the point $t_1^* = t_d$.

Step(5) To find the minimum $TC(t_1)$, we find a $\min TC(t_1) = \min\{TC_{21}(t_1), TC_{22}(t_1), TC_{23}(t_1), TC_{24}(t_1)\}$ and accordingly select the optimal value of $t_1 = t_1^*$ and total relevant cost $TC(t_1)$.

7. Numerical Examples

Example 7.1. In order to illustrate the solution procedure, let us consider an inventory system with the following data -

$A = 250$, $h = 1$, $s = 25$, $O = 30$, $p = 80$, $D_0 = 1000$, $I_p = 0.15$, $I_e = 0.12$, $p_1 = 85$, $\mu = 1$ weeks, $\theta = 0.01$, $\delta = 0.56$, $M = 0.4$, $t_d = 0.5$ weeks, $a = 0.01$ and $T = 30$ weeks. Since, here $t_d < M$ thus, applying the algorithm given in Section (6.1) we find that $t_1^* = 3.19$ weeks. Then $TC_{11} = 2.77259 \times 10^6$ \$ per unit and the optimum order quantity $Q^* = 9.96 \times 10^6$ \$ per unit.

Example 7.2. In order to illustrate the solution procedure, let us consider an inventory system with the following data -
$A = 250$, $h = 1$, $s = 25$, $O = 30$, $p = 80$, $D_0 = 1000$, $I_p = 0.15$, $I_e = 0.12$, $p_1 = 85$, $\mu = 1$ weeks, $\theta = 0.01$, $\delta = 0.56$, $M = 0.9$ weeks, $t_d = 0.5$ weeks, $a = 0.01$ and $T = 30$ weeks. Since this is the Case $t_d < M$. Applying the algorithm given in Section (6.1), we found that $t_1^* = 2.79$ weeks. Then $TC_{12} = 84675.4$ \$ per unit and the optimum order quantity $Q^* = 157082$ \$ per unit.

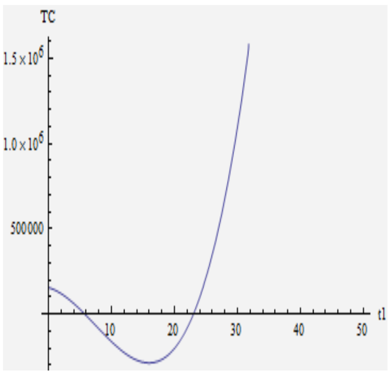

Figure 9: Graphical representation of TC_{14} with respect to t_1

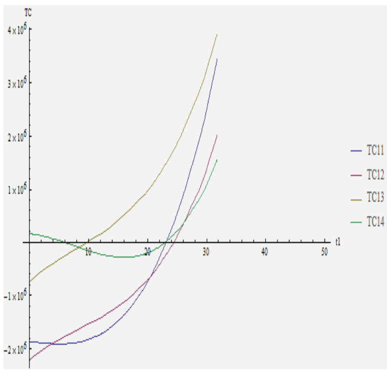

Figure 10: Graphical representation of $TC_{11}, TC_{12}, TC_{13}, TC_{14}$ with respect to t_1

Example 7.3. In order to illustrate the solution procedure, let us consider an inventory system with the following data
$A = 250$, $h = 1$, $s = 25$, $O = 30$, $p = 80$, $D_0 = 1000$, $I_p = 0.15$, $I_e = 0.12$, $p_1 = 85$, $\mu = 1$ weeks, $\theta = 0.01$, $\delta = 0.56$, $M = 1.2$ weeks, $t_d = 0.5$ weeks, $a = 0.01$ and $T = 30$ weeks. Since this is the Case $t_d < M$. Applying the algorithm given in Section (6.1), we found that $t_1^* = 3.00$ weeks. Then $TC_{13} = 78287.1$ \$ per unit and the optimum order quantity $Q^* = 157317$ \$ per unit.

Example 7.4. In order to illustrate the solution procedure, let us consider an inventory system with the following data
$A = 250$, $h = 1$, $s = 25$, $O = 30$, $p = 80$, $D_0 = 1000$, $I_p = 0.15$, $I_e = 0.12$, $p_1 = 85$, $\mu = 1$ weeks, $\theta = 0.01$, $\delta = 0.56$, $M = 24$ weeks, $t_d = 0.5$ weeks, $a = 0.01$ and $T = 30$ weeks. Now, applying algorithm Step (4), we see that, $\Delta_{141} \leq 0 \leq \Delta_{142}$, does not hold and $\Delta_{141} \geq 0$. Thus we set $t_1^* = \mu = 1$ weeks. Then $TC_{14} = 69291.9$ \$ per unit and the optimum order quantity is $Q^* = 155208$ \$ per unit.

Thus, the total annual cost $(TC) = \min\{TC_{11}, TC_{12}, TC_{13}, TC_{14}\}$. Then the total minimum cost $TC = TC_{14} = 69291.9$ \$ per unit and the optimum order quantity $Q^* = 155208$ \$ per unit. One can understand this through graphically.

Example 7.5. In order to illustrate the solution procedure, let us consider an inventory system with the following data
$A = 150$, $h = 0.6$, $a = 0.01$, $s = 3$, $0 = 6$, $p = 80$, $D_0 = 1000$, $I_p = 0.15$, $I_e = 0.12$, $p_1 = 85$, $\theta = 0.01$, $\delta = 0.56$, $\mu = 0.5$ weeks, $M = 0.4$ weeks, $T = 30$ weeks, $a = 0.01$ and $t_d = 1.5$ weeks. Since this is the case $M > t_d$. Applying the algorithm given in Section (6.2), find that $t_1^* = 3.67$ weeks. Then $TC_{21} = 29504.8$ \$ per unit and the optimum order quantity $Q^* = 4936.62$ \$ per unit.

Example 7.6. In order to illustrate the solution procedure, let us consider an inventory system with the following data
$A = 150, h = 5, s = 3, 0 = 6, p = 80, D_0 = 1000, I_p = 0.15, I_e = 0.12, p_1 = 85, \theta = 0.01$, $\delta = 0.56$, $\mu = 0.5$ weeks, $M = 1$ weeks, $T = 30$ weeks, $a = 0.01$ and $t_d = 1.5$ weeks. Since this is the case $t_d < M$. Applying the algorithm given in Section (6.1) we find that $t_1^* = 3.80$ weeks. Then $TC_{22} = 29136.2$ \$ per unit and the optimum order quantity $Q^* = 4928.6$ \$ per unit.

Example 7.7. In order to illustrate the solution procedure, let us consider an inventory system with the following data
$A = 150, h = 5, s = 3, 0 = 6, p = 80, D_0 = 1000, I_p = 0.15, I_e = 0.12, p_1 = 85, \theta = 0.01$, $\delta = 0.56$, $\mu = 0.5$ weeks, $M = 2.5$ weeks, $T = 30$ weeks, $a = 0.01$ and $t_d = 1.5$ weeks. Since this is the case $t_d < M$. Applying the algorithm given in Section (6.1) we find that $t_1^* = 4.82$ weeks. Then $TC_{23} = 27379$ \$ per unit and the optimum order quantity $Q^* = 4869.46$ \$ per unit.

Example 7.8. In order to illustrate the solution procedure, let us consider an inventory system with the following data
$A = 150, h = 5, s = 3, 0 = 6, p = 80, D_0 = 1000, I_p = 0.15, I_e = 0.12, p_1 = 85, \theta = 0.01$,

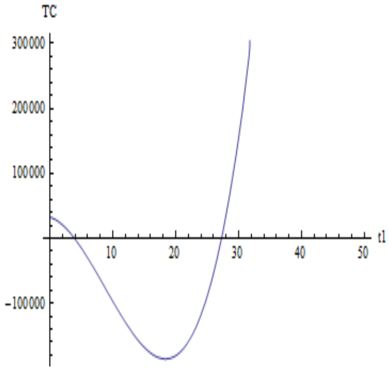

Figure 11: Graphical representation of TC_{14} with respect to t_1

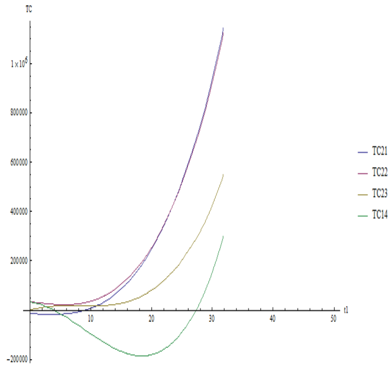

Figure 12: Graphical representation of $TC_{11}, TC_{12}, TC_{13}, TC_{14}$ with respect to t_1

$\delta = 0.56$, $\mu = 0.5$ weeks, $M = 10$ weeks, $T = 30$ weeks, $a = 0.01$ and $t_d = 1.5$ weeks. Now, applying algorithm (6.2) Step (4), we see that, $\Delta_{241} \leq 0 \leq \Delta_{242}$, does not hold and $\Delta_{241} \geq 0$. Thus we set $t_1^* = t_d = 1.5$ weeks. Thus, $TC_{24} = 24722.1$ \$ per unit and the optimum order quantity is $Q^* = 5055.1$ \$ per unit.

Thus the total annual cost $(TC) = \min\{TC_{11}, TC_{12}, TC_{13}, TC_{14}\}$. Then the total minimum cost $TC = TC_{24} = 24722.1$ \$ per unit and the optimum order quantity $Q^* = 5055.1$ \$ per unit. One can understand this through graphically.

8. Conclusion

In this paper, we have presented an appropriate inventory model for non-instantaneous deteriorating items, where supplier offers to retailer a permissible delay in payment to increase its own selling price. The main purpose of this model is to develop an inventory model for such types of items, which has ramped type demand rate and non-instantaneous deterioration. Constant holding cost hasn't seemed to be a realistic assumption in some real world business problems. Here, we assumed two cases, first in which delay time period is less than to non-instantaneous deterioration rate and second, where the delay time period is greater than to non-instantaneous deterioration rate. We discussed the solution procedure, thus one can easily find the minimum total relevant costs.

This model is more applicable for some new brands of consumer goods (say, cosmetic products, seasonal products, and so on). When those types of items are entered in the market the demand rate of consumer goods are increasing at the beginning, and then remains constant for the rest period of time. For future works, one can extend the model by considering a non-zero lead time, finite replenishment, inflation, two level trade credits, partial trade credit, for non-linear demands, fuzzy sets, warehouse problems, etc.

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